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An application of Lauricella hypergeometric functions to the generalized heat equations

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Abstract

In the recent paper, we give a formal solution of a certain one dimensional time fractional homogeneous conduction heat equation. This equation and its solution impose a rise to new forms of generalized fractional calculus. The new solution involves the Lauricella hypergeometric function of the third type. This type of functions is utilized to explain the probability of thermal transmission in random media. We introduce the analytic form of the thermal distribution related to such Lauricella function.

Keywords: Fractional calculus, fractional differential equations, analytic function

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1 Introduction

The fractional heat conduction equation is studied by Povstenko in 2004 [1]. He proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative. Later he focused on the heat conduction with time and space fractional derivatives and on the theory of thermal stresses based on this equation [2, 3]. Recently, Povstenko [4] obtained a solution of these equations by applying Laplace and Weber integral transforms. Furthermore, he formulated fundamental solutions to the central symmetric space-time fractional heat conduction equation and associated thermal stresses [5].

Newly, Li et. al., described heat conduction in fractal media, such as polar bear hair, wool fibers and goose down. By employing the modified Riemann-Liouville derivative, a fractional complex transform is used to convert time-fractional heat conduction equations into ordinary differential equations, therefore, precise solutions can be easily obtained [6]. At the same time, the authors generalized the fractional complex transform to obtain accurate solutions for time-fractional differential equations with the modified Riemann-Liouville derivative [7]. Yang and Baleanu posed a local fractional variational iteration method for processing the local fractional heat conduction equation accruing in fractal heat transfer [8]. Sherief and Latief created the problem for a half-space formed of a material with variable thermal conductivity [9].

In this work, we utilize a Lauricella type function to describe the time evolution of the fractional heat equation. We find the analytic form of these equations related to such Lauricella function. The fractional calculus is taken in sense of the Caputo derivative. The advantage of Caputo fractional derivative is that the derivative of a constant is zero, whereas for the Riemann- Liouville is not. Moreover, Caputo's derivative requests higher conditions of regularity for differentiability which allows us to geometrize various physical problems with fractional order.

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Finally, the advantage of Caputo fractional derivative is that the fractional differential equations with Caputo fractional derivative use the initial conditions (including the mixed boundary conditions) on the same character as for the integer-order differential equations [10].

2 Calculus of arbitrary order

This section concerns with some preliminaries and notations regarding the Caputo operator. The Caputo fractional derivative strongly poses the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations.

Definition 2.1 The fractional order integral of the function h of order $\alpha > 0$ is defined by

$$I_a^\alpha h(t) = \int_\wp^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau) d\tau.$$

When $\wp = 0$, we write $I_\wp^\alpha h(t) = h(t) * \psi_\alpha(t)$, where $(*)$ denoted the convolution product, $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\psi_\alpha(t) = 0,\ t \leq 0$ and $\psi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2 The Riemann-Liouville fractional order derivative of the function h of order $0 \leq \alpha < 1$ is defined by

$$D_\wp^\alpha h(t) = \frac{d}{dt} \int_\wp^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} h(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} h(t).$$

Remark 2.1 From Definition 2.1 and Definition 2.2, $\wp = 0$, we have

$$D^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha},\ \ \nu > -1;\ 0 < \alpha < 1$$

and

$$I^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha},\ \ \nu > -1;\ \alpha > 0.$$

Definition 2.3 The Caputo fractional derivative of order $\alpha > 0$ is defined, for a analytic function $h(t)$ by

$${}^cD^\alpha h(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\zeta)}{(t-\zeta)^{\alpha-n+1}} d\zeta,$$

where $n = [\alpha] + 1$, (the notation $[\alpha]$ stands for the largest integer not greater than α). In the sequel, we shall use the notation

$${}^cD^\alpha h(t) := \frac{\partial^\alpha h(t)}{\partial t^\alpha}.$$

3 Generalized heat equation

Consider the two dimensional time fractional homogeneous heat conduction equation of the form

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \delta(T_{xx} + T_{yy}) \tag{3.1}$$

$$(\delta > 0,\ t > 0,\ 0 < x, y < 1,\ 0 < \alpha \leq 1),$$

where:

- $T = T(x, y, t)$ is temperature as a function of space and time
- $\dfrac{\partial^\alpha T}{\partial t^\alpha}$ is the rate of change of temperature at a point over time
- δ is the thermal diffusivity

By using the fractional complex transform [7],

$$\zeta = \frac{\phi t^\alpha}{\Gamma(1+\alpha)} + \psi x + \kappa y,$$

it was shown that the exact solution of (3.1) can be expressed as

$$T(x,y,t) = c_1 + c_2 \exp\Big(\frac{\phi\psi x}{\delta(\psi^2+\kappa^2)} + \frac{\phi\kappa y}{\delta(\psi^2+\kappa^2)} + \frac{\phi^2 t^\alpha}{\delta\Gamma(1+\alpha)(\psi^2+\kappa^2)}\Big).$$

For special case we may have a solution of the form

$$T(x,y,t) = \exp\Big(-\frac{\phi\psi x}{\delta(\psi^2+\kappa^2)} - \frac{\phi\kappa y}{\delta(\psi^2+\kappa^2)} - \frac{\phi^2 t^\alpha}{\delta\Gamma(1+\alpha)(\psi^2+\kappa^2)}\Big). \tag{3.2}$$

In [11], the authors generalized the fractional probability of extinction, by applying the Caputo fractional derivative of one parameter as follows:

$$P_\mu(k,z) = \frac{(\nu z^\mu)^k}{k!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \frac{(-\nu z^\mu)^n}{\Gamma(\mu(n+k)+1)} \tag{3.3}$$

and the probability of transmission

$$P_\mu(0,z) = \sum_{n=0}^{\infty} \frac{(-\nu z)^n}{\Gamma(\mu n+1)} = E_\mu(-\nu z^\mu),$$

where

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}$$

is the Mittag-Leffler function and its popularity increased significantly due to its important role in applications and fractional of arbitrary orders related differential and integral equations of fractional order, solutions of problems of control theory, fractional viscoelastic models, diffusion theory, continuum mechanics and fractals [10].

Newly, numerical routines for Mittag-Leffler functions have been developed, e.g., by Freed et al. [12], Gorenflo et al. [13] (with MATHEMATICA), Podlubny [14] (with MATLAB), Seybold and Hilfer [15].

Here, we generalize probability of extinction, using the fractional Poisson process of three variables as follows:

$$\begin{aligned} P_{\mu,\beta,\sigma}(k,z,w,u) &= \frac{(\nu z)^k}{k!}\frac{(\rho w)^k}{k!}\frac{(\sigma u)^k}{k!} \\ &\times \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+k)!}{n!}\frac{(j+k)!}{j!} \\ &\times \frac{(m+k)!}{m!}\frac{(-\nu z^\mu)^n}{\Gamma(\mu(n+k)+1)}\frac{(-\rho w^\beta)^j}{\Gamma(\beta(j+k)+1)} \\ &\times \frac{(-\sigma u^\gamma)^m}{\Gamma(\gamma(m+k)+1)}; \end{aligned} \tag{3.4}$$

thus the 3-D probability of transmission becomes

$$P_{\mu,\beta,\sigma}(0,z,w,u) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-\nu z^\mu)^n}{\Gamma(\mu n+1)}\frac{(-\rho w^\beta)^j}{\Gamma(\beta j+1)}\frac{(-\sigma u^\gamma)^m}{\Gamma(\gamma m+1)}$$

$$= E_{\mu,\beta,\gamma}(-\nu z^\mu, -\rho w^\beta, -\sigma u^\gamma),$$

where $E_{\mu,\beta,\gamma}(-\nu z^\mu, -\rho w^\beta, -\sigma u^\gamma)$ is a multi-index Mittag-Leffler function, which can be found in [10].

Our approach depends on the Lauricella hypergeometric function of third type of three variables, which can be defined by

$$F_A^{(3)}(a,b_1,b_2,b_3,c_1,c_2,c_3;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c_1)_{i_1}(c_2)_{i_2}(c_3)_{i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|+|x_2|+|x_3|<1$ and

$$F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3,c;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a_1)_{i_1}(a_2)_{i_2}(a_3)_{i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c)_{i_1+i_2+i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|<1,|x_2|<1,|x_3|<1$ and

$$F_C^{(3)}(a,b,c_1,c_2,c_3;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b)_{i_1+i_2+i_3}}{(c_1)_{i_1}(c_2)_{i_2}(c_3)_{i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

for $|x_1|\frac{1}{2}+|x_2|\frac{1}{2}+|x_3|\frac{1}{2}<1$ and

$$F_D^{(3)}(a,b_1,b_2,b_3,c;x_1,x_2,x_3)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{(a)_{i_1+i_2+i_3}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}}{(c)_{i_1+i_2+i_3}\,i_1!\,i_2!\,i_3!}\,x_1^{i_1}x_2^{i_2}x_3^{i_3}.$$

The notation $(x)_n$ refers to the Pochhammer symbol

$$(x)_n=x(x+1)(x+2)\cdots(x+n-1)$$

or in gamma function

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)},\quad (x)_n=\frac{(-1)^n\Gamma(1-x)}{\Gamma(1-x-n)}.$$

For special case, we obtain

$$F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{\alpha})=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3}\frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{\alpha})^{i_3}}{i_1!\,i_2!\,i_3!},$$

where $\widehat{\psi}$, $\widehat{\kappa}$ and $\widehat{\phi}$ are the coefficients in Eq.(3.2).

Now we proceed to define a Lauricella hypergeometric functions in the positive semi-space in order to introduce the probability of heat distribution. For non negative variables X,Y,T, we may describe the following distribution:

$$\Omega(x,y,t):=P(X\le x,Y\le y,T\le t)=1-F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha}),\tag{3.5}$$

$$(t,x,y\ge 0,\ \alpha\in(0,1]).$$

Also we define

$$\begin{aligned}\Theta(x,y,t)&:=P(X>x,Y>y,T>t)=1-\Omega(x,y,t)\\&:=\overline{F}_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha}).\end{aligned}\tag{3.6}$$

Physically, the above equations correspond to the probability of heat transmission. We impose the following result

Theorem 3.1 Assume Ω and Θ as in (3.5) and (3.6) respectively. Then Eq.(3.1) has a solution in terms of Lauricella hypergeometric functions.

Proof. The probability density function corresponding to (3.1) can be written by

$$\omega(x,y,t) = \frac{\partial^3}{\partial x \partial y \partial t}\Omega(x,y,t)$$

with

$$\int_D \omega(x,y,t)dxdydt = 1, \quad D \in \mathbb{R}^3.$$

The probability of the transmission, during time t in 2-dimensional space (x,y), of object can be related to (x',y',t') to be between (x,y,t) and $(x+dx, y+dy, t+dt)$. It is read by

$$\partial^3 P(x,y,t) = |\frac{\partial^3\Theta(x,y,t)}{\partial x \partial y \partial t}|(\partial x \partial y \partial t).$$

Thus we pose that

$$\begin{aligned}
\frac{\partial^3\Theta(x,y,t)}{\partial x\partial y\partial t}(\partial x\partial y\partial t) &= \frac{\partial}{\partial x\partial y\partial t}\Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3} \\
&\times \frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{1/\alpha})^{i_3}}{i_1!\,i_2!\,i_3!}\Big)\partial x\partial y\partial t \\
&= -\frac{\widehat{\psi}\widehat{\kappa}\widehat{\phi}\,t^{\frac{1}{\alpha}-1}(\alpha)_3}{\alpha}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(\alpha)_{i_1+i_2+i_3} \\
&\times \frac{(-\widehat{\psi}x)^{i_1}(-\widehat{\kappa}y)^{i_2}(-\widehat{\phi}t^{1/\alpha})^{i_3}}{i_1!\,i_2!\,i_3!}\partial x\partial y\partial t \\
&= -\widehat{\psi}\widehat{\kappa}\widehat{\phi}\,t^{\frac{1}{\alpha}-1}(\alpha+1)_2\; F_D^{(3)}(\alpha,c,c,c,c;-\widehat{\psi}x,-\widehat{\kappa}y,-\widehat{\phi}t^{1/\alpha})\;\partial x\partial y\partial t
\end{aligned} \tag{3.7}$$

Now for a function $f(x,y,z)$ has a series expansion of the form

$$f(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_1(i)\lambda_2(j)\lambda_3(k)\frac{(-1)^i x^i}{i!}\frac{(-1)^j y^j}{j!}\frac{(-1)^k z^k}{k!},$$

with

$$\lambda_1(0) \neq 0, \quad \lambda_2(0) \neq 0, \quad \lambda_3(0) \neq 0,$$

then

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{\alpha_1-1}y^{\alpha_2-1}z^{\alpha_3-1}f(x,y,z)dxdydz = \\
&\qquad\qquad \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\lambda_1(-\alpha_1)\lambda_2(-\alpha_2)\lambda_3(-\alpha_3).
\end{aligned} \tag{3.8}$$

The last assertion is called the generalized Ramanujan Master Theorem. By applying (3.8) in (3.7), where

$$\lambda_1 = (\alpha)_{i_1}, \quad \lambda_2 = (\alpha)_{i_2}, \quad \lambda_3 = (\alpha)_{i_3}$$

and that

$$\lambda_1(0) = 1, \quad \lambda_2(0) = 1, \quad \lambda_3(0) = 1,$$

we have a solution of (3.1) which is in terms of Lauricella hypergeometric functions. □

4 Conclusion

Distributions of barriers do not elaborate in some physical states, e.g. in media with locative interconnections between particles. In this work, we utilized the 3- D fractional derivative Poisson process which can be viewed as a utility tool to put into account long domain interconnections between particles in the medium. In addition, we applied the generalized Lauricella hypergeometric functions to give various methods of the probability of the heat transmission, depending on the renewal process. Our main result showed a solution of 2- D fractional heat equation in terms of the Lauricella hypergeometric functions based on the generalized Ramanujan Master Theorem.

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