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Curvature tensor of almost $C(\lambda)$ manifolds

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Abstract

The present paper deals with certain characterization of curvature conditions on Pseudo-projective and Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds. The main object of the paper is to study the flatness of the Pseudo-projective, Quasi-conformal curvature tensor, ξ-Pseudo-projective, ξ-Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds.

Keywords: Almost $C(\lambda)$ manifolds, Pseudo-projective curvature tensor, Quasi-conformal curvature tensor, ξ-Pseudo-projectively flat, ξ-Quasi-conformally flat, η-Einstein.

2010 MSC: 53C15, 53C20, 53C21, 53C25, 53D10.

1 Introduction

In 1981, D. Janssen and L. Vanhecke [6] have introduced the notion of almost $C(\lambda)$ manifolds. Further Z. Olszak and R. Rosca [11] investigated such manifolds. Again S. V. Kharitonava [8] studied conformally flat almost $C(\lambda)$ manifolds. In the paper [2] the author studied Ricci tecsor and quasi-conformal curvature tensor of almost $C(\lambda)$ manifolds. In the paper [1] the authors have studied on quasi-conformally flat spaces. Also in paper [4] the authors have studied on pseudo projective curvature tensor on a Riemannian manifold and in the paper [3] the authors are studied on the Conharmonic and Concircular curvature tensors of almost $C(\lambda)$ manifolds. Our present work is motivated by these works.

2 Preliminaries

Let M be a n-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [5].

$$\eta(\xi) = 1, \tag{2.1}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X, \xi) = \eta(X), \tag{2.4}$$
$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$
$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X. \tag{2.6}$$

If an almost contact Riemannian manifold M satisfies the condition

$$S = ag + b\eta \otimes \eta \tag{2.7}$$

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for some functions a and b in $C^{\infty}(M)$ and S is the Ricci tensor, then M is said to be an $\eta-$Einstein manifold. If, in particular, $a = 0$ then this manifold will be called a special type of $\eta-$Einstein manifold.
An almost contact manifold is called an almost $C(\lambda)$ manifold if the Riemannian curvature R satisfies the following relation [8]

$$R(X,Y)Z = R(\phi X, \phi Y)Z - \lambda[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y, Z) + g(\phi X, Z)\phi Y] \tag{2.8}$$

where, $X, Y, Z \in TM$ and λ is a real number.

Remark 2.1. *A C(l)-curvature tensor is a Sasakian curvature tensor, a C(O)-curvature tensor is a co-Kahler or CK-curvature tensor and a C(l)-curvature tensor is a Kenmotsu curvature tensor.*

From [9] we have,

$$R(X,Y)\xi = R(\phi X, \phi Y)\xi - \lambda[X\eta(Y) - \eta(X)Y] \tag{2.9}$$

On an almost $C(\lambda)$ manifold, we also have [2]

$$QX = AX + B\eta(X)\xi. \tag{2.10}$$

wher, $A = -\lambda(n-2)$, $B = -\lambda$ and Q is the Ricci-operator.

$$\eta(QX) = (A+B)\eta(X), \tag{2.11}$$
$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{2.12}$$
$$r = -\lambda(n-1)^2, \tag{2.13}$$
$$S(X,\xi) = (A+B)\eta(X), \tag{2.14}$$
$$S(\xi,\xi) = (A+B), \tag{2.15}$$
$$g(QX,Y) = S(X,Y). \tag{2.16}$$

3 Quasi-conformally flat almost $C(\lambda)$ manifolds

Definition 3.1. *The Quasi-conformal curvature tensor $\tilde{C}$ of type $(1,3)$ on a Riemannian manifold (M, g) of dimension n is defined by [1]*

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY) \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.17}$$

for all $X, Y, Z \in \chi(M)$, where Q is the Ricci-operator.

If $\tilde{C}$ vanishes identically then we say that the manifold is Quasi-conformally flat, where $a, b \neq 0$ are constants.
Thus for a Quasi-conformally flat $C(\lambda)$ manifold, we get from (3.17)

$$\begin{aligned}aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \\ &\quad -b(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY).\end{aligned} \tag{3.18}$$

By virtue of(2.10) and (2.12), (3.18) takes the form

$$\begin{aligned}aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] - b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X \\ &\quad -Ag(X,Z)YB\eta(X)\eta(Z)Y + Ag(Y,Z)X + B\eta(X)g(Y,Z) \\ &\quad -Ag(X,Z)Y - B\eta(Y)g(X,Z)].\end{aligned} \tag{3.19}$$

In view of (2.8) we get from (3.19)

$$\begin{aligned}aR(\phi X, \phi Y)Z &= \lambda a[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y, Z) + g(\phi X, Z)\phi Y] \\ &\quad + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] - b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X \\ &\quad - Ag(X,Z)YB\eta(X)\eta(Z)Y + Ag(Y,Z)X + B\eta(X)g(Y,Z) \\ &\quad - Ag(X,Z)Y - B\eta(Y)g(X,Z)]\end{aligned} \tag{3.20}$$

Putting $Y=\xi$ and using the value of A and B in (3.20) we get

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+2\lambda b(n-2)\right][X\eta(Z)-g(X,Z)\xi]=0. \tag{3.21}$$

Taking inner product of (3.21) with a vector field ξ, we get

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+2\lambda b(n-2)\right][\eta(X)\eta(Z)-g(X,Z)]=0. \tag{3.22}$$

Putting $X=QX$ in (3.22) we get

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+2\lambda b(n-2)\right][\eta(QX)\eta(Z)-g(QX,Z)]=0. \tag{3.23}$$

Using (2.15),(2.11) and by the virtue of (2.13) in (3.24) we get

$$\left[\lambda a+\lambda b-\frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1}+2b\right)+2\lambda b(n-2)\right][(A+B)\eta(X)\eta(Z)-S(X,Z)]=0. \tag{3.24}$$

Therefore, either

$$\lambda=0\ \ (or)\ \ S(X,Z)=A+B\eta(X)\eta(Y) \tag{3.25}$$

Thus we can state the following:

Theorem 3.1. *For a Quasi-conformally flat almost $C(\lambda)$ manifold, either $\lambda=0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (3.25) and remark (2.1). □

4 ξ-Quasi-conformally flat almost $C(\lambda)$ manifolds

Definition 4.1. *The Quasi-conformal curvature tensor $\tilde{C}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n will be defined as ξ-quasi-conformally flat [1] if $\tilde{C}(X,Y)\xi=0$ for all $X,Y\in TM$.*
Thus for a ξ-quasi-conformally flat almost $C(\lambda)$ manifolds we get from (3.17)

$$\begin{aligned} aR(X,Y)\xi &= \frac{r}{n}\left(\frac{a}{n-1}+2b\right)[\eta(Y)X-\eta(X)Y] \\ &- b(S(Y,\xi)X-S(X,\xi)Y+\eta(Y)QX-\eta(X)QY) \end{aligned} \tag{4.26}$$

In the view of (2.9). Taking $Y=\xi$, by virtue of (2.10), (2.14)and (2.15), putting the value A and B taking inner product with respect to vector field V we get from (4.26).

$$\left[\lambda a+\lambda b-\frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1}+2b\right)+2b\lambda(n-2)\right][g(X,V)-\eta(X)\eta(V)]=0 \tag{4.27}$$

Taking $X=QX$ in (4.27) we get

$$\left[\lambda a+\lambda b-\frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1}+2b\right)+2b\lambda(n-2)\right][g(QX,V)-\eta(QX)\eta(V)]=0 \tag{4.28}$$

Using (2.11) and (2.16) in (4.28)

$$\left[\lambda a+\lambda b-\frac{\lambda(n-1)^2}{n}\left(\frac{a}{n-1}+2b\right)+2b\lambda(n-2)\right][S(X,V)-(A+B)\eta(X)\eta(V)]=0 \tag{4.29}$$

Therefore, either

$$\lambda=0\ \ (or)\ \ S(X,Z)=A+B\eta(X)\eta(Y) \tag{4.30}$$

Thus we can state the following:

Theorem 4.1. *For a ξ-Quasi-conformally flat almost $C(\lambda)$ manifold, either $\lambda=0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (4.30) and remark (2.1). □

5 Pseudo-projectively curvature flat almost $C(\lambda)$ manifolds

Definition 5.1. *The Pseudo-projective curvature tensor $\tilde{P}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n is defined by [4]*

$$\begin{aligned} \tilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.31}$$

for all $X,Y,Z \in \chi(M)$. If $\tilde{p}$ vanishes identically then we say that the manifold is Pseudo-projectively curvature flat, where $a,b \neq 0$ are constants.

Thus for a Pseudo-projectively curvature flat $C(\lambda)$ manifold, we get from(5.31)

$$aR(X,Y)Z = \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] - b[S(Y,Z)X - S(X,Z)Y]. \tag{5.32}$$

By virtue of (2.12), (5.32) takes the form

$$\begin{aligned} aR(X,Y)Z &= \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \\ &- b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X - Ag(X,Z)Y - B\eta(X)\eta(Z)Y]. \end{aligned} \tag{5.33}$$

In view of (2.8) we get from (5.33)

$$\begin{aligned} aR(\phi X,\phi Y)Z &= \lambda a[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y] \\ &+ \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \\ &- b[Ag(Y,Z)X + B\eta(Y)\eta(Z)X - Ag(X,Z)Y - B\eta(X)\eta(Z)Y]. \end{aligned} \tag{5.34}$$

Putting $Y = \xi$, using the value of A and B, taking inner product with a vector field ξ in (5.34)we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\lambda(n-2)\right][\eta(X)\eta(Z) - g(X,Z)] = 0. \tag{5.35}$$

Taking $X = QX$ and by the virtue of (2.13) in (5.35) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\lambda(n-2)\right][\eta(QX)\eta(Z) - g(QX,Z)] = 0. \tag{5.36}$$

Using (2.15) and (2.11) in (5.36) we get

$$\left[\lambda a + \lambda b + \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\lambda(n-2)\right[(A+B)\eta(X)\eta(Z) - S(X,Z)] = 0. \tag{5.37}$$

Therefore, either

$$\lambda = 0 \ \ (or) \ \ S(X,Z) = A + B\eta(X)\eta(Y) \tag{5.38}$$

Thus we can state the following:

Theorem 5.1. *For a Pseudo-projectively curvature flat almost $C(\lambda)$ manifold, either $\lambda = 0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (5.38) and remark (2.1). □

6 $\xi-$Pseudo-projectively curvature flat almost $C(\lambda)$ manifolds

Definition 6.1. *The Pseudo-projectively curvature tensor $\tilde{P}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension n will be defined as ξ-Pseudo-projectively flat [4] if $\tilde{P}(X,Y)\xi$=0 for all $X,Y \in TM$.*
Thus for a ξ-Pseudo-projectively flat almost $C(\lambda)$ manifolds we get from (5.31)

$$\begin{aligned} aR(X,Y)\xi &= \frac{r}{n}\left(\frac{a}{n-1}+2b\right)[\eta(Y)X-\eta(X)Y] \\ &- b(S(Y,\xi)X - S(X,\xi)Y + \eta(Y)QX - \eta(X)QY) \end{aligned} \tag{6.39}$$

In the view of (2.9). Taking $Y=\xi$, by virtue of (2.10), (2.14)and (2.15), putting the value A and B taking inner product with respect to vector field V we get from (6.39)

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][g(X,V)-\eta(X)\eta(V)]=0. \tag{6.40}$$

Taking $X=QX$ in (6.40) we get

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][g(QX,V)-\eta(QX)\eta(V)]=0. \tag{6.41}$$

Using (2.11),(2.16) and by the virtue of (2.13) in (6.41)

$$\left[\lambda a+\lambda b+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)+b\lambda(n-2)\right][S(X,V)-(A+B)\eta(X)\eta(V)]=0. \tag{6.42}$$

Therefore, either

$$\lambda = 0 \ \ (or) \ \ S(X,Z) = A + B\eta(X)\eta(Y) \tag{6.43}$$

Thus we can state the following:

Theorem 6.1. *For a ξ-Pseudo-projectively flat almost $C(\lambda)$ manifold, either $\lambda = 0$ i.e. $C(\lambda)$ is cosymplectic. or the manifold is special type of $\eta-$Einstein.*

Proof. Follows form (6.43) and remark (2.1). □

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