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Existence results for fractional differential equations with infinite delay and interval impulsive conditions

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Abstract

This paper is mainly concerned with the existence and uniqueness of mild solutions for nonlocal fractional infinite delay differential equations with interval impulses. The results are obtained by using fixed point theorem.

Keywords: Fractional derivative, fractional differential equations, impulsive conditions, fixed point theorem.

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1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems. In the consequence, fractional differential equations have been of great interest. For details, see the monographs of Kilbas et al. [7], Lakshmikantham et al. [8], Miller and Ross [11], Podlubny [12], Anguraj et al. [1], [2] and the references there in.

On the other hand, the theory of impulsive differential equations is also an important area of research which has been investigated in the last few years by great number of mathematicians. We recall that the impulsive differential equations may better model phenomena and dynamical processes subject to a great changes in short times issued for instance in biotechnology, automatics, population dynamics, economics and robotics. To learn more about this kind of problems, we refer to the books [9], [13].

Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [5], [13]. Balachandran and Trujillo [3], [4] investigated the non-local cauchy poblem for non-linear fractional integro differential equations in Banach spaces. Xianmin Zhang [14] studied the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. In most of the impulsive differential equations studied so for, the impulses occur instantaneously. But there are some situations in which the impulsive action starts abruptly and stays active on a finite time interval. Eduardo Hernandez and Donal O'Regan [6] established on a new class of abstract impulsive differential equations for which the impulses are not instantaneous.

Motivated by [6], we consider the following fractional infinite delay differential equations with interval

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impulsive

$$D_t^q x(t) = f(t, x_t), \qquad t \in (s_i, t_{i+1}],\ i = 0, 1, ..., N, \tag{1.1}$$

$$x(t) = g_i(t, x(t)), \qquad t \in (t_i, s_i],\ i = 1, 2, ..., N, \tag{1.2}$$

$$x(0) + k(x) = \phi, \qquad \phi \in B_\vartheta, \tag{1.3}$$

where $0 < q < 1$ and the state $x(.)$ belongs to Banach space X endowed with the norm $\| . \|$. D_t^q is the Caputo fractional derivative and f is a suitable function. $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < ...t_N \le s_N \le t_{N+1} = b$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, ..., N$. Let $x_t(.)$ denote $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. The impulses starts abruptly at the point t_i and their action continue on the interval $[t_i, s_i]$.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we study the existence and the uniqueness of solutions for the impulsive fractional system (1.1)-(1.3).

2 Preliminaries

In this section, we shall introduce some basic definitions, notations, lemmas and theorem which are used throughout this paper.

Assume that $\vartheta : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfying $\ell = \int_{-\infty}^0 \vartheta(t)dt < +\infty$. The Banach space $(B_\vartheta, ||.||_{B_\vartheta})$ induced by the function ϑ is defined as follows

$$B_\vartheta = \begin{cases} \varphi : (-\infty, 0] \longrightarrow X :\ for\ any\ c > 0,\ \varphi(\theta)\ is\ a\ bounded\ and \\ measurable\ function\ on\ [-c, 0]\ and\ \int_{-\infty}^0 \vartheta(t) sup_{t\le\theta\le 0} \|\varphi(\theta)\|\, dt < +\infty \end{cases}$$

endowed with the norm $\|\varphi\|_{B_\vartheta} = \int_{-\infty}^0 \vartheta(s) sup_{s\le\theta\le 0} \|\varphi(\theta)\|\, ds$.
Let us define the space

$$B'_\vartheta = \begin{cases} \varphi : (-\infty, b] \to X : \varphi_k \in C(J_k, X),\ k = 0, 1, 2, ..., N\ and\ there\ exist \\ \varphi(t_k^-) and\ \varphi(t_k^+)\ with\ \varphi(t_k) = \varphi(t_k^-),\ \varphi(t) = g_k(t, x(t)),\ t \in (t_k, s_k], \\ k = 1,\ 2, ..., N,\ \varphi_0 = \varphi(0) + k(\varphi) = \phi \in B_\vartheta \end{cases}$$

where φ_k is the restriction of φ to J_k, $J_0 = [0, t_1]$, $J_k = [s_k, t_{k+1}]$, $k = 1, 2, ..., N$.
Denote by $||.||_{B'_\vartheta}$, a seminorm in the space B'_ϑ, which is defined by
$||\varphi||_{B'_\vartheta} = ||\varphi||_{B_\vartheta} + max\left\{||\varphi_k||_{J_k},\ k = 1, 2, ..., N\right\}$ where $||\varphi_k||_{J_k} = sup_{s\in J_k} ||\varphi_k(s)||$.

For the impulsive conditions, we consider the space $PC(X)$ which is formed by all the functions $x : [0, b] \to X$ such that $x(.)$ is continuous at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i = 1, 2, ..., N$, endowed with the uniform norm on $[0, b]$ denoted by $\|x\|_{PC(X)}$. It is easy to see that $PC(X)$ is a Banach space. For a function $x \in PC(X)$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\tilde{x}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\tilde{x}_i(t) = \begin{cases} x(t), & for\ t \in (t_i, t_{i+1}] \\ x(t_i^+), & for\ t = t_i. \end{cases} \tag{2.1}$$

In addition, for $B \subseteq PC(X)$ we use the notation $\tilde{B}_i$ for the set $\tilde{B}_i = \left\{x_i : x \in B\right\}$ and $i \in \{0, 1, ..., N\}$.

Lemma 2.1. *A set $B \subseteq PC(X)$ is relatively compact in $PC(X)$ if and only if each set $\tilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], X)$.*

Theorem 2.1. *(Schauder's Theorem) Suppose that D is a closed bounded convex subset of the Banach space X and A is completely continuous function from D into D. Then there is a point $z \in D$ such that $Az = z$.*

Definition 2.1. *A function $x : (-\infty, b] \to X$ is called a mild solution of the problem $(1.1) - (1,3)$ if $x(0) + k(x) = \phi \in B_\vartheta$, $x(t) = g_i(t, x(t))$ for all $t \in (t_i, s_i]$, each $i = 1, 2, ..., N$, the restriction of $x(.)$ to the interval $J_k (k = 0, 1, 2, ..., N)$ is continuous, and the following integral equation holds*

$$x(t) = \phi(0) - k(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_s) ds, \qquad for\ all\ \ t \in [0, t_1]\ \ and$$

$$x(t) = g_i(s_i, x(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, x_s) ds,\ for\ all\ t \in [s_i, t_{i+1}]\ and\ every\ i = 1, 2, ..., N.$$

Definition 2.2. *The Riemann - Liouville fractional integral operator of order $q \geq 0$ of function $f \in L^1(R^+)$ is defined as*

$$I^q_{0^+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \ \ t > 0$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.3. *The Caputo fractional derivative of order $q \geq 0$, $n-1 < q < n$, is defined as*

$$D^q_{0^+} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{(n-q-1)} f^{(n)}(s) ds, \quad t > 0$$

where the function f(t) has absolutely continuous derivatives up to order (n-1).

If $0 < q < 1$, then

$$D^q_{0^+} f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{(-q)} f^{(1)}(s) ds$$

where $f^{(1)}(s) = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

Lemma 2.2. *Assume that $x \in B'_\vartheta$ then, for $t \in [0,b]$, $x_t \in B_\vartheta$. Moreover*

$$\ell ||x(t)|| \leq ||x_t||_{B_\vartheta} \leq ||\phi||_{B_\vartheta} + \ell \ \ sup_{s\in[0,t]} ||x(s)||.$$

3 Main results

For $\phi \in B_\vartheta$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \phi(0), & t \in [0, t_1] \\ 0, & t \in (t_1, b] \end{cases}$$

then $\hat{\phi} \in B'_\vartheta$.

Let $x(t) = y(t) + \hat{\phi}(t), \ -\infty < t < b$. It is evident that y satisfies $y_0 = 0, \ t \in (-\infty, 0]$,

$$y(t) = -k(y+\hat{\phi}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, \qquad forall \ \ t \in [0, t_1],$$

$$y(t) = g_i(t, (y+\hat{\phi})(t)), \qquad for\ all\ t \in (t_i, s_i] \ and\ each\ i = 1, 2, ..., N,$$

and

$$y(t) = g_i(s_i, (y+\hat{\phi})(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds,$$
$$for\ all\ t \in [s_i, t_{i+1}] \ and\ every\ i = 1, 2, ..., N$$

if and only if x satisfies

$$x(t) = \phi(t), \qquad t \in (-\infty, 0],$$

$$x(t) = \phi(0) - k(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_s) ds, \qquad for\ all \ \ t \in [0, t_1],$$

$$x(t) = g_i(t, x(t)), \qquad for\ all\ t \in (t_i, s_i] \ and\ each\ i = 1, 2, ..., N,$$

and

$$x(t) = g_i(s_i, x(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, x_s) ds,$$
$$for\ all\ t \in [s_i, t_{i+1}] \ and\ every\ i = 1, 2, ..., N.$$

To prove our main results, we introduce the following conditions:

(H_1) $f : [0,b] \times B_\vartheta \to X$ is continuous and there exist two positive constants $K_1,\ K_2$ such that $||f(t,\phi_1) - f(t,\phi_2)|| \leq K_1||\phi_1 - \phi_2||_{B_\vartheta},\ K_2 = sup_{t\in[0,b]}||f(t,0)||$.

(H_2) The functions $g_i : (t_i, s_i] \times X \to X$ are continuous and there are positive constants L_{g_i} such that $\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i} \|x - y\|$ for all $x,\ y \in X, t \in (t_i, s_i]$ and each $i = 0,1,...,N$.

(H_3) $k : B'_\vartheta \to X$ is continuous and there exist some positive constant$\delta_1,\ \delta_2$ such that $||k(x) - k(y)|| \leq \delta_1||x - y||_{B'_\vartheta}$ and $||k(x)|| \leq \delta_1||x||_{B'_\vartheta} + \delta_2$.

(H_4) $\Omega = max_i \left\{ L_{g_i} + \frac{K_1 \ell b^q}{\Gamma(q+1)} + \delta_1 \right\} < 1,\ i = 1,2,...,N.$

Theorem 3.1. *Suppose that the conditions $(H_1) - (H_4)$ are satisfied then there exists a unique mild solution of the problem (1.1)-(1.3).*

Proof. Define $\Theta : B'_\vartheta \to B'_\vartheta$ by

$$\Theta y(t) = 0, \qquad t \in (-\infty, 0]$$

$$\Theta y(t) = -k(y + \hat{\phi}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, \qquad for\ all\ \ t \in [0, t_1],$$

$$\Theta y(t) = g_i(t, (y + \hat{\phi})(t)), \qquad for\ all\ t \in (t_i, s_i]\ and\ each\ i = 1,2,...,N,$$

and

$$\Theta y(t) = g_i(s_i, (y + \hat{\phi})(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, \qquad for\ all\ t \in [s_i, t_{i+1}]\ and\ every\ i = 1,2,...,N.$$

Clearly, y is a fixed point of Θ then $y + \hat{\phi}$ is a solution of the system (1.1)-(1.3). We shall show that Θ satisfies the hypotheses of Theorem 2.1.

Define the Banach space $(B''_\vartheta, ||.||_{B'_\vartheta})$ induced by B'_ϑ ,

$B''_\vartheta = \{y \in B'_\vartheta : y_0 = 0 \in B_\vartheta\}$ with norm $||y||_{B'_\vartheta} = sup\{||y(s)|| : s \in [0,b]\}$, set $B_r = \{y \in B''_\vartheta : ||y||_{B'_\vartheta} \leq r\}$ for some $r > 0$.

For any $y \in B_r$, $t \in [0,b]$ and by Lemma 2.2, we have

$$|y_t + \hat{\phi}_t||_{B_\vartheta} \leq ||\phi||_{B_\vartheta} + \ell[r + \|\phi(0)\|],$$
$$||y + \hat{\phi}||_{B'_\vartheta} \leq r + ||\phi||_{B_\vartheta} + \|\phi(0)\|.$$

From the assumption it is easy to see that Θ is well defined. Moreover, for $y_1, y_2 \in B'_\vartheta$, i $\in \{1,2,...,N\}$, and t $\in [s_i, t_{i+1}]$ we get

$$\begin{aligned}
||\Theta y_1(t) - \Theta y_2(t)|| &\leq ||g_i(s_i, (y_1 + \hat{\phi})(s_i)) - g_i(s_i, (y_2 + \hat{\phi})(s_i))|| \\
&\quad + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} ||f(s, y_{1_s} + \hat{\phi}_s) - f(s, y_{2_s} + \hat{\phi}_s)|| ds \\
&\leq L_{g_i} ||y_1 - y_2||_{B'_\vartheta} + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} k_1 \ell ||y_1 - y_2||_{B'_\vartheta} ds \\
&\leq [L_{g_i} + \frac{K_1 \ell b^q}{\Gamma(q+1)}] ||y_1 - y_2||_{B'_\vartheta}
\end{aligned}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C([s_i, t_{i+1}];X)} \leq \Omega \, \|y_1 - y_2\|_{B'_\vartheta}, \ \ i = 1,2,...,N.$$

proceeding the same manner for the interval $[0,t_1]$, we obtain that

$$\begin{aligned} ||\Theta y_1(t) - \Theta y_2(t)|| &\leq || - k(y_1 + \hat{\phi}) + k(y_2 + \hat{\phi})|| \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ||f(s, y_{1_s} + \hat{\phi}_s) - f(s, y_{2_s} + \hat{\phi}_s)||ds \\ &\leq [\delta_1 + \frac{K_1 \ell b^q}{\Gamma(q+1)}]||y_1 - y_2||_{B'_\vartheta} \end{aligned}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C([0,t_1];X)} \leq \Omega \, \|y_1 - y_2\|_{B'_\vartheta}$$

Moreover, for $t \in (t_i, s_i]$ we have

$$||\Theta y_1(t) - \Theta y_2(t)|| \leq L_{g_i} ||y_1 - y_2||_{B'_\vartheta}$$

hence

$$\|\Theta y_1 - \Theta y_2\|_{C((t_i,s_i];X)} \leq \Omega \, \|y_1 - y_2\|_{B'_\vartheta}, \;\; i = 1, 2, ..., N$$

From the above we have that $\|\Theta y_1 - \Theta y_2\| \leq \Omega \, \|y_1 - y_2\|_{B'_\vartheta}$. Therefore Θ is a contraction and there exists a unique mild solution of (1.1)-(1.3). This completes the proof. □

Next, we establish the existence of a mild solution using a fixed point criteria for completely continuous maps.

Theorem 3.2. *Assume the hypotheses $(H_1) - (H_4)$ are satisfied and the functions $g_i(.,0)$ are bounded then the system (1.1)-(1.3) has a mild solution.*

Proof. We divide the proof into five steps.
Step 1: To prove $\Theta B_r \subset B_r$.

There exists a positive integer r such that B_r is clearly a closed bounded convex set in B'_ϑ. If $\Theta B_r \subset B_r$ is not true then for each positive integer r, there exist y $\in B_r$ and $t \in (-\infty, b]$ such that $||\Theta(y)(t)|| > r$, where t is depending upon r.
However, on the other hand for i ≥ 1, let y $\in B_r$ and t $\in (t_i, s_i]$ we have

$$\begin{aligned} r &< ||\Theta y(t)|| \\ &\leq ||g_i(t, (y + \hat{\phi})(t))|| \\ &\leq L_{g_i} ||y + \hat{\phi}||_{B'_\vartheta} + ||g_i(t, 0)|| \\ &\leq L_{g_i}(r + ||\phi||_{B_\vartheta}) + ||g_i(.,0)||_{C((t_i,s_i];X)} \end{aligned}$$

Dividing on both sides by r and taking the lower limit as $r \to +\infty$, weget $1 \leq L_{g_i}$. This is a contradiction to (H_4). Therefore $||\Theta y||_{C((t_i,s_i];X)} \leq r$ for $i \geq 1$.
Proceeding as above for $t \in [s_i, t_{i+1}]$ and $t \in [0, t_1]$, $i \geq 1$ we obtain that
$1 \leq L_{g_i} + \frac{K_1 \ell b^q}{\Gamma(q+1)}$ and $1 \leq \delta_1 + \frac{K_1 \ell b^q}{\Gamma(q+1)}$, which gives a contradiction to (H_4). Hence, for some positive integer r, $\Theta B_r \subset B_r$.

Next, we introduce the decomposition $\Theta = \Theta_1 + \Theta_2 = \sum_{i=0}^{N} \Theta_i^1 + \sum_{i=0}^{N} \Theta_i^2$ where $\Theta_i^j : B_r \to B_r, i = 1, 2, ..., N$, $j = 1, 2$ are given by

$$\Theta_i^1 y(t) = \begin{cases} 0, & for \;\; t \in (-\infty, 0], \\ -k(y + \hat{\phi}), & for \;\; t \in [0, t_1], \\ g_i(t, (y + \hat{\phi})(t)), & for \;\; t \in (t_i, s_i], \;\; i \geq 1, \\ g_i(s_i, (y + \hat{\phi})(s_i)), & for \;\; t \in (s_i, t_{i+1}], \;\; i \geq 1, \\ 0, & for \;\; t \notin (t_i, t_{i+1}], \;\; i \geq 0. \end{cases}$$

$$\Theta_i^2 y(t) = \begin{cases} 0, & for \ \ t \in (-\infty, 0], \\ \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds, & for \ \ t \in (s_i, t_{i+1}], \quad i \geq 0, \\ 0, & for \ \ t \notin (s_i, t_{i+1}], \quad i \geq 0. \end{cases}$$

Step 2: The map $\Theta_1 = \sum_{i=0}^{N} \Theta_i^1$ is a contraction on B_r.
Take $y_1, y_2 \in B_r$ arbitrarily. Then, for each $t \in (-\infty, b]$ and from (H_2) to (H_4), we have

$$||\Theta_i^1 y_1(t) - \Theta_i^1 y_2(t)|| \leq \delta_1 ||y_1 - y_2||_{B'_\vartheta} + L_{g_i} ||y_1 - y_2||_{B'_\vartheta}$$

Which implies that $\left\| \sum_{i=0}^{N} \Theta_i^1 y_1 - \sum_{i=0}^{N} \Theta_i^1 y_2 \right\| \leq \Omega \left\| y_1 - y_2 \right\|_{B'_\vartheta}$.

This proves that Θ_1 is a contraction on B_r.

Next, we use the notation $\Theta_i^2 B_r(t) = \left\{ \Theta_i^2 y(t) : B_r \right\}$.

Step 3: For $i = 0, 1, ..., N$ and $s_i < s < t \leq t_{i+1}$, the set $\cup_{\tau \in [s,t]} \Theta_i^2 B_r(\tau)$ is relatively compact in B'_ϑ. Let $s_i < \mu < s$. For $\epsilon > 0$ we choose $0 < \lambda < \frac{s-\mu}{2}$ such that $\frac{\lambda^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \leq \epsilon$ for all interval $I \subset [0, b]$ with $Diam(I) \leq \lambda$.

Then, for $\tau \in [s, t]$ and $y \in B_r$ we get

$$\begin{aligned} \Theta_i^2 y(\tau) &= \frac{1}{\Gamma(q)} \int_{s_i}^{\tau - \lambda} (\tau - \lambda - s)^{q-1} f(s, y_s + \hat{\phi}_s) ds + \frac{1}{\Gamma(q)} \int_{\tau - \lambda}^{\tau} (\tau - s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \\ &\in B_{r_1} + B_{r_1, \epsilon}, \end{aligned}$$

where $r_1 = \frac{b^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]$, $r_1, \epsilon = \frac{\lambda^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]$, which implies that $\cup_{\theta \in [s,t]} \Theta_i^2 B_r(\theta) \subset B_{r_1} + B_{r_1, \epsilon}$. Since B_{r_1} is relatively compact and $Diam(B_\epsilon) \to 0$ as $\epsilon \to 0$, it follows that $\cup_{\theta \in [s,t]} \Theta_i^2 B_r(\theta)$ is relatively compact in B'_ϑ.

In the next step we use the notation introduced in (2.1).

Step 4: The set of functions $\left\{ \Theta_i^2 \widetilde{B_r} \right\}_i$, $i = 0, 1, ..., N$, is a equicontinuous subset of $C([t_i, t_{i+1}]; X)$.

It is clear that $\left\{ \Theta_i^2 \widetilde{B_r} \right\}_i$ is right equicontinuous on $[t_i, s_i)$ and left equicontinuous on $(t_i, s_i]$. Let $t \in (s_i, t_{i+1})$, since the set $\Theta_i^2 B_r(t)$ is relatively compact in B'_ϑ. Then, for $y \in B_r$ and $0 < h < \lambda < t_{i+1} - t$ we get

$$\begin{aligned} \left\| \tilde{\Theta_i^2} y(t+h) - \tilde{\Theta_i^2} y(t) \right\| &= \left\| \Theta_i^2 y(t+h) - \Theta_i^2 y(t) \right\| \\ &= || \frac{1}{\Gamma(q)} \int_{s_i}^{t+h} (t+h-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \\ &\quad - \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds || \\ &\leq \frac{1}{\Gamma(q)} \int_{t}^{t+h} (t+h-s)^{q-1} ||f(s, y_s + \hat{\phi}_s)|| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{s_i}^{t} ||(t+h-s)^{q-1} - (t-s)^{q-1}|| ||f(s, y_s + \hat{\phi}_s)|| ds \\ &\leq \frac{h^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \\ &\quad + \frac{1}{\Gamma(q)} \int_{s_i}^{t} ||(t+h-s)^{q-1} - (t-s)^{q-1}|| [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] ds \end{aligned}$$

The right-hand side is independent of $y \in B_r$ and tends to zero as $h \to 0$. This shows that $\left\{ \widetilde{\Theta_i^2 B_r} \right\}_i$ is right equicontinuous at t.

In the simillar manner we proceed for $t = s_i$ and $h > 0$ with $s_i + h < t_{i+1}$we have that

$$\begin{aligned} \left\| \tilde{\Theta_i^2} y(s_i + h) - \tilde{\Theta_i^2} y(s_i) \right\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{s_i}^{s_i + h} (t+h-s)^{q-1} f(s, y_s + \hat{\phi}_s) ds \right\| \\ &\leq \frac{h^q}{\Gamma(q+1)} [k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \end{aligned}$$

which implies that $\left\{\widetilde{\Theta_i^2 B_r}\right\}_i$ is right equicontinuous at s_i.

Now for $t \in (s_i, t_{i+1}]$. Let $\mu \in (s_i, t]$. Since $\cup_{s\in[\mu,t]}\Theta_i^2 B_r(s)$ is relatively compact in B'_ϑ, we select $0 < \lambda < \frac{t-\mu}{2}$ then for $0 < h \leq \lambda$ and $y \in B_r$ we get,

$$
\begin{aligned}
\left\|\tilde{\Theta_i^2} y(t-h) - \tilde{\Theta_i^2} y(t)\right\| &= \left\|\Theta_i^2 y(t-h) - \Theta_i^2 y(t)\right\| \\
&\leq \frac{1}{\Gamma(q)}\int_{t-h}^{t}(t-s)^{q-1}||f(s, y_s + \hat{\phi}_s)||ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{s_i}^{t-h}||(t-s)^{q-1} - (t-h-s)^{q-1}||||f(s, y_s + \hat{\phi}_s)||ds \\
&\leq \frac{h^q}{\Gamma(q+1)}[k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2] \\
&\quad + \frac{1}{\Gamma(q)}\int_{s_i}^{t}||(t-s)^{q-1} - (t-h-s)^{q-1}||[k_1(\ell r + ||\phi||_{B_\vartheta}) + k_2]ds
\end{aligned}
$$

which shows that $\{\widetilde{\Theta_i^2 B_r}\}_i$ is left equicontinuous at $t \in (s_i, t_{i+1}]$.

This completes the proof that the set $\{\widetilde{\Theta_i^2 B_r}\}_i$ is equicontinuous.

Step 5: For $i \neq j$, the set $\{\widetilde{\Theta_i^2 B_r}\}_j$ is a equicontinuous subset of $C([t_j, t_{j+1}]; X)$.

From the above steps and Lemma 2.1 it follows that, the map Θ_1 is a contraction and the maps Θ_2 are completely continuous. Thus, $\Theta = \Theta_1 + \Theta_2$ is a condensing operator. Finally, from [[10], Theorem 4.3.2]. we assert that there exists a mild solution of (1.1)-(1.3). □

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