Malaya J. Mat. 2(1)(2014) 1-9

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Robinson-Schensted correspondence for party algebras

A. Vidhya[a] and A. Tamilselvi[b,*]

[a] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai-600 005, Tamil Nadu, India.

[b] Anna University, MIT Campus, Chennai - 600 044, Tamil Nadu, India.

Abstract

In this paper, we construct a bijective proof of the identity $n^k = \sum_{[\tilde{\lambda}]\in\Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$, where $m_k^{[\tilde{\lambda}]}$ is the multiplicity of the irreducible representation of $\mathbb{Z}_r \wr S_n$ module indexed by $[\tilde{\lambda}] \in \Lambda_n^k$, $f^{[\tilde{\lambda}]}$ is the degree of the corresponding representation indexed by $[\tilde{\lambda}] \in \Lambda_n^k$ and $\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^k i|\lambda^{(i)}| = k\}$. We give the proof of Robinson-Schensted correspondence for the party algebras which gives the bijective proof of party diagrams and the pairs of vacillating tableaux.

Keywords: Partition, Bratteli diagram, Robinson-Schensted correspondence.

2010 MSC: 05E10, 05A05, 20C99.

1 Introduction

Let G be the group of linear transformations on a n-dimensional vector space V. Suppose that G acts diagonally on the k-fold tensor space $V^{\otimes k}$. Then the k-fold tensor space $V^{\otimes k}$ decomposes into irreducible representation of G as centraliser algebra $End_G\left(V^{\otimes k}\right)$. This work was successfully done in Partition algebra $End_{S_n}\left(V^{\otimes k}\right)$, Brauer algebra $End_{O(n)}\left(V^{\otimes k}\right)$ where $O(n)$ is the orthogonal group of degree n and so on.

The party algebra $\mathbb{C}L_k$ is the subalgebra of the partition algebra which is generated by S_n and the diagram corresponding to the set partition $\{\{1,2,1',2'\}\{3,3'\}\ldots\{k,k'\}\}$

Masashi Kosuda defined the irreducible representations of the party algebras. There exists a surjective homomorphism from $\mathbb{C}L_k$ to $End_{G(r,1,n)}\left(V^{\otimes k}\right)$. Moreover if $n \geq k$ and $r > n$, this homomorphism is injective and thus forms an irreducible representations of party algebras.

The number of standard Young tableaux of shape $[\lambda]$ is $f^{[\lambda]}$ which is the degree of the corresponding representation of the group $G(r,1,n)$. In this paper, we develop a Robinson-Schensted correspondence for the party algebras which gives the bijection between the diagrams in $\mathbb{C}L_k$ and the pairs of vacillating tableaux $\left(P_{[\lambda]}, Q_{[\lambda]}\right)$ in Γ_k. We also develop the bijection proof for the identity $n^k = \sum_{[\tilde{\lambda}]\in\Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$, where $m_k^{[\tilde{\lambda}]}$ is the multiplicity of the irreducible representation of $\mathbb{Z}_r \wr S_n$ module indexed by $[\tilde{\lambda}]$, by constructing a bijection between the sequences $(i_1, i_2, \ldots, i_k)$, $1 \leq i_j \leq n$ and the pair $\left(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}\right)$ where $T_{[\tilde{\lambda}]}$ is a standard tableau of shape $[\tilde{\lambda}]$ and $P_{[\tilde{\lambda}]}$ is the vacillating tableaux of shape $[\tilde{\lambda}]$.

2 Preliminaries

Definition 2.1. *[7] A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a non-increasing sequence of positive integers, that is $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_k| = n$. It is denoted by $\lambda \vdash n$.*

*Corresponding author.

E-mail address: tamilselvi.riasm@gmail.com (A. Tamilselvi)

Definition 2.2. *[4] A multipartition* $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ *such that each* $\lambda^{(i)}$ *is a partition and* $\sum_i |\lambda^{(i)}| = n$. *We say that* $\lambda^{(i)}$ *is the* $i-$*th component of* $[\lambda]$.

Definition 2.3. *[4] A diagram of a partition* λ *is an array of boxes in which first row contains* λ_1 *number of boxes, second row contains* λ_2 *number of boxes and so on.*

Definition 2.4. *[4] Let* $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ *be a multipartition of* n. *A* $[\lambda]$ *-tableau* $t = (t^{(1)}, t^{(2)}, \ldots, t^{(k)})$ *is obtained by filling the boxes of the diagram from* $\{1, 2, ..., n\}$.

1. *A* $[\lambda]-$*tableau* t *is said to be row standard if the entries in each row of each component is strictly increasing*
2. *A* $[\lambda]-$*tableau* t *is said to be standard if the entries in each row and in each column of each component is strictly increasing.*

Definition 2.5. *[7] A rim hook is a connected skew shape containing no* 2×2 *square.*

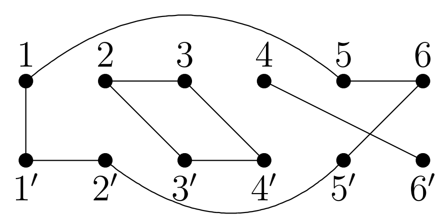
2.1 Party algebras

For $k \in \mathbb{Z}$, let

$$\begin{aligned} A_k &= \{ \text{ set partitions of } \{1, 2, \ldots, k, 1^{'}, 2^{'}, \ldots, k^{'}\}\} \text{ and} \\ A_{k+\frac{1}{2}} &= \{d \in A_{k+1} | (k+1) \text{ and } (k+1)^{'} \text{ are in the same block}\}. \end{aligned}$$

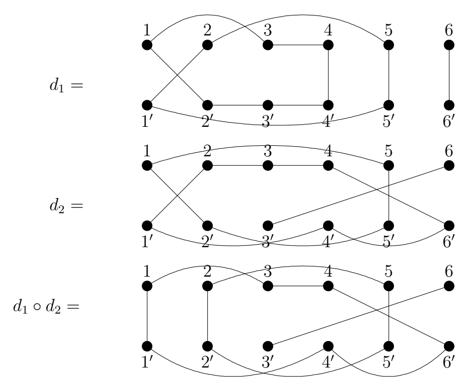
For a set partition $d = \{B_1, B_2, \ldots, B_s\} \in A_k$ and $B_i \in d$, let $N(B_i) = \#(B_i \cap \underline{k})$ and $M(B_i) = \#(B_i \cap \underline{k}')$.

For $k > 0$, let $L_k = \{d \in A_k | N(B_i) = M(B_i) \text{ for all } B_i \in d\}$. Represent $d \in L_k$ as a graph with two rows of k vertices, the first row of k vertices is labeled by $1, 2, \ldots, k$ and the second row of k vertices is labeled by $1', 2', \ldots, k'$. For example,



Definition 2.6. *Let* $d_1, d_2 \in L_k$, *the multiplication of diagrams* $d_1 \circ d_2$ *is obtained by placing* d_2 *below* d_1 *and identifying each vertex* i' *in the bottom row of* d_1 *with the each vertex* i *in the top row of* d_2 *and by removing any component that lie entirely in the middle row.*

For example,



For $k \in \mathbb{N}$, the party algebra $\mathbb{C}L_k$ is an associative subalgebra of the partition algebra $\mathbb{C}A_k$ with basis L_k.

3 Schur Weyl Duality between $\mathbb{Z}_r \wr S_n$ and $\mathbb{C}L_k$

The irreducible representations of $\mathbb{Z}_r \wr S_n$ are indexed by the multi partition $[\lambda]$ of n. If $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ where each $\lambda^{(i)}$ is the partition of the $i-$ th component and $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. Let V be the n dimensional representation of the group $\mathbb{Z}_r \wr S_n$. Consider $S_{n-1} \subseteq S_n$ as the subgroup of permutations that fix n. Let $V^{\otimes k}$ be the k fold tensor representation of V. Let $V^{[\lambda]}$ be the irreducible representation of $\mathbb{Z}_r \wr S_n$ indexed by $[\lambda] \vdash n$ where $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r-1)})$ and $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. The induction and restriction rules for $\mathbb{Z}_r \wr S_n$ are as follows:

If $[\lambda] \vdash n$, $Res^{\mathbb{Z}_r \wr S_n}_{\mathbb{Z}_r \wr S_{n-1}} V^{[\lambda]}$ denotes the irreducible representation obtained from restricting the multi partition $[\lambda] \vdash n$ to the multi partition $[\mu] \vdash n-1$ by removing a box from any one of the residues in $[\lambda]$. $Ind^{\mathbb{Z}_r \wr S_{n-1}}_{\mathbb{Z}_r \wr S_n} V^{[\mu]}$ denotes the irreducible representation obtained from inducing the multi partition $[\mu] \vdash n-1$ to the multi partition $[\lambda] \vdash n$ by adding a box in the $\lambda^{(l+1)}$ if the box is removed from $\mu^{(l)}$ while restriction.

$$Res^{\mathbb{Z}_r \wr S_n}_{\mathbb{Z}_r \wr S_{n-1}} V^{[\lambda]} \cong \bigoplus_{[\mu] \vdash n-1, [\mu] \subseteq [\lambda]} V^{[\mu]},$$

for $[\lambda] \vdash n$.

$$Ind^{\mathbb{Z}_r \wr S_{n-1}}_{\mathbb{Z}_r \wr S_n} V^{[\mu]} \cong \bigoplus_{[\mu] \vdash n, [\lambda] \subseteq [\mu]} V^{[\lambda]}.$$

for $[\mu] \vdash n-1$.

Suppose that $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r-1)})$ and $[\mu] = (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r-1)})$ are multi partitions of n. we say that $[\mu] \subseteq [\lambda]$ if,

$$\sum_{i=1}^{m-1} |\mu^i| + \sum_{i=1}^{j} \mu_j^{(m)} \leq \sum_{i=1}^{m-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda_i^{(m)}.$$

Starting with the trivial representation $\underbrace{(n, \emptyset, \ldots, \emptyset)}_{r\text{tuples}}$ and iterating the restriction and induction rules. We see the irreducible $\mathbb{Z}_r \wr S_n$ representation that appears in $V^{\otimes k}$ are labeled by the partition in Λ_n^k. If $[\lambda] = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)} \vdash n$ and if $[\tilde{\lambda}] = (n-t, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}, \emptyset^{(k+1)}, \ldots, \emptyset^{(r-1)})$, where $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}) \vdash t$, $1 \leq t \leq k$ and $r > k$.

$$\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^{k} i|\lambda^{(i)}| = k\}$$

and the irreducible $\mathbb{Z}_r \wr S_{n-1}$ representation that appear in $V^{\otimes k}$ are labeled by the partitions in Λ_{n-1}^k. If $[\tilde{\lambda}] \vdash n-1, 0 \leq t \leq k$.

$$\Lambda_{n-1}^k = \{[\tilde{\lambda}] \vdash n-1 | \sum_{i=1}^{k} i|\lambda^{(i)}| \leq k\}.$$

There is an action of $\mathbb{C}L_k$ on $V^{\otimes k}$ [6] that commutes with $\mathbb{Z}_r \wr S_n$ and maps surjectively onto centralizer of $End_{\mathbb{Z}_r \wr S_n} V^{\otimes k}$. Furthermore when $n \geq k$ and $r > k$ we have

$$\mathbb{C}L_k \cong End_{\mathbb{Z}_r \wr S_n} V^{\otimes k} \text{ and } \mathbb{C}L_{k+\frac{1}{2}} \cong End_{\mathbb{Z}_r \wr S_{n-1}} V^{\otimes k}.$$

The Bratelli diagram for $\mathbb{C}L_k$ consists of rows of vertices with the rows labeled by $0, \frac{1}{2}, \ldots, k$ such that the vertices in row i are labeled by Λ_n^i and the vertices in row $i + \frac{1}{2}$ are Λ_{n-1}^i. Two vertices are connected by an edge if they are in consecutive rows and they differ by exactly one box. The irreducible representations of $\mathbb{C}L_k$ are indexed by Λ_n^k, so we let $M_k^{[\tilde{\lambda}]}$ denote the irreducible representation of $\mathbb{C}L_k$ indexed by $[\tilde{\lambda}] \in \Lambda_n^k$.

The decomposition of $V^{\otimes k}$ as an $\mathbb{Z}_r \wr S_n \times \mathbb{C}L_k$ bimodule is given by

$$V^{\otimes k} \cong \bigoplus_{[\tilde{\lambda}] \in \Lambda_n^k} V^{[\lambda]} \otimes M_k^{[\tilde{\lambda}]}. \tag{3.1}$$

The dimension of $M_k^{[\tilde{\lambda}]}$ equals the multiplicity of $V^{[\lambda]}$ in $V^{\otimes k}$.

$$m_k^{[\tilde{\lambda}]} = \dim(M_k^{[\tilde{\lambda}]}) = \{\text{the number of paths from the top of the Bratelli diagram to } [\tilde{\lambda}]\}.$$

4 Vacillating Tableaux

Let $[\lambda] \in \Lambda_n^k$. A vacillating tableaux of shape λ and length $2k$ is a sequence of partitions,

$$\left((n, \emptyset, \dots, \emptyset) = [\lambda]^{(0)}, [\lambda]^{\left(\frac{1}{2}\right)}, [\lambda]^{(1)}, [\lambda]^{1\left(\frac{1}{2}\right)}, \dots, [\lambda]^{\left(k-\frac{1}{2}\right)}, [\lambda]^{(k)} = [\lambda]\right),$$

satisfying, for each i,

1. $[\lambda]^{(i)} \in \Lambda_n^i$, $[\lambda]^{(i+\frac{1}{2})} \in \Lambda_{n-1}^i$,
2. $[\lambda]^{(i)} \supseteq [\lambda]^{(i+\frac{1}{2})}$ and $|[\lambda]^{(i)}/[\lambda]^{(i+\frac{1}{2})}| = 1$,
3. $[\lambda]^{(i+\frac{1}{2})} \subseteq [\lambda]^{(i+1)}$ and $|[\lambda]^{(i+1)}/[\lambda]^{(i+\frac{1}{2})}| = 1$.

The vacillating tableaux of shape $[\lambda]$ correspond exactly with paths from the top of the Brattelli diagram to $[\lambda]$.

For $n \geq k$ and $r > n$, the sets

$$\Lambda_n^k = \{[\tilde{\lambda}] \vdash n | \sum_{i=1}^{k} i|\lambda^{(i)}| = k\}$$

and if $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$,

$$\Gamma_k = \{[\lambda] \vdash t | \sum_{i=1}^{k} i|\lambda^{(i)}| = k \text{ and } 0 \leq t \leq k\}$$

are in bijection with one another. For example, the following sequence represents the same vacillating tableaux $P_{[\lambda]}$.

$P_{[\lambda]}$=$\left((\Box\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \Box, \emptyset, \emptyset), (\Box\Box, \emptyset, \emptyset, \emptyset), (\Box\Box, \emptyset, \Box, \emptyset), (\Box, \emptyset, \Box, \emptyset), (\Box, \Box, \Box, \emptyset)\right)$

$\left((\emptyset, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset), (\Box, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset), (\emptyset, \Box, \emptyset), (\emptyset, \Box, \emptyset), (\Box, \Box, \emptyset)\right)$

Bratelli diagram for $End_{\mathbb{Z}_5 \wr S_4}(V^{\otimes 3})$

Thus, if we let $VT_k([\tilde{\lambda}])$ denote the set of vacillating tableaux of shape $[\tilde{\lambda}]$ and length k, then

$$m_k^{[\tilde{\lambda}]} = \dim(M_k^{[\tilde{\lambda}]}) = |VT_k([\tilde{\lambda}])|.$$

5 A Bijective Proof of $n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$

Comparing dimensions on both sides of equation 3.1, gives

$$n^k = \sum_{[\tilde{\lambda}] \in \Lambda_n^k} f^{[\tilde{\lambda}]} m_k^{[\tilde{\lambda}]}$$

where $f^{[\tilde{\lambda}]}$ is the number of standard Young tableaux of shape $[\tilde{\lambda}]$.

Bratelli diagram for $\mathbb{C}L_k$

We now give the combinatorial proof of the above equality. Let $T_{[\tilde{\lambda}]}$ be the standard Young tableau of shape $[\tilde{\lambda}]$ and $P_{[\tilde{\lambda}]}$ be the vacillating tableau of shape $[\tilde{\lambda}]$. Let $SYT([\tilde{\lambda}])$ be the set of all standard Young tableau of shape $[\tilde{\lambda}]$.

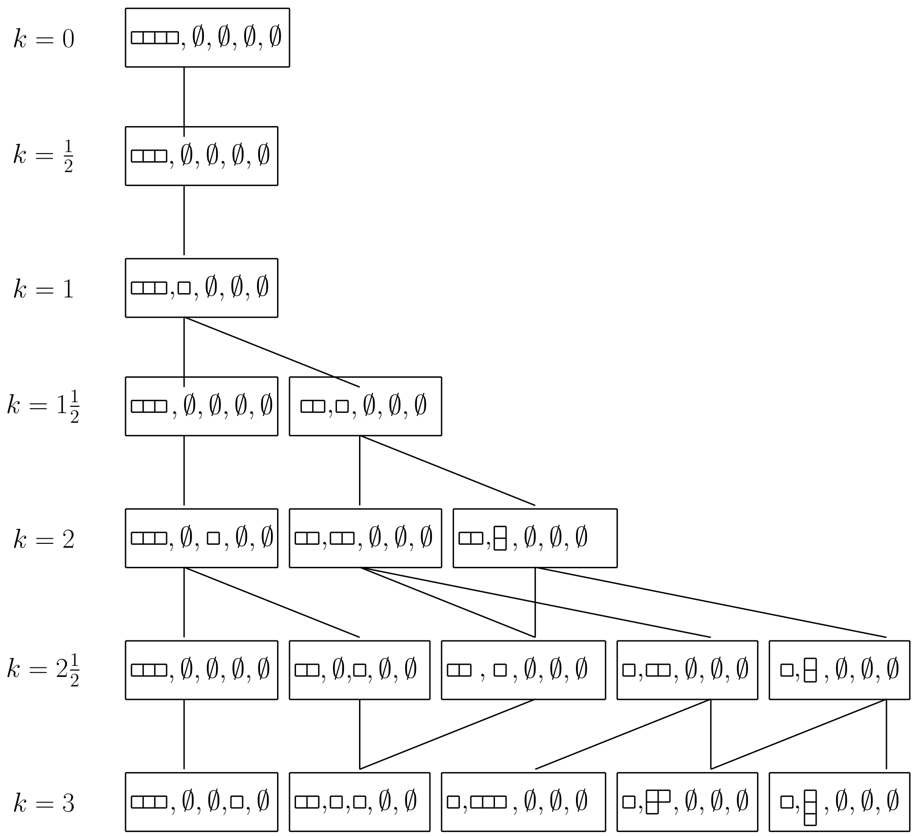
Theorem 5.1. *The map $(i_1, i_2, \dots, i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$ is bijective where $\{(i_1, i_2, \dots, i_k) || 1 \leq i_j \leq n\}$ and the pair $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}) \in SYT([\tilde{\lambda}]) \times VT_k([\tilde{\lambda}])$.*

Proof. To prove $(i_1, i_2, \dots, i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$, we first initiate

$$T^{(0)} = \left(\boxed{1 \mid 2 \mid \cdots \mid n}^{\,(0)}, \emptyset^{(1)}, \emptyset^{(2)}, \dots, \emptyset^{(r-1)} \right)$$

Then recursively define standard tableaux $T^{j+\frac{1}{2}}$ and T^{j+1} as

$$T^{(j+\frac{1}{2})} = \left(i_{j+1} \longleftarrow T^{(j)}\right)$$



$$T^{(j+1)} = \left(i_{j+1} \longrightarrow T^{(j+\frac{1}{2})}\right)$$

$i_{j+1} \longleftarrow T^{(j)}$ means delete i_{j+1} using Jeu-de-taquin [see [7], p.113] from the corresponding residue where it lies. $i_{j+1} \longrightarrow T^{(j+\frac{1}{2})}$ means insert i_{j+1} in the $\lambda^{(l+1)}$ using RSK insertion [see [7], p.92] if i_{j+1} is removed from $\lambda^{(l)}$.

Let $[\tilde{\lambda}]^{(j)} \in \Lambda_n^j$ be the shape of $T^{(j)}$ and $[\tilde{\lambda}]^{(j+\frac{1}{2})} \in \Lambda_{n-1}^j$ be the shape of $T^{(j+\frac{1}{2})}$. Then let

$$P_{[\tilde{\lambda}]} = \left([\tilde{\lambda}]^{(0)}, [\tilde{\lambda}]^{\frac{1}{2}}, \ldots, [\tilde{\lambda}]^{(k-\frac{1}{2})}, [\tilde{\lambda}]^{(k)}\right)$$

and $T_{[\tilde{\lambda}]} = T^{(k)}$.

This insertion and deletion process produces the vacillating tableaux $P_{[\tilde{\lambda}]}$ of shape $[\tilde{\lambda}] = [\tilde{\lambda}]^{(k)} \in \Lambda_n^k$ and the standard tableau $T_{[\tilde{\lambda}]}$ of the same shape $[\tilde{\lambda}]$. Hence $(i_1, i_2, \ldots i_k) \mapsto (T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$.

To prove $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]}) \longleftarrow (i_1, i_2, \ldots, i_k)$.

Given $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$ of shape $[\tilde{\lambda}] = [\tilde{\lambda}]^{(k)} \in \Lambda_n^k$. We use RSK reverse insertion [see [7], p.94] to obtain the sequence $(i_1, i_2, \ldots, i_k)$ from the given pair $(T_{[\tilde{\lambda}]}, P_{[\tilde{\lambda}]})$. $\square$

For example, consider the sequence (i_1, i_2, i_3) as $(4, 2, 3)$.

j	i_j		$T^{(j)}$
0			$\left(\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}^{(0)}, \varnothing^{(1)}, \varnothing^{(2)}, \varnothing^{(3)}, \varnothing^{(4)}\right)$
$\frac{1}{2}$	4	$\longleftarrow$	$\left(\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}^{(0)}, \varnothing^{(1)}, \varnothing^{(2)}, \varnothing^{(3)}, \varnothing^{(4)}\right)$
1	4	$\longrightarrow$	$\left(\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 4 \\ \hline\end{array}^{(1)}, \varnothing^{(2)}, \varnothing^{(3)}, \varnothing^{(4)}\right)$

j	i_j		$T^{(j)}$
$1\frac{1}{2}$	2	$\longleftarrow$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 3\\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 4\\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
2	2	$\longrightarrow$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 3\\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 2\\ \hline 4\\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$2\frac{1}{2}$	3	$\longleftarrow$	$\left(\begin{array}{\|c\|}\hline 1\\ \hline\end{array}^{(0)}, \begin{array}{\|c\|}\hline 2\\ \hline 4\\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
3	3	$\longrightarrow$	$\left(\begin{array}{\|c\|}\hline 1\\ \hline\end{array}^{(0)}, \begin{array}{\|c\|c\|}\hline 2 & 3\\ \hline 4 \\ \cline{1-1}\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$

6 $R-S$ correspondence for Party algebra

Represent $d \in L_k$ in a single row with vertices labeled by $1, 2, \ldots, 2k$ where we relate vertex j' with the label $2k-j+1$. Connect vertices i and j in $d \in L_k$ as the graph represented in a single row by the standard representation with $i \leq j$ if and only if i and j are related in d and there does not exist k related to i and j with $i < k < j$. Each vertex is connected only to its nearest neighbours in its block.

We label each edge e of the diagram d represented in the standard representation by $2k+1-l$ where l is the right vertex of e.

Define the insertion sequence of a diagram to be the sequence $E = \{E_j\}$ indexed by j in the sequence $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots, 2k$ where

$$E_j = \begin{cases} a, & \text{if vertex } j \text{ is the left endpoint of edge } a \\ \emptyset, & \text{if vertex } j \text{ is not a left endpoint of edge } a. \end{cases}$$

$$E_{j-\frac{1}{2}} = \begin{cases} a, & \text{if vertex } j \text{ is the right endpoint of edge } a \\ \emptyset, & \text{if vertex } j \text{ is not a right endpoint of edge } a. \end{cases}$$

For example, the standard representation and insertion sequence of $d \in L_k$ is as

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
E_j	$\emptyset$	2	$\emptyset$	6	6	3	$\emptyset$	4	4	$\emptyset$	3	1	2	$\emptyset$	1	$\emptyset$

For a given $d \in L_k$, with insertion sequence $\{E_j\}$, we will produce a pair of vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ of shape $[\lambda] \in \Gamma_k$. Begin with the empty tableau,

$$T^{(0)} = (\emptyset, \emptyset, \ldots, \emptyset)$$

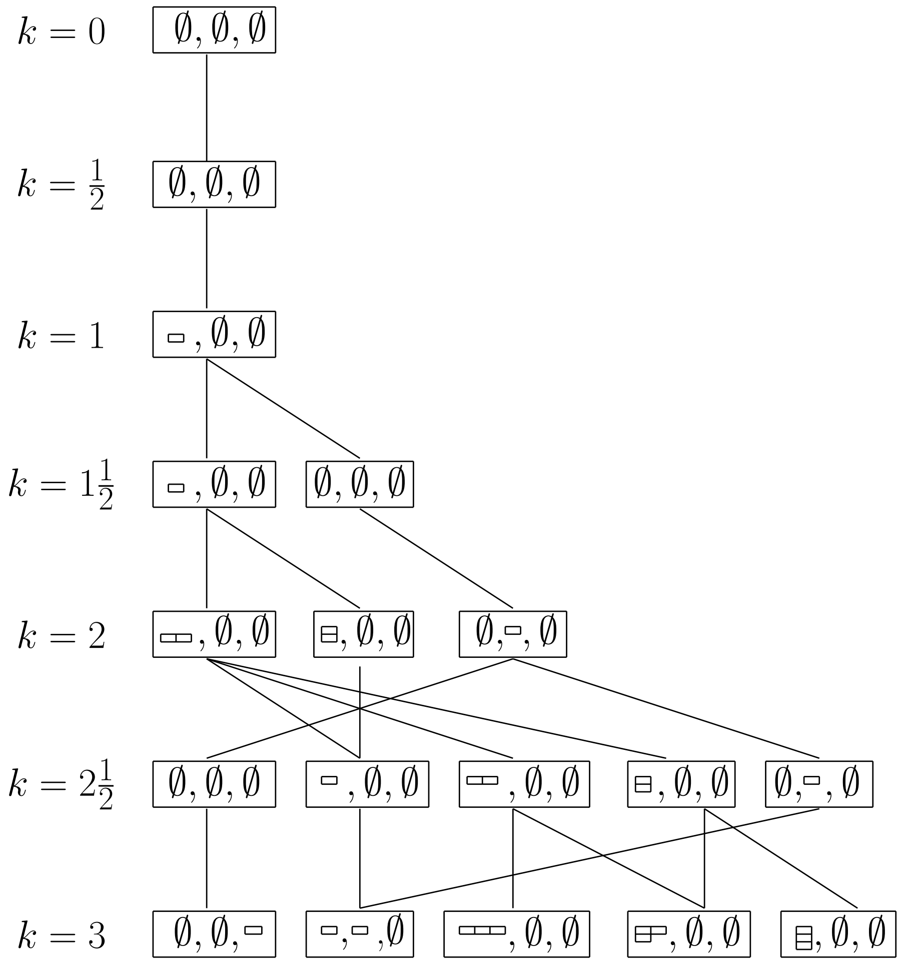
Deleting and inserting, $E_j, j = 1, \ldots, 2k$ in the corresponding residues $1, \ldots, k$, are the two procedures involved in this algorithm, then we successively deleting $E_{j-\frac{1}{2}}$ and inserting E_j as follows.

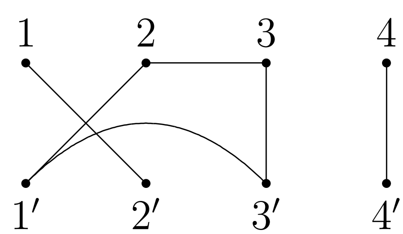
$$T^{(0)} = (\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)})$$

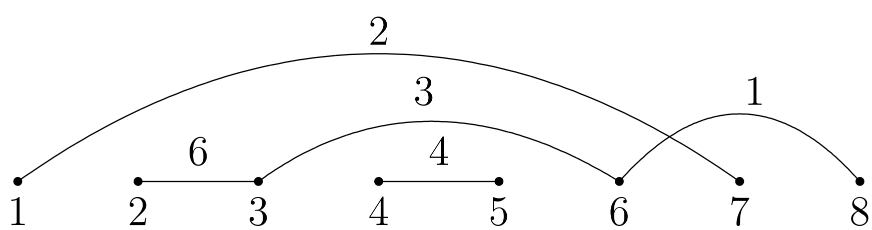
$$T^{(j-\frac{1}{2})} = \begin{cases} T^{(j-1)} & \text{if } E_{j-\frac{1}{2}} = \emptyset, \\ E_{j-\frac{1}{2}} \longleftarrow T^{(j-1)} & \text{if } E_{j-\frac{1}{2}} \neq \emptyset, \end{cases}$$

$E_{j-\frac{1}{2}} \longleftarrow T^{(j-1)}$ means that, delete $E_{j-\frac{1}{2}}$ in $T^{(j-1)}$ using jeu-de-taquin from where it lies.

$$T^{(j)} = \begin{cases} T^{(j-\frac{1}{2})} & \text{if } E_j = \emptyset, \\ E_j \longrightarrow T^{(j-\frac{1}{2})} & \text{if } E_j \neq \emptyset, \end{cases}$$







$E_j \longrightarrow T^{(j-\frac{1}{2})}$ means that the insertion of E_j into $T^{(j-\frac{1}{2})}$ in the following way:

1. For $j = 1, \ldots, k$, insert E_j into $\lambda^{(1)}$ if $E_{j-\frac{1}{2}} = \emptyset$, else insert E_j in $\lambda^{(l+1)}$ if $E_{j-\frac{1}{2}}$ is deleted from $\lambda^{(l)}$.

2. For $j = k+1, \ldots, 2k$, insert E_j into the $\lambda^{(1)}$ if $E_{j-\frac{1}{2}} = \emptyset$, else insert E_j in $\lambda^{(l-1)}$ if $E_{j-\frac{1}{2}}$ is deleted from $\lambda^{(l)}$.

Let $[\lambda]^{(i)}$ be the shape of $T^{(i)}$, let $[\lambda]^{(i+\frac{1}{2})}$ be the shape of $T^{(i+\frac{1}{2})}$ and let $[\lambda] = [\lambda]^{(k)}$. Define

$$Q_{[\lambda]} = \left(\emptyset, [\lambda]^{(\frac{1}{2})}, \ldots, [\lambda]^{(k-\frac{1}{2})}, [\lambda]^{(k)}\right)$$

.

$$P_{[\lambda]} = \left([\lambda]^{(2k)}, [\lambda]^{(2k-\frac{1}{2})}, \ldots, [\lambda]^{(k)}\right).$$

In this insertion process, every edge of the diagram is inserted when we come to its left endpoint and deleted when we come to its right endpoint. so the final shape is $[\lambda]^{(2k)} = \emptyset$. So we associate a pair of vacillating tableaux $(P_{[\lambda]}, Q_{[\lambda]})$ to $d \in L_k$. Denote this process by $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$.

j	E_j		$T^{(j)}$
0			$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
1	2	$\longrightarrow$	$\left(\boxed{2}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$1\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\boxed{2}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
2	6	$\longrightarrow$	$\left(\begin{array}{\|c\|c\|}\hline 2 & 6 \\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$2\frac{1}{2}$	6	$\longleftarrow$	$\left(\boxed{2}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
3	3	$\longrightarrow$	$\left(\boxed{2}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$3\frac{1}{2}$	$\emptyset$	$\longleftarrow$	$\left(\boxed{2}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
4	4	$\longrightarrow$	$\left(\begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline\end{array}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$4\frac{1}{2}$	4	$\longleftarrow$	$\left(\boxed{2}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
5	$\emptyset$	$\longrightarrow$	$\left(\boxed{2}^{(1)}, \boxed{3}^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$5\frac{1}{2}$	3	$\longleftarrow$	$\left(\boxed{2}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
6	1	$\longrightarrow$	$\left(\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$6\frac{1}{2}$	2	$\longleftarrow$	$\left(\boxed{1}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
7	$\emptyset$	$\longrightarrow$	$\left(\boxed{1}^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
$7\frac{1}{2}$	1	$\longleftarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$
8	$\emptyset$	$\longrightarrow$	$\left(\emptyset^{(1)}, \emptyset^{(2)}, \emptyset^{(3)}, \emptyset^{(4)}\right)$

Theorem 6.2. *The map* $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$ *provides a bijection between the set of* $d \in L_k$ *and the pairs of vacillating tableaux of shape* $[\lambda]$, $[\lambda] \in \Gamma_k$,

Proof. From the above example it is clear that, for a given $d \in L_k$ we can construct a pair of vacillating tableau $(P_{[\lambda]}, Q_{[\lambda]})$ of shape $[\lambda]$. We prove the theorem by constructing the inverse of $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. First we use $Q_{[\lambda]}$ followed by $P_{[\lambda]}$ in reverse order to construct the sequence $\left[\lambda^{(\frac{1}{2})}\right], \left[\lambda^{(1)}\right], \ldots, \left[\lambda^{(2k)}\right]$. We initialize $T^{(2k)} = \emptyset$.

We now show how to construct $T^{(i)}$ and E_{i+1} so that $T^{(i+1)} = \left(\ E_{i+1} \longrightarrow T^{(i)}\ \right)$. If $[\lambda]^{(i+1)}/[\lambda]^{(i)}$ is a box containing a, and we use reverse algorithm [see [7], p.94] on the value in the box containing a to produce $T^{(i)}$ and $I_{i+1}^{(d,n)}$ such that $T^{(i+1)} = \left(\ E_{i+1} \longrightarrow T^{(i)}\ \right)$. Since we remove the value in position a by using reverse RS insertion [see [7], p.92], we know that $T^{(i)}$ has shape $[\lambda]^{(i)}$.

We then show how to construct $T^{(i)}$ and E_{i+1} so that $T^{(i+1)} = \left(\ E_{i+1} \longleftarrow T^{(i)}\ \right)$. If $[\lambda]^{(i)}/[\lambda]^{(i+1)}$ is a box containing a. Let $T^{(i)}$ be the tableau of shape $\lambda^{(i)}$ with the same entries as $T^{(i+1)}$ and having the entry $2k - i$ in box containing a. Let $E_{i+1} = 2k - i$. At any given step i, $2k - i$ is the largest entry added to the tableau thus far, so $T^{(i)}$ is standard. Furthermore, $T^{(i+1)} = \left(\ E_{i+1} \longleftarrow T^{(i)}\ \right)$, since $E_{i+1} = 2k - i$ is already in the rim hook and thus simply delete it.

Proceeding in this manner, we will produce $E_{2k}, E_{2k-1}, \ldots, E_1$ which completely determines d. By the way we have constructed d, we have $d \longrightarrow (P_{[\lambda]}, Q_{[\lambda]})$. □

References

[1] S. Ariki and K. Koike, A Hecke Algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr_k$ and Construction of Its Irreducible Representations, *Advances in Mathematics,* 106(1994), 216-243.

[2] T. Halverson and A. Ram, Partition algebras, *European Journal of Combinatorics,* 26(2005), 869-921.

[3] T. Halverson and T. Lewandowski, RSK insertion for set partitions and diagram algebras, *Electronic J.Combinatorics,* 11(2)(2004-2005) R24.

[4] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, Addison Wesley Publishing Company, 1981.

[5] M.Kosuda, Characterization for the party algebras, *Ryukyu Math.J.,* 13(2000), 7-22.

[6] M. Kosuda, Irreducible representations of the party algebras, *Osaka J.Math.,* 43(2)(2006), 431-474.

[7] B.E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, Second edition. Graduate Texts in Mathematics, 203. Springer-Verlag, New York, 2001.

[8] K. Tanabe, On the centralizer algebra of the unitary reflection group G(m, p, n), *Nagoya Math. J.,* 148(1997), 113 - 126.

Received: March 1, 2013; *Accepted*: October 11, 2013

UNIVERSITY PRESS

Website: http://www.malayajournal.org/