Variational homotopy perturbation method for the approximate solution of the foam drainage equation with time and space fractional derivatives

A. Bouhassoun[a,*], M. Hamdi Cherif[b] and M. Zellal[c]

[a,b,c]*Laboratory (LAMAP), Faculty of exact and applied sciences, University of Oran, P.O. Box, 1524, Oran, Algeria.*

Abstract

In this paper, variational homotopy perturbation method (VHPM) is applied for solving the foam drainage equation with time and space-fractional derivatives. Numerical solutions are obtained for various values of the time and space-order derivative in (0,1]. For the first-order time and space derivative, compared with the exact solution, the result showed that the proposed method could be used as an alternative method for obtaining an analytic and approximate solution for different types of differential equations.

Keywords: Caputo fractional derivative, variational homotopy perturbation method, foam drainage equation, fractional differential equations.

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1 Introduction

Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems [13]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Many powerful methods have been presented for solving such kind of problems. Among them, the Adomian decomposition method [1, 2] (ADM), the variational iteration method (VIM) [7], and the homotopy perturbation method (HPM) [8].

In this paper, we consider the following foam drainage equation with time and space fractional derivatives of the form

$$\begin{cases} {}^{c}D_t^{\alpha}u = \frac{1}{2}uu_{xx} - 2u^2 \ {}^{c}D_x^{\beta}u + ({}^{c}D_x^{\beta}u)^2 \quad ; \quad 0 < \alpha, \beta \leq 1 \\ u(x,0) = g(x) \ . \end{cases} \tag{1.1}$$

When $\alpha = \beta = 1$, this fractional equation is reduced to the foam drainage equation of the form

$$u_t = \frac{1}{2}uu_{xx} - 2u^2 \ u_x + (u_x)^2. \tag{1.2}$$

Notice that Eq. (1.2) is the reduced form obtained by putting $\quad \Psi(x,t) = u^2(x,t)$ in the original one [15] defined as

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x}(\Psi^2 - \frac{\sqrt{\Psi}}{2}\frac{\partial \Psi}{\partial x}) = 0. \tag{1.3}$$

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*Corresponding author.

E-mail addresses: a.bouhassoun@yahoo.fr (Abdelkader Bouhassoun), mountassir27@yahoo.fr (Mountassir Hamdi Cherif) and m.zellal@yahoo.fr (Mohamed Zellal).

The term foam drainage originally described the process by which fluid flows out of a foam, such as liquid draining out of a soap froth [4, 17]. Since then many technological applications have been developed for foams, which include cleansing, water purification, and minerals extraction as well as production of cushions, food stuffs, and ultra-lightweight structural materials [14]. Foams are metastable dispersions of gas in liquid that are evolving in time, which complicates precise measurements and obfuscates experimental trends.

Here α and β are the parameters standing for the order of the fractional time and space derivatives, and they satisfy $0 < \alpha, \beta \leq 1$. In fact, different response equations can be obtained when at least one of the parameters varies. In recent years, Eq. (1.1), has attracted many authors and has been studied from various methods. For example, Dahmani et al. [5] used the ADM method for solving Eq. (1.1) and then the VIM method for solving the same equation (see [6]). Later Yildirim and Koçak [18] employed VIM and HPM method for solving Eq. (1.1). The result is obtained in the form of power series convergent to the exact solution.

In what follows, we will use the variational homotopy perturbation method (VHPM) [9, 11], to solve the foam drainage equation of the form (1.1). The main characteristic of this proposed method is to avoid calculating the Adomian polynomials as in [5], and instead of using separately VIM and HPM as in [18], we will use the combined one between VIM and HPM. The result is obtained in the form of convergent series, and this method will be proved to be very useful to accelerate the convergence. Furthermore, the exact solution for $\alpha = \beta = 1$ will be used to compare those obtained by the VHPM method.

2   Basic Definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ (see [13]). The most commonly used definitions are the Riemann–Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper .

Definition 2.1. *A real function $f(t), t > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in C[0, \infty)$, and it is said to be in the space $C_\mu^n, n \in \mathbb{N}$, if $f^{(n)} \in C_\mu$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$(J^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \ \alpha > 0, t > 0 \tag{2.4}$$

$$(J^0 f)(t) = f(t).$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.3. *Let $u \in C_{-1}^n$, $n \in \mathbb{N}^*$. Then the (left sided) Caputo fractional derivative of u is defined as*

$$^c D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, \ n-1 < \alpha < n, \ n \in \mathbb{N}^*, t > 0, \\ \frac{\partial^n u(x,t)}{\partial t^n}, \ \alpha = n \in \mathbb{N}. \end{cases} \tag{2.5}$$

According to (2.5), we can obtain:

$$^c D^\alpha K = 0, \ K \text{ is a constant, and } ^c D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \ \beta > \alpha - 1 \\ 0, \ \beta \leq \alpha - 1. \end{cases}$$

3   Variational Homotopy Perturbation Method

To illustrate the basic idea of the VHPM, we consider the following general differential equation [9, 11]:

$$^c D_t^\alpha u(x,t) + R[u(x,t)] + N[u(x,t)] = g(x,t), \tag{3.6}$$

where $^c D_t^\alpha$ is the Caputo fractional derivative, R is a linear operator, N is a nonlinear operator, $g(x,t)$ is an in homogeneous term, and $m - 1 < \alpha \leq m, m \in \mathbb{N}^*$. According to the variational iteration method [16], we

can construct a correct functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) \left\{ \frac{\partial \, _\tau^m u_n}{\partial \, \tau^m}(x,\tau) + R[u_n(x,\tau)] + N[\tilde{u}_n \, (x,\tau)] - g(x,\tau) \right\} d\tau, \qquad (3.7)$$

where λ is a general Lagrange multiplier. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta \tilde{u}_n(t) = 0$ and (3.7) is called a correct functional. Now, we apply the homotopy perturbation method [3, 12]

$$\sum_{i=0}^\infty p^i u_i = u_0 + p \int_0^t \lambda(\tau) \left\{ \sum_{i=0}^\infty p^i \frac{\partial \, _\tau^\alpha u_i(x,\tau)}{\partial \, \tau^\alpha} + R\left[\sum_{i=0}^\infty p^i u_i(x,\tau) \right] + N\left[\sum_{i=0}^\infty p^i u_i(x,\tau) \right] - g(x,\tau)) \right\} ds, \quad (3.8)$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and He's polynomials [9]. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$f = \sum_{i=0}^{+\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \dots. \qquad (3.9)$$

If $p \to 1$, then (3.9) becomes the approximate solution of the form

$$u = \lim_{p \to 1} f = u_0 + u_1 + u_2 + \dots. \qquad (3.10)$$

A comparison of like powers of p gives solutions of various orders.

4 Application of the VHPM for the time-fractional derivative

We consider the foam drainage equation with time fractional derivative:

$$^c D_t^\alpha u = \frac{1}{2} u u_{xx} - 2u^2 \, u_x + (u_x)^2; \quad 0 < \alpha \leq 1, \qquad (4.11)$$

subject to the initial condition

$$u(x,0) = g(x) = -\sqrt{c} \tanh \sqrt{c}(x), \qquad (4.12)$$

where c is the velocity of wavefront [15].

The exact solution of (4.11) for the special case $\alpha = 1$ is

$$\begin{cases} u(x,t) = -\sqrt{c} \tanh \, [\sqrt{c}(x - ct)] \; ; \; x \leq ct. \\ \qquad 0 \qquad\qquad\qquad ; \; x > ct. \end{cases} \qquad (4.13)$$

According to the VIM method, the correction variational functional of equation (4.11) can be expressed as follows

$$u_{k+1} = u_k + \int_0^t \lambda(\tau) \left\{ \frac{\partial \, _\tau^\alpha u_k}{\partial \, \tau^\alpha} - \frac{1}{2} u_k \, (u_k)_{xx} + 2u_k^2 \, (u_k)_x - ((u_k)_x)^2 \right\} d\tau. \qquad (4.14)$$

Since $\alpha \in (0,1]$, the calcul of the Lagrange multiplier optimally via variational theory yields the stationary conditions $\left\{ \begin{array}{c} \lambda' = 0 \\ \lambda + 1 = 0 \end{array} \right\}$, and hence, the general Lagrange multiplier can be readily identified as $\lambda = -1$.

Substituting this value of the Lagrangian multiplier into functional (4.14) gives the iteration formula

$$u_{k+1} = u_k - \int_0^t \left\{ \frac{\partial \, _\tau^\alpha u_k}{\partial \, \tau^\alpha} - \frac{1}{2} u_k \, (u_k)_{xx} + 2u_k^2 \, (u_k)_x - ((u_k)_x)^2 \right\} d\tau. \qquad (4.15)$$

While applying the variational homotopy perturbation method, one obtains

$$u_0 + pu_1 + p^2 u_2 + \ldots = u_0 - p \int_0^t \left\{ \begin{array}{l} \dfrac{\partial_\tau^\alpha}{\partial \tau^\alpha} \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right) - \dfrac{1}{2} \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right) \dfrac{\partial^2}{\partial^2 x} \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right) \\[2mm] + 2 \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right)^2 \dfrac{\partial}{\partial x} \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right) - \left(\dfrac{\partial}{\partial x} \left(\sum_{i=0}^\infty p^i \ u_i(x,\tau) \right) \right)^2 \end{array} \right\} d\tau. \tag{4.16}$$

Comparing the coefficients of like powers of p one obtains the following set of linear partial differential equations

$$p^0 \quad : \quad u_0(x,t) = g(x) \tag{4.17}$$

$$p^1 \quad : \quad u_1(x,t) = \int_0^t \left\{ \begin{array}{l} -\dfrac{\partial_\tau^\alpha u_0}{\partial \tau^\alpha}(x,\tau) + \dfrac{1}{2} u_0(x,\tau) \dfrac{\partial^2 u_0}{\partial^2 x}(x,\tau) \\[2mm] -2u_0^2 \dfrac{\partial u_0}{\partial x}(x,\tau) + \left(\dfrac{\partial u_0}{\partial x}(x,\tau) \right)^2 \end{array} \right\} d\tau$$

$$p^2 \quad : \quad u_2(x,t) = \int_0^t \left\{ \begin{array}{l} -\dfrac{\partial_\tau^\alpha u_1}{\partial \tau^\alpha} + \dfrac{1}{2} u_0 \dfrac{\partial^2 u_1}{\partial^2 x} + \dfrac{1}{2} u_1 \dfrac{\partial^2 u_0}{\partial^2 x} \\[2mm] -2u_0^2 \dfrac{\partial u_1}{\partial x} - 4u_0 u_1 \dfrac{\partial u_0}{\partial x} + 2 \dfrac{\partial u_0}{\partial x} \dfrac{\partial u_1}{\partial x} \end{array} \right\} d\tau$$

$$p^3 \quad : \quad u_3(x,t) = \int_0^t \left\{ \begin{array}{l} -\dfrac{\partial_\tau^\alpha u_2}{\partial \tau^\alpha} + \dfrac{1}{2} u_1 \dfrac{\partial^2 u_1}{\partial^2 x} + \dfrac{1}{2} u_2 \dfrac{\partial^2 u_0}{\partial^2 x} + \dfrac{1}{2} u_0 \dfrac{\partial^2 u_2}{\partial^2 x} \\[2mm] -4u_0 u_2 \dfrac{\partial u_0}{\partial x} - 4u_0 u_1 \dfrac{\partial u_1}{\partial x} + 2 \dfrac{\partial u_0}{\partial x} \dfrac{\partial u_2}{\partial x} \\[2mm] -2(u_1)^2 \dfrac{\partial u_0}{\partial x} - 2(u_0)^2 \dfrac{\partial u_2}{\partial x} + (\dfrac{\partial u_1}{\partial x})^2 \end{array} \right\} d\tau, \tag{4.18}$$

and so on, in the same manner the rest of components can be obtained using the Maple package. Consequently, while taking the initial value $u_0(x,t) = -\sqrt{c}\tanh\sqrt{c}(x)$, and according to Eqs. (4.17)– (4.18), the first few components of the variational homotopy perturbation solution for Eq. (4.11) are derived as follows

$$u_0(x,t) = -\sqrt{c}\tanh\sqrt{c}(x),$$

$$u_1(x,t) = \frac{c^2}{\cosh(\sqrt{c}x)^2} t,$$

$$u_2(x,t) = -c^{7/2} \left(-1 + (\tanh(\sqrt{c}x))^2 \right) t^2 \tanh(\sqrt{c}x) + \frac{c^2 \left(-1 + (\tanh(\sqrt{c}x))^2 \right) t^{2-\alpha}}{\Gamma(2-\alpha)(2-\alpha)},$$

$$u_3(x,t) = \frac{c^2}{(\cosh(\sqrt{c}x))^2 \Gamma(4-2\alpha)} t^{3-2\alpha} - 4 \frac{c^{7/2} \sinh(\sqrt{c}x)}{(\cosh(\sqrt{c}x))^3 \Gamma(4-\alpha)} t^{3-\alpha}$$

$$+ \frac{1}{3} \frac{\left(2(\cosh(\sqrt{c}x))^2 - 3 \right) c^5}{(\cosh(\sqrt{c}x))^4} t^3$$

$$\ldots.$$

The other components of the (VHPM) can be determined in a similar way. Finally, the approximate solution of Eq. (4.11) in a series form is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

Consequently, the third-order approximation solution of Eq. (4.11) is given by

$$u(x,t) \quad : \quad = c^2 t \mathrm{Sech}\left[\sqrt{c}x\right]^2 + \frac{c^2 t^{3-2\alpha} \mathrm{Sech}\left[\sqrt{c}x\right]^2}{\Gamma[4-2\alpha]} + \frac{c^2 t^{2-\alpha} \mathrm{Sech}\left[\sqrt{c}x\right]^2}{(-2+\alpha)\Gamma[2-\alpha]} \tag{4.19}$$

$$+ \frac{1}{3} c^5 t^3 \left(-3 + 2\mathrm{Cosh}\left[\sqrt{c}x\right]^2 \right) \mathrm{Sech}\left[\sqrt{c}x\right]^4 - \sqrt{c}\mathrm{Tanh}\left[\sqrt{c}x\right]$$

$$+ c^{7/2} t^2 \mathrm{Sech}\left[\sqrt{c}x\right]^2 \mathrm{Tanh}\left[\sqrt{c}x\right] - \frac{4c^{7/2} t^{3-\alpha} \mathrm{Sech}\left[\sqrt{c}x\right]^2 \mathrm{Tanh}\left[\sqrt{c}x\right]}{\Gamma[4-\alpha]}$$

4.1   Numerical results

For $\alpha = 1$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u(x,t) \quad = \quad \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + \frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375}$$

$$-\frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}}.$$

Now, an expansion of the exact solution (4.13) in Taylor series over $t = 0$ to order 3 gives:

$$u(x,t) \quad = \quad -\frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{1}{25}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 t + \frac{\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]t^2}{125\sqrt{5}} \tag{4.20}$$

$$+\frac{\left(-2+\text{Cosh}\left[\frac{2x}{\sqrt{5}}\right]\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4 t^3}{9375} + O[t]^4 \tag{4.21}$$

This confirms the accuracy of the method.
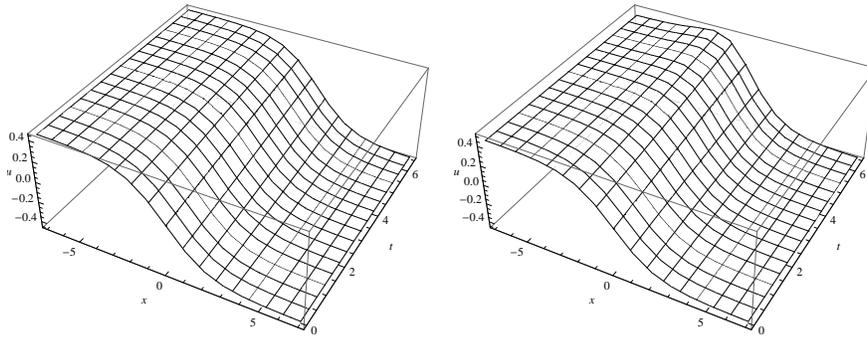
Figure 1: (Left): Exact solution (4.13) for Eqs. (4.11)-(4.12); (Right): Series approximation solution of Eqs. (4.11)-(4.12) by VHPM method for $\alpha = 1$ with four terms.

So, for $\alpha = \frac{1}{2}$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u(x,t) = \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 - \frac{4t^{3/2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2}{75\sqrt{\pi}} + \frac{1}{50}t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2$$

$$+\frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375} - \frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}}$$

$$+\frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}} - \frac{32t^{5/2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{1875\sqrt{5\pi}}$$

For $\alpha = 0.9$ and $c = \frac{1}{5}$, while inserting in (4.19), one obtains the approximation

$$u_{0.9}(x,t) \quad = \quad \frac{1}{25}t\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 - 0.0382232t^{1.1}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2 + 0.0363041t^{1.2}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2$$

$$+\frac{t^3\left(-3+2\text{Cosh}\left[\frac{x}{\sqrt{5}}\right]^2\right)\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^4}{9375} - \frac{\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{\sqrt{5}}$$

$$+\frac{t^2\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]}{125\sqrt{5}} - 0.00651197t^{2.1}\text{Sech}\left[\frac{x}{\sqrt{5}}\right]^2\text{Tanh}\left[\frac{x}{\sqrt{5}}\right]$$

These figures represent the graphs of Eq. (1.1) for various values of α. For example, Fig. 1 (left) represents the graph of the exact solution (4.13) of the initial value problem (4.11)–(4.12). Fig. 1 (right) is the graph
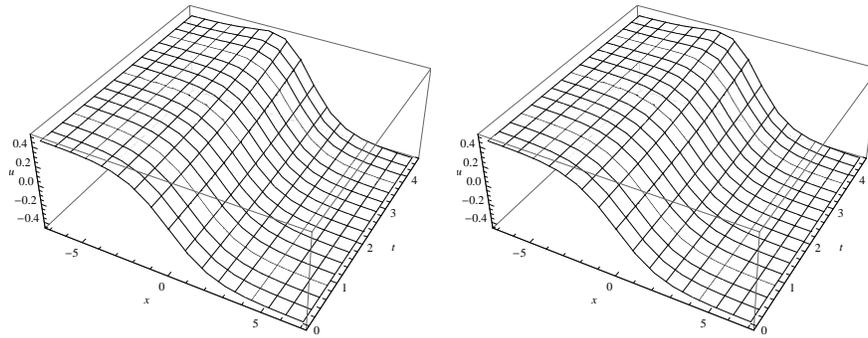
Figure 2: Series approximation solution of Eqs. (4.11)-(4.12) by VHPM method with four terms (Right: $\alpha = 1/2$; Left: $\alpha = 0.9$).

of the numerical one obtained by VHPM for $\alpha = 1$ with four terms in the series solution. One observes that there is a similarity between the two figures and this leads to say that the method employed could be used as an alternative method for obtaining an analytic and approximate solution for different types of differential equations. In Fig. 2, one has represented the graphs of Eq. (1.1) for $\alpha = 1/2$ (left) and $\alpha = 0.9$ (right) respectively.

5 Application of the VHPM for the space-fractional derivative

We next consider the following space-fractional foam drainage equation

$$\begin{cases} u_t = \frac{1}{2}uu_{xx} - 2u^2 \, {}^cD_x^\beta u + ({}^cD_x^\beta u)^2 & ; \quad 0 < \beta \le 1 \\ u(x,0) = x^2. \end{cases} \tag{5.22}$$

This initial condition is taken as polynomial to avoid heavy calculations of fractional differentiation.
According to the VHPM method, one obtains

$$u_0 + pu_1 + p^2u_2 + \dots = u_0 - p \int_0^t \left\{ \begin{array}{c} \dfrac{\partial}{\partial \tau}\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right) \\[2mm] -\dfrac{1}{2}\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right)\dfrac{\partial^2}{\partial^2 x}\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right) \\[2mm] +2\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right)^2\dfrac{\partial^\beta}{\partial x^\beta}\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right) \\[2mm] -\left(\dfrac{\partial^\beta}{\partial x^\beta}\left(\sum\limits_{i=0}^\infty p^i \, u_i(x,\tau)\right)\right)^2 \end{array} \right\} d\tau.$$

Comparing the coefficients of like powers of p one obtains the following set of linear partial differential equations

$$\frac{\partial u_0}{\partial t} = \frac{\partial v_0}{\partial t}, u_0(x,0) = x^2 \tag{5.23}$$

$$\frac{\partial u_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \frac{1}{2}u_0\frac{\partial^2 u_0}{\partial x^2} - 2u_0{}^2\frac{\partial^\beta u_0}{\partial x^\beta} + \left(\frac{\partial^\beta u_0}{\partial x^\beta}\right)^2, u_1(x,0) = 0 \tag{5.24}$$

$$\frac{\partial u_2}{\partial t} = \frac{1}{2}(u_0\frac{\partial^2 u_1}{\partial x^2} + u_1\frac{\partial^2 u_0}{\partial x^2}) - \frac{\partial u_1}{\partial t} - 2\frac{\partial^\beta u_1}{\partial x^\beta}(u_0{}^2\frac{\partial^\beta u_1}{\partial x^\beta} - \frac{\partial^\beta u_0}{\partial x^\beta}\frac{\partial^\beta u_1}{\partial x^\beta}) \\ -4u_0u_1\frac{\partial^\beta u_0}{\partial x^\beta}, u_2(x,0) = 0 \tag{5.25}$$

$$\frac{\partial u_3}{\partial t} = \frac{1}{2}u_2\frac{\partial^2 u_0}{\partial x^2} - 4u_0u_1\frac{\partial^\beta u_1}{\partial x^\beta} + \left(\frac{\partial^\beta u_1}{\partial x^\beta}\right)^2 + \frac{1}{2}u_0\frac{\partial^2 u_2}{\partial x^2} + \frac{1}{2}u_1\frac{\partial^2 u_1}{\partial x^2} \\ -4u_0u_2\frac{\partial^\beta u_0}{\partial x^\beta} - 2u_0^2\frac{\partial^\beta u_2}{\partial x^\beta} + 2\frac{\partial^\beta u_0}{\partial x^\beta}\frac{\partial^\beta u_2}{\partial x^\beta} - 2u_1^2\frac{\partial^\beta u_0}{\partial x^\beta} - 2\frac{\partial u_2}{\partial t}, u_2(x,0) = 0. \tag{5.26}$$

Selecting the initial value $u(x,0) = x^2$ and using equations (5.23)-(5.26) one obtains the following successive approximations

$$u_0(x,t) = x^2 \tag{5.27}$$

$$u_1(x,t) = \frac{t\left(x^2\left(\Gamma(3-\beta)\right)^2 - 4\,x^{6-\beta}\Gamma(3-\beta) + 4\,x^{4-2\,\beta}\right)}{\left(\Gamma(3-\beta)\right)^2}, \tag{5.28}$$

and so on, in the same manner the rest of components can be obtained using the iteration formula (5.23)-(5.26). Hence, for $\beta = 1$, and the initiale condition $u(x,0) = x^2$, one obtains the third order approximation of the initial value problem (5.22) as

$$u(x,t) = x^2 + 20tx^2\left(2 - 5x^3 + 2x^6\right) + t^2x^2\left(5 - 42x^3 + 16x^6\right). \tag{5.29}$$

When $\beta = 1/2$, one obtains the approximate solution for the initial value problem (5.22) as

$$\begin{aligned}
u(x,t) \;=\; & x^2 + \frac{1}{2160\pi^2}t^2x^2\left(\begin{array}{c} 2160\pi^2 + 30720\pi x - 88560\pi^{3/2}x^{7/2} \\ -81920\sqrt{\pi}x^{9/2} + 61440\pi x^7 \end{array}\right) \\
& + \frac{1}{2160\pi^2}tx^2\left(\begin{array}{c} 30720\pi x + 262144x^2 - 11520\pi^{3/2}x^{7/2} \\ -98304\sqrt{\pi}x^{9/2} - 83160\pi^{3/2}x^{9/2} + 31185\pi^2 x^7 \end{array}\right)
\end{aligned}$$
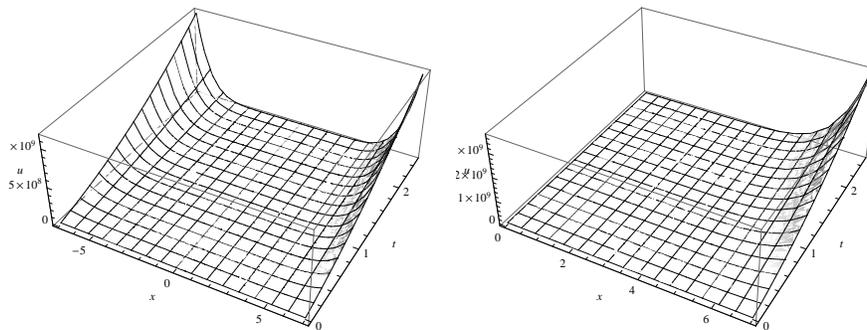
Figure 3: Series approximation solution of Eq. (5.22) by VHPM method with four terms for $\beta = 1$ (Left), and $\beta = 1/2$ (Right).

6   Conclusion

In this paper, based on the VIM and HPM, the variational homotopy perturbation method VHPM is considered for solving the time and space-fractional foam drainage partial differential equation. The numerical results obtained with different values of the time and space derivatives showed that the VHPM is a powerful and reliable method for finding the approximate analytical solutions of the time and space-fractional foam drainage. The current work illustrates that the VHPM is indeed a powerful analytical technique for most types of nonlinear problems and several such problems in scientific studies and engineering may be solved by this method.

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