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# Functional equation originating from sum of higher powers of arithmetic progression using difference operator is stable in Banach space: direct and fixed point methods

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**Abstract**

In this paper, the authors has proved the solution of a new type of functional equation

$$f\left(\sum_{j=1}^{k} j^p\, x_j\right) = \sum_{j=1}^{k} \left(j^p\, f(x_j)\right), \qquad k, p \geq 1$$

which is originating from sum of higher powers of an arithmetic progression. Its generalized Ulam - Hyers stability in Banach space using direct and fixed point methods are investigated. An application of this functional equation is also studied.

*Keywords:* Additive functional equations, stirling numbers, polynomial factorial, difference operator, generalized Ulam - Hyers stability, fixed point.

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## 1 Introduction

During the last seven decades, the perturbation problems of several functional equations have been extensively investigated by a number of authors [1, 2, 12, 13, 20, 21, 26, 28] . The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [5, 8, 9, 10, 14, 15, 16, 22, 23, 24, 27].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

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The solution and stability of the following various additive functional equations

$$f(2x-y)+f(x-2y)=3f(x)-3f(y), \tag{1.2}$$
$$f(x+y-2z)+f(2x+2y-z)=3f(x)+3f(y)-3f(z), \tag{1.3}$$
$$f(m(x+y)-2mz)+f(2m(x+y)-mz)=3m[f(x)+f(y)-f(z)]\ m\geq 1, \tag{1.4}$$
$$f\left(a\sum_{i=1}^{n-1}x_i-2ax_n\right)+f\left(2a\sum_{i=1}^{n-1}x_i-ax_n\right)=3a\left(\sum_{i=1}^{n-1}f(x_i)-f(x_n)\right)\ n\geq 3, \tag{1.5}$$
$$f(2x\pm y\pm z)=f(x\pm y)+f(x\pm z) \tag{1.6}$$
$$f(qx\pm y\pm z)=f(x\pm y)+f(x\pm z)+(q-2)f(x),\ \ q\geq 2 \tag{1.7}$$

were discussed by D.O. Lee [11], K. Ravi, M. Arunkumar [25], M. Arunkumar [3, 4].

Also M. Arunkumar et. al., [7] investigated the generalized Ulam-Hyers stability of a functional equation

$$f\,(y)=\frac{f\,(y+z)+f\,(y-z)}{2}$$

which is originating from arithmetic mean of consecutive terms of an arithmetic progression using direct and fixed point methods. Infact M. Arunkumar et. al.,[6] has proved the solution and generalized Ulam - Hyers - Rassias stability of a $n$ dimensional additive functional equation

$$f(x)=\sum_{\ell=1}^{n}\left(\frac{f\,(x+\ell y_\ell)+f\,(x-\ell y_\ell)}{2\,\ell}\right)$$

where $n$ is a positive integer, which is originating from arithmetic mean of $n$ consecutive terms of an arithmetic progression.

In this paper, the authors established the solution and the generalized Ulam - Hyers stability of a new type of additive functional equation

$$f\left(\sum_{j=1}^{k}j^p\,x_j\right)=\sum_{j=1}^{k}\left(j^p\,f(x_j)\right),\qquad k,p\geq 1 \tag{1.8}$$

which is originating from sum of higher powers of an arithmetic progression. An application of this functional equation is also studied.

In Section 2, some basic preliminaries about difference operator is discussed. In Section 3, the general solution of the functional equation (1.8) is given. In Section 4 and 5, the generalized Ulam - Hyers stability of the additive functional equation (1.8) using direct and fixed point methods are respectively proved. An application of the additive functional equation (1.8) is discussed in Section 6.

## 2 Basic preliminaries on difference operator

**Definition 2.1.** *[18] If $\{y_k\}$ is a sequence of numbers, then we define the difference operator $\Delta$ as*

$$\Delta(y_k)\ =\ y_{k+1}-y_k. \tag{2.1}$$

**Lemma 2.1.** *[18] From (2.1) and the shift relation, $E(y_k)\ =\ y_{k+1}$, we obtain*

$$E=\Delta+1. \tag{2.2}$$

**Definition 2.2.** *[18] If $n$ is positive integer, then the positive polynomial factorial is defined as*

$$k^{(n)}=k(k-1)(k-2)...(k-(n-1)). \tag{2.3}$$

**Lemma 2.2.** *[18] If $S_r^n$'s are the Stirling numbers of second kind, then*

$$k^n=\sum_{r=1}^{n}S_r^n k^{(r)}. \tag{2.4}$$

**Definition 2.3.** *[18] For the positive integer $n$, the inverse operators are defined as if*

$$\Delta^n(z_k) \;=\; y_k, \text{ then } z_k \;=\; \Delta^{-n}(y_k). \tag{2.5}$$

**Lemma 2.3.** *[18] If $m, k$ are positive integers and $k > m$, then*

$$\Delta^{-1}k^{(m)} \;=\; \frac{k^{(m+1)}}{(m+1)} + c, \qquad \text{where } c \text{ is constant}. \tag{2.6}$$

**Theorem 2.1.** *[18] If $k$ is positive integer, then*

$$\Delta^{-1}(y_k) \;=\; \sum_{r=1}^{k} y_{(k-r)} + c, \qquad \text{where } c \text{ is constant}. \tag{2.7}$$

**Theorem 2.2.** *If $k$ and $p$ are positive integers then*

$$\sum_{r=1}^{p}(k+1-r)^p = \sum_{r=1}^{n} S_r^n k^{(p)} \frac{[k+1]^{(r+1)}}{[r+1]}. \tag{2.8}$$

*Proof.* The proof follows by Lemmas 2.2, 2.3 and Theorem 2.1. □

## 3 General solution of the functional equation(1.8)

In this section, the general solution of the functional equation (1.8) is given.

**Theorem 3.3.** *Let $X$ and $Y$ be real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1.1) for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (1.8) for all $x_1, x_2, \cdots, x_k \in X$.*

*Proof.* The proof follows by the additive property. □

Hereafter though out this paper, let us consider $X$ and $Y$ to be a normed space and a Banach space, respectively.

## 4 Stability results: Direct method

In this section, the generalized Hyers - Ulam - Rassias stability of the additive functional equation (1.8) is provided.

**Theorem 4.4.** *Let $i \in \{-1, 1\}$ and $\alpha : X^k \to [0, \infty)$ be a function such that*

$$\sum_{t=0}^{\infty} \frac{\alpha\left[\wp^{ti}x_1, \wp^{ti}x_2, \cdots, \wp^{ti}x_k\right]}{\wp^{ti}} \quad \text{converges in } \mathbb{R} \tag{4.1}$$

*for all $x_1, x_2, x_3 \cdots, x_k \in X$. Let $f : X \to Y$ be a function satisfying the inequality*

$$\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k}\left[j^p\, f[x_j]\right] \right\| \le \alpha\left[x_1, x_2, x_3 \cdots, x_k\right] \tag{4.2}$$

*for all $x_1, x_2, x_3 \cdots, x_k \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equation (1.8) and*

$$\|f[x] - A[x]\| \le \frac{1}{\wp} \sum_{s=\frac{1-i}{2}}^{\infty} \frac{\alpha[\wp^{si}x, \wp^{si}x, \cdots, \wp^{si}x]}{\wp^{si}} \tag{4.3}$$

*where*

$$\wp = \sum_{r=1}^{p} S_r^p \; \frac{[k+1]^{(r+1)}}{[r+1]} \tag{4.4}$$

*for all $x \in X$. The mapping $A[x]$ is defined by*

$$A[x] = \lim_{t\to\infty} \frac{f[\wp^{ti}x]}{\wp^{ti}} \tag{4.5}$$

*for all $x \in X$.*

*Proof.* Assume $i = 1$. Replacing $[x_1, x_2, \cdots, x_k]$ by $[x, x, \cdots, x]$ in (4.2), we get

$$\|f\,[[1^p + 2^p + \cdots + k^p]\, x] - [1^p + 2^p + \cdots + k^p]\, f[x]\| \le \alpha\,[x, x, \cdots, x] \tag{4.6}$$

for all $x \in X$. The above equation can be rewritten as

$$\left\| f\,\left[\left\{\sum_{r=1}^{k}[k+1-r]^p\right\}\, x\right] - \left[\sum_{r=1}^{k}[k+1-r]^p\right] f[x]\right\| \le \alpha\,[x, x, \cdots, x] \tag{4.7}$$

for all $x \in X$. Using Theorem 2.2, we have

$$\left\| f\,\left[\left\{\sum_{r=1}^{p} S_r^p\ \frac{[k+1]^{(r+1)}}{[r+1]}\right\}\, x\right] - \left[\sum_{r=1}^{p} S_r^p\ \frac{[k+1]^{(r+1)}}{[r+1]}\right] f[x]\right\| \le \alpha\,[x, x, \cdots, x] \tag{4.8}$$

for all $x \in X$. Define

$$\wp = \sum_{r=1}^{p} S_r^p\ \frac{[k+1]^{(r+1)}}{[r+1]}$$

in the above equation and re modifying, we arrive

$$\left\| \frac{f[\wp\, x]}{\wp} - f[x]\right\| \le \frac{\alpha\,[x, x, \cdots, x]}{\wp} \tag{4.9}$$

for all $x \in X$. Now replacing $x$ by $\wp x$ and dividing by $\wp$ in (4.9), we get

$$\left\| \frac{f[\wp x]}{\wp} - \frac{f[\wp^2 x]}{\wp^2}\right\| \le \frac{\alpha\,[\wp x, \wp x, \cdots, \wp x]}{\wp^2} \tag{4.10}$$

for all $x \in X$. From (4.8) and (4.10), we obtain

$$\begin{aligned} \left\| f[x] - \frac{f[\wp^2 y]}{\wp^2}\right\| &\le \left\| f[x] - \frac{f[\wp x]}{\wp}\right\| + \left\| \frac{f[\wp x]}{\wp} - \frac{f[\wp^2 x]}{\wp^2}\right\| \\ &\le \frac{1}{\wp}\left\{\alpha\,[x, x, \cdots, x] + \frac{\alpha\,[\wp x, \wp x, \cdots, \wp x]}{\wp}\right\} \end{aligned} \tag{4.11}$$

for all $x \in X$. In general for any positive integer $t$ , we get

$$\begin{aligned} \left\| f[x] - \frac{f[\wp^t x]}{\wp^t}\right\| &\le \frac{1}{\wp}\sum_{s=0}^{t-1} \frac{\alpha\,[\wp^s x, \wp^s x, \cdots, \wp^s x]}{\wp^s} \\ &\le \frac{1}{\wp}\sum_{s=0}^{\infty} \frac{\alpha\,[\wp^s x, \wp^s x, \cdots, \wp^s x]}{\wp^s} \end{aligned} \tag{4.12}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{\frac{f[\wp^t x]}{\wp^t}\right\},$$

replace $x$ by $\wp^l x$ and dividing by $\wp^l$ in (4.12), for any $t, l > 0$ , we deduce

$$\begin{aligned} \left\| \frac{f[\wp^l x]}{\wp^l} - \frac{f[\wp^{l+t} x]}{\wp^{[l+t]}}\right\| &= \frac{1}{\wp^l}\ \left\| f[\wp^l x] - \frac{f[\wp^t \cdot \wp^l x]}{\wp^t}\right\| \\ &\le \frac{1}{\wp}\sum_{s=0}^{t-1} \frac{\alpha\left[\wp^{s+l} x, \wp^{s+l} x, \cdots, \wp^{s+l} x\right]}{\wp^{s+l}} \\ &\le \frac{1}{\wp}\sum_{s=0}^{\infty} \frac{\alpha\left[\wp^{s+l} x, \wp^{s+l} x, \cdots, \wp^{s+l} x\right]}{\wp^{s+l}} \\ &\to 0 \quad as\ \ l \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{\frac{f[\wp^t x]}{\wp^t}\right\}$, is a Cauchy sequence. Since $Y$ is complete, there exists a mapping $A : X \to Y$ such that

$$A[x] = \lim_{t\to\infty} \frac{f[\wp^t x]}{\wp^t} \quad \forall \ x \in X.$$

Letting $t \to \infty$ in (4.12) we see that (4.3) holds for all $x \in X$.

To show that $A$ satisfies (1.8), replacing $[x_1, x_2, x_3 \cdots, x_k]$ by $\left[\wp^t x_1, \wp^t x_2, \cdots, \wp^t x_k\right]$ and dividing by $\wp^t$ in (4.2) and using the definition of $A(x)$ , and then letting $t \to \infty$, we see that $A$ satisfies (1.8) for all $x_1, x_2, \cdots, x_k \in X$. To prove that $A$ is unique, let $B[x]$ be another additive mapping satisfying (1.8) and (4.3), then

$$\begin{aligned}
\|A[x] - B[x]\| &= \frac{1}{\wp^t} \left\|A[\wp^t x] - B[\wp^t x]\right\| \\
&\leq \frac{1}{\wp^t} \left\{\left\|A[\wp^t x] - f[\wp^t x]\right\| + \left\|f[\wp^t x] - B[\wp^t x]\right\|\right\} \\
&\leq \sum_{s=0}^{\infty} \frac{2\,\alpha[\wp^{t+s}x, \wp^{t+s}x, \cdots, \wp^{t+s}x]}{\wp \cdot \wp^{t+s}} \\
&\to 0 \quad as \ \ s \to \infty
\end{aligned}$$

for all $x \in X$. Hence $A$ is unique.

For $i = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.4 concerning the Ulam-Hyers [13], Ulam-Hyers-Rassias [21], Ulam-Gavruta-Rassias [20] and Ulam-JRassias [26] stabilities of (1.8).

**Corollary 4.1.** *Let $\lambda$ and $q$ be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$\begin{aligned}
&\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k}\left[j^p\, f[x_j]\right]\right\| \\
&\leq \begin{cases}
\lambda, & \\
\lambda \left\{\sum_{j=1}^{k} ||x_j||^q\right\}, & q < 1 \quad or \quad q > 1; \\
\lambda \prod_{j=1}^{k} ||x_j||^q, & q < \frac{1}{k} \quad or \quad q > \frac{1}{k}; \\
\lambda \left\{\prod_{j=1}^{k} ||x_j||^q + \left\{\sum_{j=1}^{k} ||x_j||^{kq}\right\}\right\}, & q < \frac{1}{k} \quad or \quad q > \frac{1}{k};
\end{cases}
\end{aligned} \tag{4.13}$$

*for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f[x] - A[x]\| \leq \begin{cases}
\frac{\lambda}{|\wp - 1|}, \\
\frac{k\lambda||x||^q}{|\wp - \wp^q|}, \\
\frac{\lambda||x||^{kq}}{|\wp - \wp^{kq}|} \\
\frac{(k+1)\lambda||x||^{kq}}{|\wp - \wp^{kq}|}
\end{cases} \tag{4.14}$$

*for all $x \in X$.*

## 5 Stability results: Fixed point method

In this section, we apply a fixed point method for achieving stability of the additive functional equation (1.8).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [17] for fixed point theory.

**Theorem 5.5.** *[17] Suppose that for a complete generalized metric space $(\Omega, \delta)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

*or there exists a natural number $n_0$ such that*
*(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$ ;*
*(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point $y^*$ of T*
*(FP3) $y^*$ is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*
*(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1.8).

**Theorem 5.6.** *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^k \to [0, \infty)$ with the condition*

$$\sum_{t=0}^{\infty} \frac{\alpha\left[\mu_i^t x_1, \mu_i^t x_2, \cdots, \mu_i^t x_k\right]}{\mu_i^t} \quad \text{converges} \quad \text{in} \quad \mathbb{R} \tag{5.1}$$

*where $\mu_i = \wp$ if $i = 0$ and $\mu_1 = \frac{1}{\wp}$ if $i = 1$ such that the functional inequality*

$$\left\| f\left[\sum_{j=1}^{k} j^p\, x_j\right] - \sum_{j=1}^{k} \left[j^p\, f[x_j]\right] \right\| \leq \alpha\left[x_1, x_2, x_3 \cdots, x_k\right] \tag{5.2}$$

*for all $x_1, x_2, x_3 \cdots, x_k \in X$. If there exists $L = L(i) < 1$ such that the function*

$$x \to \gamma[x] = \frac{1}{\wp}\, \alpha\left[\frac{x}{\wp}, \frac{x}{\wp}, \cdots, \frac{x}{\wp}\right],$$

*has the property*

$$\gamma[x] = L\, \mu_i\, \gamma\left[\mu_i x\right]. \tag{5.3}$$

*Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equation (1.8) and*

$$\|f[x] - A[x]\| \leq \frac{L^{1-i}}{1-L} \gamma[x] \tag{5.4}$$

*for all $x \in X$.*

*Proof.* Consider the set

$$\Omega = \{p/p : X \to Y,\ p[0] = 0\}$$

and introduce the generalized metric on $\Omega$,

$$d(p, q) = \inf\{K \in (0, \infty) : \| p[x] - q[x] \| \leq K\gamma[x], x \in X\}.$$

It is easy to see that $(\Omega, d)$ is complete. Define $T : \Omega \to \Omega$ by

$$Tp[x] = \frac{1}{\mu_i} p[\mu_i x],$$

for all $x \in E$. Now $p, q \in \Omega$,

$$\begin{aligned} d(p, q) \leq K \Rightarrow & \| p[x] - q[x] \| \leq K\gamma[x], x \in X. \\ & \Rightarrow \left\| \frac{1}{\mu_i} p[\mu_i x] - \frac{1}{\mu_i} q[\mu_i x] \right\| \leq \frac{1}{\mu_i} K\gamma[\mu_i x], x \in X, \\ & \Rightarrow \left\| \frac{1}{\mu_i} p[\mu_i x] - \frac{1}{\mu_i} q[\mu_i x] \right\| \leq LK\gamma[x], x \in X, \\ & \Rightarrow \| Tp[x] - Tq[x] \| \leq LK\gamma[x], x \in X, \\ & \Rightarrow d(p, q) \leq LK. \end{aligned}$$

This implies $d(Tp,Tq) \leq Ld(p,q)$, for all $p,q \in \Omega$. i.e., $T$ is a strictly contractive mapping on $\Omega$ with Lipschitz constant $L$.
From (4.9), we arrive

$$\left\| \frac{f[\wp\, x]}{\wp} - f[x] \right\| \leq \frac{\gamma[x]}{\wp} \tag{5.5}$$

for all $x \in X$. Using (5.3) for the case $i = 0$ it reduces to

$$\left\| \frac{f[\wp x]}{\wp} - f[x] \right\| \leq \ L\gamma[x]$$

for all $x \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq L \Rightarrow d(f,Tf) \leq L = L^1 < \infty.$$

Again replacing $x = \dfrac{x}{\wp}$ in (5.5), we get,

$$\left\| f[x] - \wp f\left[\frac{x}{\wp}\right] \right\| \leq \gamma\left[\frac{x}{\wp}\right] \tag{5.6}$$

for all $x \in X$. Using (5.3) for the case $i = 1$ it reduces to

$$\left\| f[x] - \wp f\left(\frac{x}{\wp}\right) \right\| \leq \ \gamma[x]$$

for all $x \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq 1 \Rightarrow d(f,Tf) \leq 1 = L^0 < \infty.$$

In above cases, we arrive

$$d(f,Tf) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point $A$ of $T$ in $\Omega$ such that

$$A[x] = \lim_{t\to\infty} \frac{f[\mu_i^t x]}{\mu_i^t} \qquad \forall \ x \in X. \tag{5.7}$$

To order to prove $A : X \to Y$ is additive, replacing $[x_1, \cdots, x_k]$ by $\left[\mu_i^t x_1, \cdots, \mu_i^t x_k\right]$ and dividing by $\mu_i^t$ in (5.2) and using the definition of $A(x)$ , and then letting $t \to \infty$, we see that $A$ satisfies (1.8) for all $x_1, \cdots, x_k \in X$.

By (FP3), $A$ is the unique fixed point of $T$ in the set $\Delta = \{A \in \Omega : d(f,A) < \infty\}$, $A$ is the unique function such that

$$\|f[x] - A[x]\| \leq K\gamma[x]$$

for all $x \in X$ and $K > 0$. Finally by (FP4), we obtain

$$d(f,A) \leq \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f[x] - A[x]\| \leq \frac{L^{1-i}}{1-L}\gamma[x]$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.6 concerning the Ulam-Hyers [13], Ulam-Hyers-Rassias [21], Ulam-Gavruta-Rassias [20] and Ulam-JRassias [26] stabilities of (1.8).

**Corollary 5.2.** *Let $f : X \to Y$ be a mapping and there exits real numbers $\lambda$ and $q$ such that*

$$\left\| f\left[\sum_{j=1}^{k} j^p\ x_j\right] - \sum_{j=1}^{k}\left[j^p\ f[x_j]\right]\right\| \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda\left\{\sum_{j=1}^{k}||x_j||^q\right\}, & q<1 \ \ or \ \ q>1; \\ (iii) & \lambda\prod_{j=1}^{k}||x_j||^q, & q<\frac{1}{k} \ \ or \ \ q>\frac{1}{k}; \\ (iv) & \lambda\left\{\prod_{j=1}^{k}||x_j||^q+\left\{\sum_{j=1}^{k}||x_j||^{kq}\right\}\right\}, & q<\frac{1}{k} \ \ or \ \ q>\frac{1}{k}; \end{cases} \tag{5.8}$$

*for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f[x]-A[x]\| \leq \begin{cases} (i) & \dfrac{\lambda}{|\wp-1|}, \\ (ii) & \dfrac{k\lambda||x||^q}{|\wp-\wp^q|}, \\ (iii) & \dfrac{\lambda||x||^{kq}}{|\wp-\wp^{kq}|} \\ (iv) & \dfrac{(k+1)\lambda||x||^{kq}}{|\wp-\wp^{kq}|} \end{cases} \tag{5.9}$$

*for all $x \in X$.*

*Proof.* Setting

$$\alpha[x_1,x_2,\cdots,x_k] = \begin{cases} \lambda, \\ \lambda\left\{\sum_{j=1}^{k}||x_j||^q\right\}, \\ \lambda\prod_{j=1}^{k}||x_j||^q, \\ \lambda\left\{\prod_{j=1}^{k}||x_j||^q+\left\{\sum_{j=1}^{k}||x_j||^{kq}\right\}\right\}, \end{cases}$$

for all$x_1, x_2, \cdots, x_k \in X$.. Now,

$$\frac{1}{\mu_i^t}\alpha[\mu_i^t x_1,\mu_i^t x_2,\cdots,\mu_i^t x_k] = \begin{cases} \frac{\lambda}{\mu_i^t}, \\ \frac{\lambda}{\mu_i^t}\left\{\sum_{j=1}^{k}||\mu_i^t x_j||^q\right\}, \\ \frac{\lambda}{\mu_i^t}\prod_{j=1}^{k}||\mu_i^t x_j||^q, \\ \frac{\lambda}{\mu_i^t}\left\{\prod_{j=1}^{k}||\mu_i^t x_j||^q+\left\{\sum_{j=1}^{k}||\mu_i^t x_j||^{kq}\right\}\right\}, \end{cases} = \begin{cases} \to 0 \text{ as } \ t\to\infty, \\ \to 0 \text{ as } \ t\to\infty, \\ \to 0 \text{ as } \ t\to\infty, \\ \to 0 \text{ as } \ t\to\infty. \end{cases}$$

Thus, (5.1) is holds.

But we have $\gamma[x] = \frac{1}{\wp}\ \gamma\ [x]$ has the property $\gamma[x] = L \cdot \mu_i\ \gamma\ [\mu_i x]$ for all $x \in X$. Hence

$$\gamma[x] = \frac{1}{\wp}\alpha[x,x,\cdots,x] = \begin{cases} \dfrac{\lambda}{\wp} \\ \dfrac{k\lambda}{\wp}||x||^q, \\ \dfrac{\lambda}{\wp}||x||^{kq}, \\ \dfrac{(k+1)\lambda}{\wp}||x||^{kq}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma[\mu_i x] = \begin{cases} \dfrac{\lambda}{\wp\mu_i} \\ \dfrac{k\lambda}{\wp\mu_i}||\mu_i x||^q, \\ \dfrac{\lambda}{\wp\mu_i}||\mu_i x||^{kqs}, \\ \dfrac{(k+1)\lambda}{\wp\mu_i}||\mu_i x||^{kq}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\dfrac{\lambda}{\wp}, \\ \mu_i^{q-1}\dfrac{k\lambda}{\wp}||x||^q, \\ \mu_i^{kq-1}\dfrac{\lambda}{\wp}||x||^{kq}, \\ \mu_i^{kq-1}\dfrac{(k+1)\lambda}{\wp}||x||^{kq}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\gamma[x], \\ \mu_i^{q-1}\gamma[x], \\ \mu_i^{kq-1}\gamma[x], \\ \mu_i^{kq-1}\gamma[x]. \end{cases}$$

Hence the inequality (5.3) holds either, $L = \wp^{-1}$ for $q = 0$ if $i = 0$ and $L = \frac{1}{\wp^{-1}}$ for $q = 0$ if $i = 1$. Now from (5.4), we prove the following cases for condition $(ii)$.

**Case:1** $L = \wp^{-1}$ for $q = 0$ if $i = 0$

$$\|f[x] - A[x]\| \le \frac{\left(\wp^{(-1)(0-1)}\right)^{1-0}}{1-\wp^{(-1)(0-1)}}\frac{\lambda}{\wp} = \frac{\wp}{1-\wp}\frac{\lambda}{\wp} = \frac{\lambda}{1-\wp}$$

**Case:2** $L = \frac{1}{\wp^{-1}}$ for $q = 0$ if $i = 1$

$$\|f[x] - A[x]\| \le \frac{\left(\frac{1}{\wp^{(-1)(0-1)}}\right)^{1-1}}{1-\frac{1}{\wp^{(-1)(0-1)}}}\frac{\lambda}{\wp} = \frac{\wp}{\wp-1}\frac{\lambda}{\wp} = \frac{\lambda}{\wp-1}$$

Also the inequality (5.3) holds either, $L = \wp^{q-1}$ for $q < 1$ if $i = 0$ and $L = \frac{1}{\wp^{q-1}}$ for $q > 1$ if $i = 1$. Now from (5.4), we prove the following cases for condition $(ii)$.

**Case:1** $L = \wp^{q-1}$ for $q < 1$ if $i = 0$

$$\|f[x] - A[x]\| \le \frac{\left(\wp^{(q-1)}\right)^{1-0}}{1-\wp^{(q-1)}}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}\cdot\wp}{\wp-\wp^s}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}k\lambda||x||^q}{\wp-\wp^q}$$

**Case:2** $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f[x] - A[x]\| \le \frac{\left(\frac{1}{\wp^{(q-1)}}\right)^{1-1}}{1-\frac{1}{\wp^{(q-1)}}}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}\cdot\wp}{\wp^s-\wp}\frac{k\lambda}{\wp}||x||^q = \frac{\wp^{(q-1)}k\lambda||x||^q}{\wp^s-\wp}$$

Again, the inequality (5.3) holds either, $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$ and $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$. Now from (5.4), we prove the following cases for condition $(iii)$.

**Case:1** $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$

$$\|f[x] - A[x]\| \le \frac{\left(\wp^{(kq-1)}\right)^{1-0}}{1-\wp^{(kq-1)}}\frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\cdot\wp}{\wp-\wp^{kq}}\frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\lambda||x||^{kq}}{\wp-\wp^{kq}}$$

**Case:2** $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$

$$\|f[x] - A[x]\| \leq \frac{\left(\frac{1}{\wp^{(kq-1)}}\right)^{1-1}}{1 - \frac{1}{\wp^{(kq-1)}}} \frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)} \cdot \wp}{\wp^{kq} - \wp} \frac{\lambda}{\wp}||x||^{kq} = \frac{\wp^{(kq-1)}\lambda||x||^{kq}}{\wp^{kq} - \wp}$$

Finally the inequality (5.3) holds either, $L = \wp^{kq-1}$ for $q < \frac{1}{k}$ if $i = 0$ and $L = \frac{1}{\wp^{kq-1}}$ for $q > \frac{1}{k}$ if $i = 1$. The proof of condition $(iv)$ is similar lines to that of condition $(iii)$. Hence the proof is complete. □

## 6 Application of the functional equation (1.8)

We know that the following sums of powers arithmetic progression

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k-1)}{2}$$

In general, using Stirling numbers of second kind, one can arrive

$$1^p + 2^p + 3^p + \cdots + k^p = \sum_{r=1}^{p} S_r^p \frac{[k+1]^{(r+1)}}{[r+1]}.$$

With the help of the above discussion, the authors transform the sum of $p^{th}$ power of first $k$ natural numbers as a functional equation

$$f\left(\sum_{j=1}^{k} j^p \, x_j\right) = \sum_{j=1}^{k} \left(j^p \, f(x_j)\right), \qquad k, p \geq 1$$

having additive solution.

# References


[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables,* Cambridge Univ, Press, 1989.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64-66.

[3] M. Arunkumar, *Solution and stability of Arun-additive functional equations,* International Journal Mathematical Sciences and Engineering Applications, Vol 4, No. 3, August 2010, 33-46.

[4] M. Arunkumar, G. Ganapathy, S. Murthy, S. Karthikeyan, *Stability of the generalized Arun-additive functional equation in Instutionistic fuzzy normed spaces*, International Journal Mathematical Sciences and Engineering Applications Vol.4, No. V, December 2010, 135-146.

[5] M. Arunkumar, C. Leela Sabari, *Solution and stability of a functional equation originating from a chemical equation*, International Journal Mathematical Sciences and Engineering Applications Vol. 5 No. II (March, 2011), 1-8.

[6] M. Arunkumar, S. Hema latha, C. Devi Shaymala Mary, *Functional equation originating from arithmetic Mean of consecutive terms of an arithmetic Progression are stable in banach space: Direct and fixed point method,* JP Journal of Mathematical Sciences, Volume 3, Issue 1, 2012, Pages 27-43.

[7] M. Arunkumar, G. Vijayanandhraj, S. Karthikeyan, *Solution and Stability of a Functional Equation Originating From n Consecutive Terms of an Arithmetic Progression,* Universal Journal of Mathematics and Mathematical Sciences, Volume 2, No. 2, (2012), 161-171.

[8] M. Arunkumar, P. Agilan, *Additive functional equation and inequality are Stable in Banach space and its applications*, Malaya Journal of Matematik (MJM), Vol 1, Issue 1, 2013, 10-17.

[9] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., 57, (1951), 223- 237.

[10] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[11] D.O. Lee, *Hyers-Ulam stability of an addtiive type functional equation*, J. Appl. Math. and Computing, 13 (2003) no.1-2, 471-477.

[12] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings* , J. Math. Anal. Appl., 184 (1994), 431-436.

[13] D.H. Hyers, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, 1998.

[15] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[16] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[17] B.Margoils and J.B.Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. 126 74 (1968), 305-309.

[18] Ronald E.Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York, 1990.

[19] V.Radu , *The fixed point alternative and the stability of functional equations, in: Seminar on Fixed Point Theory Cluj-Napoca*, Vol. IV, 2003, in press.

[20] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46, (1982) 126-130.

[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[22] Th.M. Rassias and P. Semrl, *On the behavior of mappings which do not satisfy Hyers- Ulam stability*, Proc. Amer. Math. Soc. 114 (1992), 989-993.

[23] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta. Appl. Math. 62 (2000), 23-130.

[24] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acadamic Publishers, Dordrecht, Bostan London, 2003.

[25] K. Ravi, M. Arunkumar, *On a $n-$dimensional additive Functional Equation with fixed point Alternative*, Proceedings of ICMS 2007, Malaysia.

[26] K. Ravi, M. Arunkumar and J.M. Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[27] G. Toader and Th.M. Rassias, *New properties of some mean values*, Journal of Mathematical Analysis and Applications 232, (1999), 376-383.

[28] S.M. Ulam, *Problems in Modern Mathematics,* Science Editions, Wiley, NewYork, 1964.

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