(e)-Convergence for double sequences

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Abstract. We define the notion of (e)-summability of double sequences and series of complex numbers. We also obtain a criteria for this summability method with regards to Berezin symbols of an diagonal operator, and show regularity of (e)-summability method for double sequences.

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Contents

1. Introduction and Background

A double sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ is called the convergent in Pringsheim's sense [14] provided that there exists a number a such that a_{mn} converges to a as both m and n approach to infinity independently of one another

$$\lim_{m,n\to\infty} a_{mn} = a,$$

that is if for every $\varepsilon > 0$ there exists $K = K(\varepsilon) \in \mathbb{N}$ such that $|a_{mn} - a| < \varepsilon$ for every $m, n \geq K$ and also a is said to the Pringsheim's limit of a_{mn}. It is obvious that $\{a_{mn}\}$ is convergent in Pringsheim's sense if and only if for every $\varepsilon > 0$ there exists an integer $K = K(\varepsilon) \in \mathbb{N}$ such that $|a_{mn} - a_{ij}| < \varepsilon$ for $\min\{m, n, i, j\} \geq K$. A double sequence $\{a_{mn}\}$ is bounded provided that there exists a positive number N such that $|a_{mn}| \leq N$ for every m and n, i.e., $\sup_{m,n} |a_{mn}| < \infty$.

A double sequence $\{a_{mn}\}$ is said to be convergent regularly provided that it is convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m,n\to\infty} a_{mn} = x_m \ (m = 1, 2, ...),$$
$$\lim_{m,n\to\infty} a_{mn} = x_n \ (n = 1, 2, ...).$$

It is well known that a convergent double sequence in Pringsheim's sense fails in general to be bounded. The concept of regular convergence, which was introduced by Hardy in [7], lacks this advantage. Moreover, the regular convergence requires the convergence of rows and columns of a double sequence. (For more information about several type convergence for double sequences, see [18] and its references.)

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A reproducing kernel Hilbert space (shorty, RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ on some set Ω is a Hilbert space of functions on Ω such that for every $\lambda \in \Omega$ the linear functional (evaluation functional) $f \to f(\lambda)$ is bounded on $\mathcal{H}$. If $\mathcal{H}$ is RKHS on set Ω, then by the classical Riesz Representation Theorem for every $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ for which $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is said to be reproducing kernel at λ. We know that (see, Aronzajn [1] and Saitoh [15]) provided that $(e_j)_{j \in J}$ is an orthonormal basis for the RKHS $\mathcal{H}$,

$$k_\lambda(z) = \sum_{j \in J}^{\infty} \overline{e_j(\lambda)} e_j(z), \ \ z \in \Omega.$$

The function

$$\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}} = \frac{1}{\left(\sum_{j \in J} |e_j(\lambda)|^2 \right)^{1/2}} \sum_{j \in J} \overline{e_j(\lambda)} e_j(z)$$

is called the normalized reproducing kernel at λ.

Berezin [2, 3] introduced the concept of contravariant and covariant symbols of an operator. The contravariant symbol of a Toeplitz operator, which is the so-called Berezin symbol, was firstly used by Berger and Coburn in [4, 5].

Let A be a bounded operator on reproducing kernel Hilbert spaces. Then the function

$$\widetilde{A}(\lambda) :=< A\widehat{k}_\lambda, \widehat{k}_\lambda >, \ \lambda \in \Omega,$$

is called the Berezin symbol, which is a bounded function by the norm of the operator (see [2]). On the reproducing kernel Hilbert spaces, $\widetilde{A}_1(\lambda) = \widetilde{A}_2(\lambda)$ for all λ implies $A_1 = A_2$, that is, the Berezin symbol uniquely determines the operator. Therefore, the Berezin symbol includes many information about the operator that induces it. Prosperous applications of the Berezin symbol are up to now commonly in the study of operator theory, such as Toeplitz and Hankel operators [19]. The Berezin symbol technique is motivated by its connections with quantum physics (see, for example, [2, 3]). Readers can found more informations about Berezin symbols and its applications, for instance in [9, 13, 19].

A RKHS $\mathcal{H}(\Omega)$ is standard provided that the underlying set Ω is a subset of a topological space and the boundary of Ω is non-empty and has the property that $(k_{\mathcal{H},\lambda_n})_n$ converges weakly to 0 whenever $(\lambda_n)_n$ is a sequence in Ω that converges to a point in $\partial\Omega$. It is obvious that $\lim_{n \to \infty} \widetilde{K}(\lambda_n) = 0$ for any compact operator K on the standard RKHS $\mathcal{H}$ whenever $(\lambda_n)_n \subset \Omega$ converges to a point of $\partial\Omega$. In this case, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary (see [13]).

Karaev [11] introduced (e)-convergent for single sequences and series of complex numbers. Later, he [12] gave a Tauberian-type therom for (e)-convergent sequences. Using the Berezin symbol technique, new proofs for (L)-convergence and Abel convergence were given in [8, 16].

2. Main Results

In this section, we define a concept of (e)-convergence for double sequences and series of complex numbers. We obtain a criteria for this summability method with regards to Berezin symbols of an diagonal operator, and show regularity of (e)-summability method for double sequences.

Recall that a method is called the regular provided that it sums each convergent sequence to its ordinary limit. For instance, Abel, Cezaro and Borel methods are regular (see [6]).

Let $\mathcal{H} = \mathcal{H}(\Omega \times \Omega)$ be a reproducing kernel Hilbert space on some set $\Omega \times \Omega$, $\{e_{mn}\}_{m,n \geq 0}$ be an orthonormal basis of $\mathcal{H}$ and

$$k_{\lambda,\mu}(z,w) := \sum_{m,n \geq 0} \overline{e_{mn}(\lambda,\mu)} e_{mn}(z,w),$$

be a reproducing kernel of $\mathcal{H} = \mathcal{H}(\Omega \times \Omega)$.

Definition 2.1. *Let $\{a_{mn}\}_{m,n\geq0}$ be a double sequence of complex numbers.*
(a) The sequence $\{a_{mn}\}_{m,n\geq0}$ is (e)-convergent to L provided that

$$\sum_{m,n=0}^{\infty} a_{mn} \left|e_{mn}\left(\lambda,\mu\right)\right|^2$$

converges for all $(\lambda,\mu) \in \Omega \times \Omega$ and

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \frac{1}{\sum_{m,n=0}^{\infty} \left|e_{mn}\left(\lambda,\mu\right)\right|^2} \sum_{m,n=0}^{\infty} a_{mn} \left|e_{mn}\left(\lambda,\mu\right)\right|^2 = L,$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.
(b) The series $\sum_{m,n=0}^{\infty} a_{mn}$ is (e)-summable to L provided that

$$\sum_{m,n=0}^{\infty} a_{mn} \left|e_{mn}\left(\lambda,\mu\right)\right|^2$$

converges for each $(\lambda,\mu) \in \Omega \times \Omega$ and

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \sum_{m,n=0}^{\infty} a_{mn} \left|e_{mn}\left(\lambda,\mu\right)\right|^2 = L$$

for each $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.

It was shown that Abel and Borel summability for double sequences coincide with concept of (e)-summability for Hardy space and Fock space, respectively (see [10, 17]).

Let $\{a_{mn}\}_{m,n\geq0}$ be a double sequence of complex numbers. Diagonal operator $D_{\{a_{mn}\}}$ on $\mathcal{H}$ is defined by

$$D_{\{a_{mn}\}}e_{mn}\left(\lambda,z\right) = a_{mn}e_{mn}\left(\lambda,z\right), \; m,n = 0,1,2,...,$$

with respect to the orthonormal basis $e = \{e_{mn}\left(\lambda,z\right)\}_{m,n\geq0}$ of $\mathcal{H}$.

The following result is main theorem of this section.

Theorem 2.2. *Let $\{a_{mn}\}_{m,n\geq0}$ be a bounded double sequence of complex numbers.*
(a) The sequence $\{a_{mn}\}_{m,n\geq0}$ is (e)-convergent to L if and only if

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \widetilde{D}_{\{a_{mn}\}}\left(\lambda,\mu\right) = L$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.
(b) The series $\sum_{m,n=0}^{\infty} a_{mn}$ is (e)-summable to L if and only if

$$\lim_{(\lambda,\mu)\to(\zeta,\xi)} \left(\sum_{m,n=0}^{\infty} \left|e_{mn}\left(\lambda,\mu\right)\right|^2\right) \widetilde{D}_{\{a_{mn}\}}\left(\lambda,\mu\right) = L$$

for every $(\zeta,\xi) \in \partial\Omega \times \partial\Omega$.
(c) (e)-summability method for double sequences is regular provided that $\mathcal{H}$ is a standard functional Hilbert space.

Proof. As $\{a_{mn}\}_{m,n\geq 0}$ is a bounded double sequence, $D_{\{a_{mn}\}}$ is a bounded operator on $\mathcal{H}$. Calculating the Berezin symbol of diagonal operator, we have

$$
\begin{aligned}
\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) &= \left\langle D_{\{a_{mn}\}}\widehat{k}_{\lambda,\mu},\widehat{k}_{\lambda,\mu}\right\rangle \\
&= \frac{1}{\|k_{\lambda,\mu}\|^2}\left\langle D_{\{a_{mn}\}}\sum_{m,n=0}^{\infty}\overline{e_{mn}(\lambda,\mu)}e_n(z,w),k_{\lambda,\mu}\right\rangle \\
&= \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\left\langle \sum_{m,n=0}^{\infty}\overline{e_{mn}(\lambda,\mu)}a_{mn}e_{mn}(z,w),k_{\lambda,\mu}\right\rangle \\
&= \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{m,n}|e_{mn}(\lambda,\mu)|^2
\end{aligned}
$$

for all $(\lambda,\mu)\in\Omega\times\Omega$. Therefore

$$
\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) = \frac{1}{\sum_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{mn}|e_{mn}(\lambda,\mu)|^2,\ (\lambda,\mu)\in\Omega\times\Omega. \tag{1}
$$

As $\sup_{(\lambda,\mu)\in\Omega\times\Omega}\left|\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu)\right|\leq\left\|\widetilde{D}_{\{a_{mn}\}}\right\| = \sup_{m,n\geq 0}|a_{mn}| < \infty$, formula (1) immediately implies the claims (a) and (b) of the theorem.

Let us show the claim (c). Let $\{a_{mn}\}_{m,n=0}^{\infty}$ converges to L. Then $D_{\{a_{mn}-L\}}$ is a compact operator, and hence $\widetilde{D}_{\{a_{mn}-L\}}$ vanishes on the boundary of $\Omega\times\Omega$ (since $\mathcal{H}$ is a standard reproducing kernel Hilbert space), that is, $\widetilde{D}_{\{a_{mn}-L\}}(\lambda,\mu)\to 0$ as $(\lambda,\mu)\to(\zeta,\xi)\in\partial\Omega\times\partial\Omega$. Taking into consideration this and formula (1), we get

$$
\begin{aligned}
\lim_{(\lambda,\mu)\to(\zeta,\xi)}\widetilde{D}_{\{a_{mn}\}}(\lambda,\mu) &= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}a_{mn}|e_{mn}(\lambda,\mu)|^2 \\
&= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}(a_{mn}-L+L)|e_{mn}(\lambda,\mu)|^2 \\
&= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\frac{1}{\sum\limits_{m,n=0}^{\infty}|e_{mn}(\lambda,\mu)|^2}\sum_{m,n=0}^{\infty}(a_{mn}-L)|e_{mn}(\lambda,\mu)|^2 + L \\
&= \lim_{(\lambda,\mu)\to(\zeta,\xi)}\widetilde{D}_{\{a_{mn}-L\}} + L,
\end{aligned}
$$

which gives that (e)-$\lim_{m,n}a_{mn} = L$. So, the proof is completed. $\blacksquare$

We can obtain the following result from Theorem 1 by putting $\mathcal{H} = \mathcal{D}(\mathbb{D}^2)$ and $\mathcal{H} = \mathcal{F}(\mathbb{C}^2)$.

Corollary 2.3. *Let $\{a_{mn}\}_{m,n\geq 0}$ be a bounded double sequence of complex numbers.*
(a) If $D_{\{a_{mn}\}}$ is a diagonal operator on the Dirichlet space $\mathcal{D}(\mathbb{D}^2)$ with diagonal elements a_{mn}, $m,n\geq 0$, with respect to the orthonormal basis of $\mathcal{D}$, then the double sequence $\{a_{mn}\}_{m,n\geq 0}$ is (L)-convergent (logarithmic convergent) to L if and only if

$$
\lim_{\lambda,\mu\to 1^-}\widetilde{D}_{\{a_{mn}\}}\left(\sqrt{x},\sqrt{y}\right) = L,
$$

where $x = |\lambda|^2$ and $y = |\mu|^2$.
(b) If $D_{\{a_{mn}\}}$ is a diagonal operator on the Fock space $\mathcal{F}(\mathbb{C}^2)$ with diagonal elements a_{mn}, $m,n\geq 0$, with

respect to the orthonormal basis of $\mathcal{D}$, then the double sequence $\{a_{mn}\}_{m,n\geq 0}$ is Borel convergent to L if and only if

$$\lim_{\lambda,\mu\to\infty} \widetilde{D}_{\{a_{mn}\}}\left(\sqrt{2x},\sqrt{2y}\right) = L,$$

where $x = \dfrac{|\lambda|^2}{2}$ and $y = \dfrac{|\mu|^2}{2}$.

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