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Strongly unique best simulations approximation in linear 2-normed spaces

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Abstract

In this paper we established some basic properties of the set of strongly unique best simultaneous approximation in the context of linear 2-normed space.

Keywords: Linear 2-normed space, strongly unique best approximation, best simultaneous approximation and strongly unique best simultaneous approximation.

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1 Introduction

The problem of simultaneous approximation was studied by several authors. Diaz and McLaughlin [2,3], Dunham [4] and Ling, et al.[8] have considered the simultaneous approximation of two real-valued functions defined on a closed interval $[a, b]$. Several results related with best simultaneous approximation in the context of normed linear space under different norms were obtained by Goel, et al. [5,6], Phillips, et al. [11], Dunham [4] and Ling, et al. [8]. Strongly unique best simultaneous approximation are investigated by Laurent, et al. [7]. Pai, et al. [9,10] studied the characterization and unicity of strongly unique best simultaneous approximation in normed linear spaces. The notion of strongly unique best simultaneous approximation in the context of linear 2-normed spaces is introduced in this paper. Section 2 gives some important definitions and results that are used in the sequel. Some fundamental properties of the set of strongly unique best simultaneous approximation with respect to 2-norm are established in Section 3.

2 Preliminaries

Definition 2.1. *Let X be a linear space over real numbers with dimension greater than one and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following properties for all x, y, z in X.*

(i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, *where α is a real number,*

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Then $\|.,.\|$ is called a 2-norm and the linear space X equipped with the 2-norm is called a linear 2-normed space. It is clear that 2-norm is non-negative.

The following important property of 2-norm was established by Cho [1].

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Theorem 2.1. *[1] For any points $x, y \in X$ and any $\alpha \in \mathbb{R}$,*

$$\|x, y\| = \|x, y + \alpha x\|.$$

Definition 2.2. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best approximation to $x \in X$ from G, if there exists a constant $t > 0$ such that for all $g \in G$,*

$$\|x - g_0, k\| \leq \|x - g, k\| - t||g - g_0, k||, \quad \text{for all } k \in X \backslash [G, x].$$

Definition 2.3. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a best simultaneous approximation to $x_1, \cdots, x_n \in X$ from G if for all $g \in G$,*

$$\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\},$$

$$\text{for all } k \in X \backslash [G, x_1, \cdots, x_n].$$

The definition of strongly unique best simultaneous approximation in the context of linear 2-normed space is introduced here for the first time as follows:

Definition 2.4. *Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best simultaneous approximation to $x_1, \cdots, x_n \in X$ from G, if there exists a constant $t > 0$ such that for all $g \in G$,*

$$\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||,$$

$$\text{for all } k \in X \backslash [G, x_1, \cdots, x_n],$$

where $[G, x_1, \cdots, x_n]$ represents a linear space spanned by elements of G and $x_1, \cdots, x_n$. Let $Q_G(x_1, \cdots, x_n)$ denote the set of all elements of strongly unique best simultaneous approximations to $x_1, \cdots, x_n \in X$ from G. The subset G is called an existence set if $Q_G(x_1, \cdots, x_n)$ contains at least one element for every $x \in X$. G is called a uniqueness set if $Q_G(x_1, \cdots, x_n)$ contains at most one element for every $x \in X$. G is called an existence and uniqueness set if $Q_G(x_1, \cdots, x_n)$ contains exactly one element for every $x \in X$.

3 Some fundamental properties of $Q_G(x_1, \cdots, x_n)$

Some basic properties of strongly unique best simultaneous approximation are obtained in the following Theorems.

Theorem 3.1. *Let G be a non-empty subset of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.*

(i) $Q_G(x_1, \cdots, x_n)$ is closed if G is closed.

(ii) $Q_G(x_1, \cdots, x_n)$ is convex if G is convex.

(iii) $Q_G(x_1, \cdots, x_n)$ is bounded.

Proof. (i). Let G be closed.

Let $\{g_m\}$ be a sequence in $Q_G(x_1, \cdots, x_n)$ such that $g_m \to \tilde{g}$.

To prove that $Q_G(x_1, \cdots, x_n)$ is closed, it is enough to show that $\tilde{g} \in Q_G(x_1, \cdots, x_n)$.

Since G is closed, $\{g_m\} \in G$ and $g_m \to \tilde{g}$, we have $\tilde{g} \in G$. Since $\{g_m\} \in Q_G(x_1, \cdots, x_n)$, we have for all $k \in X \backslash [G, x_1, \cdots, x_n], g \in G$ and for some $t > 0$ that

$$\begin{aligned}
& \max\{\|x_1 - g_m, k\|, \cdots, \|x_n - g_m, k\|\} \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_m, k||. \\
\Rightarrow \quad & \max\{\|x_1 - \tilde{g}, k\| - \|g_m - \tilde{g}, k\|, \cdots, \|x_n - \tilde{g}, k\| - \|g_m - \tilde{g}, k\|\} \\
& \quad \leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_m, k|| \qquad (3.1)
\end{aligned}$$

Since $g_m \to \tilde{g}$, $g_m - \tilde{g} \to 0$. So $\|g_m - \tilde{g}, k\| \to 0$, since 0 and k are linearly dependent.

Therefore, it follows from (3.1) that

$$\max\{\|x_1 - \tilde{g}, k\|, \cdots, \|x_n - \tilde{g}, k\|\} \le \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - \tilde{g}, k||,$$
$$\text{for all } g \in G \text{ and } k \in X\backslash[G, x_1, \ldots, x_n], \quad \text{when } m \to \infty.$$

Thus $\tilde{g} \in Q_G(x_1, \cdots, x_n)$. Hence $Q_G(x_1, \cdots, x_n)$ is closed.
(ii). Let G be a convex set, $g_1, g_2 \in Q_G(x_1, \cdots, x_n)$ and $0 < \alpha < 1$. To show that $\alpha g_1 + (1-\alpha)g_2 \in Q_G(x_1, \cdots, x_n)$, let $k \in X \setminus [G, x_1, \cdots, x_n]$.
Then

$\max\{\|x_1 - (\alpha g_1 + (1-\alpha)g_2), k\|, \cdots, \|x_n - (\alpha g_1 + (1-\alpha)g_2), k\|\}$
$\le \max\{\alpha||x_1 - g_1, k|| + (1-\alpha)||x_1 - g_2, k||, \cdots, \alpha\|x_n - g_1, k\| + (1-\alpha)||x_n - g_2, k\|\}$
$\le \max\{\alpha\|x_1 - g_1, k\|, \cdots, \alpha\|x_n - g_1, k\|\} + \max\{(1-\alpha)\|x_1 - g_2, k\|, \cdots,$
$(1-\alpha)\|x_n - g_2, k\|\}$
$\le \alpha(\max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_1, k||)$
$+ (1-\alpha)(\max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_2, k||)$, for all $g \in G$ and for some $t > 0$.
$= \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t(\|\alpha g - \alpha g_1, k\| + \|(1-\alpha)g - (1-\alpha)g_2, k\|)$
$\le \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t\|\alpha g - \alpha g_1 + (1-\alpha)g - (1-\alpha)g_2, k\|$
$= \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - (\alpha g_1 + (1-\alpha)g_2), k\|.$

Thus $\alpha g_1 + (1-\alpha)g_2 \in Q_G(x_1, \cdots, x_n)$. Hence $Q_G(x_1, \cdots, x_n)$ is convex.
(iii). To prove that $Q_G(x_1, \cdots, x_n)$ is bounded, it is enough to prove for arbitrary $g_0, \tilde{g_0} \in Q_G(x_1, \cdots, x_n)$ that $\|g_0 - \tilde{g_0}, k\| < C$ for some $C > 0$, since $\|g_0 - \tilde{g_0}, k\| < C$ implies that $\sup\limits_{g_0, \tilde{g_0} \in Q_G(x_1, \ldots, x_n)} \|g_0 - \tilde{g_0}, k\|$ is finite and hence the diameter of $Q_G(x_1, \cdots, x_n)$ is finite.
Let $g_0, \tilde{g_0} \in Q_G(x_1, \cdots, x_n)$. Then there exists a constant $t > 0$ such that for all $g \in G$ and $k \in X \setminus [G, x_1, \cdots, x_n]$,
$\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \le \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||$
and
$\max\{\|x_1 - \tilde{g_0}, k\|, \cdots, \|x_n - \tilde{g_0}, k\|\} \le \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - \tilde{g_0}, k||.$
Now,

$$\begin{aligned} \|g - g_0, k\| &\le \|x_1 - g, k\| + \|x_1 - g_0, k\| \\ &\le 2\max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||. \end{aligned}$$

Thus $\|g - g_0, k\| \le \frac{2}{1+t} \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\}$ for all $g \in G$.
Hence $\|g - g_0, k\| \le \frac{2}{1+t} d$,
where $d = \inf\limits_{g \in G} \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\}$.
Similarly, $\|g - \tilde{g_0}, k\| \le \frac{2}{1+t} d$.
Therefore, it follows that

$$\begin{aligned} \|g_0 - \tilde{g_0}, k\| &\le \|g_0 - g, k\| + \|g - \tilde{g_0}, k\| \\ &\le \frac{4}{1+t} d \\ &= C. \end{aligned}$$

Hence $Q_G(x_1, \cdots, x_n)$ is bounded.
Let X be a linear 2-normed space, $x \in X$ and $[x]$ denote the set of all scalar multiplications of x.

$$\text{i.e.,} \qquad [x] = \{\alpha x : \alpha \in \mathbb{R}\}.$$

□

Theorem 3.2. *Let G be a subset of a linear 2-normed space $X, x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for all $y \in [k]$.*

(i) $g_0 \in Q_G(x_1, \cdots, x_n)$.

(ii) $g_0 \in Q_G(x_1 + y, \cdots, x_n + y)$.

(iii) $g_0 \in Q_G(x_1 - y, \cdots, x_n - y)$.

(iv) $g_0 + y \in Q_G(x_1 + y, \cdots, x_n + y)$.

(v) $g_0 + y \in Q_G(x_1 - y, \cdots, x_n - y)$.

(vi) $g_0 - y \in Q_G(x_1 + y, \cdots, x_n + y)$.

(vii) $g_0 - y \in Q_G(x_1 - y, \cdots, x_n - y)$.

(viii) $g_0 + y \in Q_G(x_1, \cdots, x_n)$.

(ix) $g_0 - y \in Q_G(x_1, \cdots, x_n)$.

Proof. The proof follows immediately by using Theorem 2.1. □

Theorem 3.3. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then*

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0),$$

$$\textit{for all } \alpha \in \mathbb{R} \text{ and } m = 0, 1, 2, \cdots$$

Proof. Claim:

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0), \text{ for all } \alpha \in \mathbb{R}.$$

Let $g_0 \in Q_G(x_1, \cdots, x_n)$. Then

$$\begin{aligned}
&\max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \\
&\leq \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t\|g - g_0, k\|, \text{ for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow\ &\max\{\|\alpha x_1 - \alpha g_0, k\|, \cdots, \|\alpha x_n - \alpha g_0, k\|\} \\
&\leq \max\{\|\alpha x_1 - \alpha g, k\|, \cdots, \|\alpha x_n - \alpha g, k\|\} - t\|\alpha g - \alpha g_0, k\|\}, \text{ for all } g \in G. \\
\Rightarrow\ &\max\{\|\alpha x_1 - \alpha g_0, k\|, \cdots, \|\alpha x_n - \alpha g_0, k\|\} \\
&\leq \max\left\{\|\alpha x_1 - \alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right), k\|, \cdots, \|\alpha x_n - \alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right), k\|\right\} \\
&\quad -t\|\alpha\left(\frac{(\alpha-1)g_0 + g}{\alpha}\right) - \alpha g_0, k\|, \text{ for all } g \in G \text{ and } \alpha \neq 0, \text{ since } \frac{(\alpha-1)g_0 + g}{\alpha} \in G. \\
\Rightarrow\ &\max\{\|\alpha x_1 + (1-\alpha)g_0 - g_0, k\|, \cdots, \|\alpha x_n + (1-\alpha)g_0 - g_0, k\|\} \\
&\leq \max\{\|\alpha x_1 + (1-\alpha)g_0 - g, k\|, \cdots, \|\alpha x_n + (1-\alpha)g_0 - g, k\|\} - t\|g - g_0, k\|.
\end{aligned}$$

$\Rightarrow g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0)$, when $\alpha \neq 0$.
If $\alpha = 0$, then it is clear that $g_0 \in Q_G(\alpha x_1 + (1-\alpha)g_0, \cdots, \alpha x_n + (1-\alpha)g_0)$.

The converse is obvious by taking $\alpha = 1$. Hence the claim is true. □

Corollary 3.1. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for all $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$*

(i) $g_0 \in Q_G(x_1, \cdots, x_n)$.

(ii) $g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(iii) $g_0 \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(iv) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(v) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(vi) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(vii) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(viii) $g_0 + y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

(ix) $g_0 - y \in Q_G(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

Proof. The proof follows immediately from simple application of Theorem 2.2 and Theorem 3.3. □

Theorem 3.4. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then*

$$g_0 \in Q_G(x_1, \cdots, x_n) \Leftrightarrow g_0 \in Q_{G+[k]}(x_1, \cdots, x_n).$$

Proof. The proof follows from a simple application of Theorem 3.2. □

A corollary similar to that of Corollary 3.4 is established next as follows:

Corollary 3.2. *Let G be a subspace of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. Then the following statements are equivalent for all $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$*

(i) $g_0 \in Q_{G+[k]}(x_1, \cdots, x_n)$.

(ii) $g_0 \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(iii) $g_0 \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(iv) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(v) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(vi) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 + y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 + y)$.

(vii) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0 - y, \cdots, \alpha^m x_n + (1-\alpha^m)g_0 - y)$.

(viii) $g_0 + y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

(ix) $g_0 - y \in Q_{G+[k]}(\alpha^m x_1 + (1-\alpha^m)g_0, \cdots, \alpha^m x_n + (1-\alpha^m)g_0)$.

Proof. The proof easily follows from Theorem 3.5 and Corollary 3.4. □

Proposition 3.1. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$, $k \in X \setminus [G, x_1, \cdots, x_n]$ and $0 \in G$.* If *$g_0 \in Q_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that*
$\max\{\| x_1 - g_0, k \|, \cdots, \| x_n - g_0, k \|\} \le \max\{\| x_1, k \|, \cdots, \| x_n, k \|\} - t||g_0, k||$.

Proof. The proof is obvious. □

Proposition 3.2. *Let G be a subset of a linear 2-normed space X, $x_1, \cdots, x_n \in X$ and $k \in X \setminus [G, x_1, \cdots, x_n]$. If $g_0 \in Q_G(x_1, \cdots, x_n)$, then there exists a constant $t > 0$ such that for all $g \in G$,*
$||g - g_0, k|| \le 2\max\{||x_1 - g, k||, \cdots, ||x_n - g, k||\} - t||g - g_0, k||$.

Proof. The proof is trivial. □

Theorem 3.5. *Let G be a subspace of a linear 2-normed space X and $x_1, \cdots, x_n \in X$. Then the following statements hold.*
(i) $Q_G(x_1 + g, \cdots, x_n + g) = Q_G(x_1, \cdots, x_n) + g$, for all $g \in G$.
(ii) $Q_G(\alpha x_1, \cdots, \alpha x_n) = \alpha Q_G(x_1, \cdots, x_n)$, for all $\alpha \in \mathbb{R}$.

Proof. (i). Let $\tilde{g}$ be an arbitrary but fixed element of G.

Let $g_0 \in Q_G(x_1, \cdots, x_n)$. It is clear that $g_0 + \tilde{g} \in Q_G(x_1, \cdots, x_n) + \tilde{g}$.

To prove that $Q_G(x_1, \cdots, x_n) + \tilde{g} \subseteq Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$, it is enough to show that $g_0 + \tilde{g} \in Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.

Now,

$$\begin{aligned}
& \max\{\|x_1 + \tilde{g} - g_0 - \tilde{g}, k\|, \cdots, \|x_n + \tilde{g} - g_0 - \tilde{g}, k\|\} \\
\le\ & \max\{\|x_1 - g, k\|, \cdots, \|x_n - g, k\|\} - t||g - g_0, k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow\ & \max\{\|x_1 + \tilde{g} - (g_0 + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g_0 + \tilde{g}), k\|\} \\
\le\ & \max\{\|x_1 + \tilde{g} - g, k\|, \cdots, \|x_n + \tilde{g} - g, k\|\} - t||g - (g_0 + \tilde{g}), k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0, \text{ since } g - \tilde{g} \in G.
\end{aligned}$$

Thus $g_0 + \tilde{g} \in Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$.

Conversely, let $g_0 + \tilde{g} \in Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g})$. To prove that $Q_G(x_1 + \tilde{g}, \cdots, x_n + \tilde{g}) \subseteq Q_G(x_1, \cdots, x_n) + \tilde{g}$, it is enough to show that $g_0 \in Q_G(x_1, \cdots, x_n)$. Let $k \in X \setminus [G, x_1, \cdots, x_n]$.

$$\begin{aligned}
\text{Then} \quad & \max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \\
= \quad & \max\{\|x_1 + \tilde{g} - (g_0 + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g_0 + \tilde{g}), k\|\} \\
\leq \quad & \max\{\|x_1 + \tilde{g} - g, k\|, \cdots, \|x_n + \tilde{g} - g, k\|\} - t||g - (g_0 + \tilde{g}), k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0. \\
\Rightarrow \quad & \max\{\|x_1 - g_0, k\|, \cdots, \|x_n - g_0, k\|\} \\
& \leq \max\{\|x_1 + \tilde{g} - (g + \tilde{g}), k\|, \cdots, \|x_n + \tilde{g} - (g + \tilde{g}), k\|\} \\
& \qquad -t||(g + \tilde{g}) - (g_0 + \tilde{g}), k||, \\
& \qquad \text{for all } g \in G \text{ and for some } t > 0, \text{ since } g + \tilde{g} \in G. \\
\Rightarrow \quad & g_0 \in Q_G(x_1, \cdots, x_n). \text{ Thus the result follows.}
\end{aligned}$$

(ii). The proof is similar to that of (i). □

Remark 3.1. *Theorem 3.9 can be restated as*

$$Q_G(\alpha x_1 + g, \cdots, \alpha x_n + g) = \alpha Q_G(x_1, \cdots, x_n) + g, \text{ for all } g \in G.$$

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