

Common fixed points for a class of multi-valued mappings and application to functional equations arising in dynamic programming

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Abstract

In this paper, we give an existence theorem for hybrid generalized multi-valued α -contractive type mappings which extends, improves and unifies the corresponding main result of Sintunavart and Kumam [W. Sintunavart, P. Kumam, Common fixed point theorem for hybrid generalized multi-valued contraction mappings, Appl. Math. Lett. 25 (2012), 52-57] and some main results in the literature. As an application, we give some existence and uniqueness results for solutions of a certain class of functional equations arising in dynamic programming to illustrate the efficiency and usefulness of our main result.

Keywords: Common fixed point, Multi-valued contraction, Dynamic programming, Functional equations.

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1 Introduction and preliminaries

Fixed point theory is one of the classical and most powerful tools which plays an essential role in nonlinear analysis. Under the mathematical point of view, this is utilized in so many nonlinear problems arising from the most applicable areas of sciences such as engineering, economics, dynamic system and physics (more precisely in the theory of Phase Transitions), etc.

The idea of combining the multi-valued mappings, Lipschitz mapping and fixed point theorems was initiated by Nadler [14]. He was the first one that gave a famous generalization of the Banach contraction principle for multi-valued mappings from metric space X into $\mathcal{CB}(X)$ using the Hausdorff metric which is defined by

$$H(A, B) := \max \left\{ \sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \right\}, \quad A, B \in \mathcal{CB}(X),$$

where $d(x, A) = \inf_{y \in A} d(x, y)$ and $\mathcal{CB}(X)$ is a collection of nonempty closed bounded subsets of X .

Definition 1.1 ([9]). Let H be the Hausdorff metric on $\mathcal{CB}(X)$, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be multi-valued mapping. We say an element $x \in X$ is a fixed point of f (resp. T) if $fx = x$ (resp. $x \in Tx$). We denote the set of all fixed points of f (resp. T) by $\text{Fix}(f)$ (resp. $\text{Fix}(T)$). A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$ and is a common fixed point of f and T if $x = fx \in Tx$.

Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a f -weakly Picard mapping if and only if for each $x \in X$ and $fy \in Tx$ ($y \in X$), there exists a sequence $\{x_n\}$ in X such that

- (i) $x_0 = x, x_1 = y$;
- (ii) $fx_{n+1} \in Tx_n$ for all $n = 0, 1, 2, \dots$;
- (iii) the sequence $\{fx_n\}$ converges to fu , where u is the coincidence point of f and T .

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Theorem 1.1 (Nadler[14]). Let (X, d) be a complete metric space and let $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping such that for a fixed constant $h \in (0, 1)$ and for each $x, y \in X$,

$$H(Tx, Ty) \leq hd(x, y).$$

Then $\text{Fix}(T) \neq \emptyset$.

Afterward, several authors have focused on generalization of Nadler's fixed point theorem in various types. For example, in 2007, M. Berinde and V. Berinde [2] presented a nice fixed point result based on so-called \mathcal{MT} -functions which recently have been characterized by Du [6].

Definition 1.2 ([4,5,7]). A function $\phi : [0, \infty) \rightarrow [0, 1)$ is said to be an \mathcal{MT} -function if it satisfies Mizoguchi-Takahashi's condition, i.e. $\limsup_{s \rightarrow t^+} \phi(s) < 1$ for all $t \in [0, \infty)$.

Theorem 1.2 ([2]). Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. If there exist an \mathcal{MT} -function $\phi : [0, \infty) \rightarrow [0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \phi(d(x, y))d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X,$$

then $\text{Fix}(T) \neq \emptyset$.

We remark that letting $L = 0$ in Theorem 1.2 we can easily obtain the following result as Mizoguchi-Takahashi's fixed point theorem [13] which is considered as a partial answer of Problem 9 in Reich [17]. Also, Yu-Qing [21] gave an affirmative answer to a fixed point problem of Reich which was raised in [16].

Theorem 1.3 ([13]). Let (X, d) be a complete metric space, $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping and $\phi : [0, \infty) \rightarrow [0, 1)$ be an \mathcal{MT} -function. Assume that

$$H(Tx, Ty) \leq \phi(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Then $\text{Fix}(T) \neq \emptyset$.

Definition 1.3 ([9]). Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be multi-valued mapping. If the sequence $\{x_n\}$ in X satisfying conditions (i) and (ii) in Definition 1.1, then the sequence $O_f(x_0) = \{fx_n : n = 1, 2, \dots\}$ is said to be an f -orbit of T at x_0 .

Recently, Sintunavart and Kumam [20] presented a generalization of main result of Kamran [8, 9] and then unified and complemented the result by introducing the notion of hybrid generalized multi-valued contraction mapping in [19] as follows.

Definition 1.4 ([19]). Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a hybrid generalized multi-valued contraction mapping if and only if there exist two functions $\phi : [0, \infty) \rightarrow [0, 1)$ satisfying $\limsup_{r \rightarrow t^+} \phi(r) < 1$ for every $t \in [0, \infty)$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$, such that

$$H(Tx, Ty) \leq \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y),$$

for all $x, y \in X$, where

$$M(x, y) := \max\{d(fx, fy), d(fy, Tx)\}$$

and

$$N(x, y) := \min\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

Theorem 1.4 ([19]). Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. Suppose that fX is a complete subspace of X and $TX \subset fX$. Then f and T have a coincidence point $z \in X$. Moreover, if $ffz = fz$, then f and T have a common fixed point.

The following lemmas are crucial for the proof of our main result.

Lemma 1.1 ([9]). Let (X, d) be a metric space, $\{A_k\}$ be a sequence in $\mathcal{CB}(X)$ and $\{x_k\}$ be a sequence in X such that $\{x_k\} \in A_{k-1}$. Let $h : [0, \infty) \rightarrow [0, 1)$ be a function satisfying $\limsup_{r \rightarrow t+} h(r) < 1$ for every $t \in [0, \infty)$. Suppose $d(x_{k-1}, x_k)$ to be a nonincreasing sequence such that

$$\begin{aligned} H(A_{k-1}, A_k) &\leq h(d(x_{k-1}, x_k))d(x_{k-1}, x_k), \\ d(x_k, x_{k+1}) &\leq H(A_{k-1}, A_k) + h^{n_k}(d(x_{k-1}, x_k)), \end{aligned} \quad (1.1)$$

where $n_1 < n_2 < \dots$ which $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cauchy sequence in X .

Lemma 1.2 ([14]). If $A, B \in \mathcal{CB}(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.

Throughout this work, we introduce the notions of a hybrid generalized multi-valued α -contractive type mapping based on Definition 1.4 and an α -admissible multi-valued mapping to obtain some fixed point theorems which either generalize or improve the corresponding recent fixed point results of Sintunavart and Kumam [19] (Theorem 1.4 of the present paper) and some ones in the literature. As an application, to show the applicability of our results, we give an existence theorem for certain class of functional equations arising in dynamic programming.

2 Main results

Denote with Ψ the family of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy

- (i) ψ is non-decreasing,
- (ii) $\lim_{n \rightarrow \infty} \psi(t_n) = 0$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$ for $t_n \in [0, \infty)$,
- (iii) ψ is subadditive, i.e., $\psi(t + s) \leq \psi(t) + \psi(s)$ for all $t, s \in [0, \infty)$,
- (iv) for any $t > 0$ there exists an $s > 0$ such that $\psi(s) \leq t$,
- (v) if $t < s$ then $t < \psi(s)$.

For example, function $\psi(t) = at$, where $a \geq 1$ is in Ψ .

In order to proceed with developing our work and obtain our results we need the following definition inspired by Definition 1.4 as follows.

Definition 2.5. Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. The mapping T is said to be a hybrid generalized multi-valued α -contractive type mapping if and only if there exist an \mathcal{MT} -function $\phi : [0, \infty) \rightarrow [0, 1)$ and function $\varphi : [0, \infty) \rightarrow [0, \infty)$, such that

$$\alpha(x, y)\psi(H(Tx, Ty)) \leq \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y),$$

for all $x, y \in X$, where $\psi \in \Psi$, $\alpha : X \times X \rightarrow [0, \infty)$ is a given function and

$$M(x, y) := \max\{d(fx, fy), d(fy, Tx)\}$$

and

$$N(x, y) := \min\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

Definition 2.6. Let $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping, $f : X \rightarrow X$ be a single-valued mapping and $\alpha : X \times X \rightarrow [0, \infty)$. We say T is α -admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \implies \Lambda_f(Tx, Ty) \geq 1$$

where $\Lambda_f : \mathcal{CB}(X) \times \mathcal{CB}(X) \rightarrow [0, \infty)$ is given by $\Lambda_f(A, B) = \inf_{a, b \in X} \{\alpha(a, b) | fa \in A, fb \in B\}$.

Example 2.1. Let $\alpha : X \times X \rightarrow [r, \infty)$ where $r \geq 1$. Then any arbitrary multi-valued mapping $T : X \rightarrow \mathcal{CB}(X)$ is an α -admissible.

We first present the following simple lemma which is significant to prove our main result.

Lemma 2.3. Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be an α -admissible multi-valued mapping. Suppose that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$ for some $x_0 \in X$ and $\{fx_k\}$ is an f -orbit of T at x_0 . Then for all $n \in \mathbb{N}$, $\alpha(x_n, x_{n+1}) \geq 1$.

Proof. According to the hypotheses let $x_0 \in X$ such that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$ and $\{fx_k\}$ be an f -orbit of T at x_0 so that $fx_n \in Tx_{n-1}$. Since T is α -admissible, we get the following implication

$$\alpha(x_0, x_1) \geq \Lambda_f(\{fx_0\}, Tx_0) \geq 1 \implies \alpha(x_1, x_2) \geq \Lambda_f(Tx_0, Tx_1) \geq 1.$$

By the induction and using the α -admissibility of T , we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}$$

which completes the proof. \square

Now we give an immediate consequence of Lemma 1.2 as follows.

Lemma 2.4. Let $A, B \in \mathcal{CB}(X)$, $a \in A$ and $\psi \in \Psi$. Then for each $\epsilon > 0$, there exists $b \in B$ such that $\psi(d(a, b)) \leq \psi(H(A, B)) + \epsilon$.

Proof. Suppose, to the contrary, that there exist $a \in A$, $\epsilon > 0$ and $\psi \in \Psi$ such that

$$\psi(d(a, b)) > \psi(H(A, B)) + \epsilon \text{ for all } b \in B.$$

Following property (iv) of ψ we can choose an $s > 0$ such that $\psi(s) \leq \epsilon$. So for all $b \in B$ we get

$$\begin{aligned} \psi(d(a, b)) &> \psi(H(A, B)) + \epsilon \\ &\geq \psi(H(A, B)) + \psi(s) \\ &\geq \psi(H(A, B) + s), \end{aligned} \tag{2.2}$$

which implies

$$d(a, b) > H(A, B) + s \text{ for all } b \in B.$$

On the other hand, it follows from Lemma 1.2 that there is an element $b' \in B$ such that

$$d(a, b') \leq H(A, B) + s,$$

which shows an obvious contradiction and completes the proof. \square

By the virtue of the proof of Lemma 1.1 and using properties (ii) and (iii) of function $\psi \in \Psi$ we can prove the lemma given below.

Lemma 2.5. Let (X, d) be a metric space and $h : [0, \infty) \rightarrow [0, 1)$ be an \mathcal{MT} -function. Suppose $d(x_{k-1}, x_k)$ to be a nonincreasing sequence such that

$$\psi(d(x_k, x_{k+1})) \leq h(d(x_{k-1}, x_k))\psi(d(x_{k-1}, x_k)) + h^{n_k}(d(x_{k-1}, x_k)),$$

where $\psi \in \Psi$ and $n_1 < n_2 < \dots$ which $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cauchy sequence in X .

The following is our main result and can be considered as a generalization of Theorem 1.4 and a series of results in the literature.

Theorem 2.5. Let (X, d) be a metric space, $f : X \rightarrow X$ be a single-valued mapping and $T : X \rightarrow \mathcal{CB}(X)$ be a hybrid generalized multi-valued α -contractive type mapping such that fX is a complete subspace of X and $TX \subset fX$. Suppose that all the following conditions hold:

(i) T is α -admissible;

(ii) there exists $x_0 \in X$ such that $\Lambda_f(\{fx_0\}, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists an n_0 such that $\alpha(x_n, x) \geq 1$ for all $n \geq n_0$.

Then f and T have a coincidence point $u \in X$. Moreover, if $fu \in \text{Fix}(f)$ and $\alpha(u, fu) \geq 1$, then fu is a common fixed point of f and T .

Proof. Let x_0 be an arbitrary element in X and $y_0 = fx_0$. We define the sequence $\{x_n\}$ and $\{y_n\}$ recursively as follows. Since $TX \subset fX$, we can choose an $x_1 \in X$ such that $y_1 = fx_1 \in Tx_0$. Since $0 \leq \phi(t) < 1$ for all $t \in [0, \infty)$ we can choose a positive integer n_1 such that

$$\phi^{n_1}(d(y_0, y_1)) \leq [1 - \phi(M(x_0, x_1))]M(x_0, x_1). \quad (2.3)$$

Following Lemma 2.4, we may select $y_2 = fx_2 \in Tx_1$ such that

$$\psi(d(y_1, y_2)) \leq \psi(H(Tx_0, Tx_1)) + \phi^{n_1}(d(y_0, y_1)).$$

Using (2.3), (ii) and the notion of a hybrid generalized multi-valued α -contractive type mapping, we get

$$\begin{aligned} \psi(d(y_1, y_2)) &\leq \psi(H(Tx_0, Tx_1)) + \phi^{n_1}(d(y_0, y_1)) \\ &\leq \alpha(x_0, x_1)\psi(H(Tx_0, Tx_1)) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\ &\leq \phi(M(x_0, x_1))M(x_0, x_1) + \varphi(N(x_0, x_1))N(x_0, x_1) + [1 - \phi(M(x_0, x_1))]M(x_0, x_1) \\ &= M(x_0, x_1) \\ &= \max\{d(fx_0, fx_1), d(fx_1, Tx_0)\} \\ &= d(y_0, y_1). \end{aligned}$$

Now, we take a positive integer n_2 which is greater than n_1 such that

$$\phi^{n_2}(d(y_1, y_2)) \leq [1 - \phi(M(x_1, x_2))]M(x_1, x_2).$$

Again following Lemma 2.4 since $TX \subset fX$, we may select $y_3 = fx_3 \in Tx_2$ such that

$$\psi(d(y_2, y_3)) \leq \psi(H(Tx_1, Tx_2)) + \phi^{n_2}(d(y_1, y_2)).$$

Similar to the previous case and using the α -admissibility of T and Lemma 2.3 we obtain the following

$$\begin{aligned} \psi(d(y_2, y_3)) &\leq \psi(H(Tx_1, Tx_2)) + \phi^{n_2}(d(y_1, y_2)) \\ &\leq \alpha(x_1, x_2)\psi(H(Tx_1, Tx_2)) + [1 - \phi(M(x_1, x_2))]M(x_1, x_2) \\ &\leq \phi(M(x_1, x_2))M(x_1, x_2) + \varphi(N(x_1, x_2))N(x_1, x_2) + [1 - \phi(M(x_1, x_2))]M(x_1, x_2) \\ &= M(x_1, x_2) \\ &= \max\{d(fx_1, fx_2), d(fx_2, Tx_1)\} \\ &= d(y_1, y_2). \end{aligned}$$

Continuing the procedure as above and applying Lemma 2.3, for all $k \in \mathbb{N}$, we can find a positive integer n_k such that

$$\phi^{n_k}(d(y_{k-1}, y_k)) \leq [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k).$$

By applying Lemma 2.4 we may choose $y_{k+1} = fx_{k+1} \in Tx_k$ such that

$$\psi(d(y_k, y_{k+1})) \leq \psi(H(Tx_{k-1}, Tx_k)) + \phi^{n_k}(d(y_{k-1}, y_k)) \quad (2.4)$$

for each $k = 1, 2, \dots$. The recent inequalities together with the notion of a hybrid generalized multi-valued α -contractive type mapping imply

$$\begin{aligned} \psi(d(y_k, y_{k+1})) &\leq \psi(H(Tx_{k-1}, Tx_k)) + \phi^{n_k}(d(y_{k-1}, y_k)) \\ &\leq \alpha(x_{k-1}, x_k)\psi(H(Tx_{k-1}, Tx_k)) + [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k) \\ &\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k) \\ &\quad + [1 - \phi(M(x_{k-1}, x_k))]M(x_{k-1}, x_k) \\ &= M(x_{k-1}, x_k) \\ &= \max\{d(fx_{k-1}, fx_k), d(fx_k, Tx_{k-1})\} \\ &= d(y_{k-1}, y_k) \end{aligned} \quad (2.5)$$

for all $k \in \mathbb{N}$. If for some $k \in \mathbb{N}$, $y_{k-1} = y_k$ then there is nothing to prove. Otherwise, we claim that $d(y_{k-1}, y_k)$ is a nonincreasing sequence of nonnegative numbers. Suppose the opposite is true. It follows from (2.5) and property (v) of ψ that $d(y_{k-1}, y_k)$ is a nonincreasing sequence of nonnegative numbers. On the other hand

$$\begin{aligned} \psi(H(Tx_{k-1}, Tx_k)) &\leq \alpha(x_{k-1}, x_k)\psi(H(Tx_{k-1}, Tx_k)) \\ &\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) + \varphi(N(x_{k-1}, x_k))N(x_{k-1}, x_k) \\ &\leq \phi(M(x_{k-1}, x_k))M(x_{k-1}, x_k) \\ &= \phi(d(y_{k-1}, y_k))d(y_{k-1}, y_k). \end{aligned}$$

This together with (2.4) imply

$$\psi(d(y_k, y_{k+1})) \leq \phi(d(y_{k-1}, y_k))d(y_{k-1}, y_k) + \phi^{n_k}(d(y_{k-1}, y_k)),$$

which shows that $\{y_k\} = \{fx_k\}$ is a Cauchy sequence in fX followed by Lemma 2.5. Hence, the sequence $\{fx_k\}$ is convergent to fu for some $u \in X$. Now following condition (iii) of assumptions and Lemma 2.3, since $\alpha(x_k, x_{k+1}) \geq 1$ we easily conclude that there exists a positive integer k_0 such that

$$\begin{aligned} \psi(d(fu, Tu)) &\leq \psi(d(fu, fx_k)) + \psi(d(fx_k, Tu)) \\ &\leq \psi(d(fu, fx_k)) + \psi(H(Tx_{k-1}, Tu)) \\ &\leq \psi(d(fu, fx_k)) + \alpha(x_{k-1}, u)\psi(H(Tx_{k-1}, Tu)) \\ &\leq \psi(d(fu, fx_k)) + \phi(M(x_{k-1}, u))M(x_{k-1}, u) + \varphi(N(x_{k-1}, u))N(x_{k-1}, u) \end{aligned} \tag{2.6}$$

for all $k \geq k_0$. On the other hand, since $fx_k \rightarrow fu$ as $k \rightarrow \infty$ and $fx_k \in Tx_{k-1}$ so $d(fu, Tx_{k-1}) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, the following equalities

$$\begin{aligned} M(x_{k-1}, u) &= \max\{d(fx_{k-1}, fu), d(fu, Tx_{k-1})\}, \\ N(x_{k-1}, u) &= \min\{d(fx_{k-1}, fu), d(fx_{k-1}, Tx_{k-1}), d(fu, Tu), d(fx_{k-1}, Tu), d(fu, Tx_{k-1})\} \end{aligned}$$

together with (2.6) and property (ii) of ψ imply $d(fu, Tu) = 0$, that is, $fu \in Tu$ since $Tu \in \mathcal{CB}(X)$. This shows that u is a coincidence point of f and T . To prove that fu is a common fixed point of f and T , using the fact that $fu \in \text{Fix}(f)$ and taking $v := fu \in Tu$ we have $fv = ffu = fu = v$ and hence

$$\begin{aligned} \psi(H(Tu, Tv)) &\leq \alpha(u, v)\psi(H(Tu, Tv)) \\ &\leq \phi(M(u, v))M(u, v) + \varphi(N(u, v))N(u, v) \\ &= \phi(\max\{d(fu, fv), d(fv, Tu)\}) \max\{d(fu, fv), d(fv, Tu)\} \\ &\quad + \varphi(\min\{d(fu, fv), d(fu, Tu), d(fv, Tu), d(fu, Tv), d(fv, Tu)\}) \\ &\quad \times \min\{d(fu, fv), d(fu, Tu), d(fv, Tv), d(fu, Tv), d(fv, Tu)\} \\ &= 0. \end{aligned}$$

Now it follows from $\psi(d(fv, Tv)) = \psi(d(fu, Tv)) \leq \psi(H(Tu, Tv)) = 0$ that $v = fv \in Tv$. Therefore, $v = fu$ is a common fixed point of f and T . □

Now we have the following immediate consequence.

Corollary 2.1. *Let (X, d) be a metric space and $f, T : X \rightarrow X$ be single-valued mappings. Suppose that T satisfies*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \phi(M(x, y))M(x, y) + \varphi(N(x, y))N(x, y),$$

where $M(x, y), N(x, y), \phi, \varphi$ and ψ are given as in Definition 2.5, fX is a complete subspace of X and $TX \subset fX$. Suppose also that

(i) T is α -admissible, that is,

$$\text{if } x, y \in X, \quad \alpha(x, y) \geq 1 \implies \Lambda_f(Tx, Ty) \geq 1$$

where $\Lambda_f : X \times X \rightarrow [0, \infty)$ is given by $\Lambda_f(x, y) = \inf_{a, b \in X} \{\alpha(a, b) | fa = x, fb = y\}$;

(ii) there exists $x_0 \in X$ such that $\Lambda_f(fx_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then there exists an n_0 such that $\alpha(x_n, x) \geq 1$ for all $n \geq n_0$.

Then f and T have a coincidence point $u \in X$. Moreover, if $fu \in \text{Fix}(f)$ and $\alpha(u, fu) \geq 1$, then fu is a common fixed point of f and T .

To prove this, considering T as a single-valued mapping in Theorem 2.5 one can easily utilize the proof of Theorem 2.5, step by step, to conclude the result.

3 Functional equations in dynamic programming

Throughout this section, to verify the practicability of our main result we prove an existence theorem for a certain class of functional equations arising in dynamic programming which has been studied by utilizing various fixed point theorems in several papers (see [3,10-12,15,18] and the references therein).

From now on, we suppose that U and V are Banach spaces, $W \subset U$ and $D \subset V$. Also let $B(W)$ denote the set of all bounded real valued functions on W which is a metric space under the usual metric

$$d_B(h, k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W).$$

Bellman and Lee [1] pointed out that the basic form of the functional equation of dynamic programming is as follows:

$$p(x) = \sup_{y \in D} H(x, y, p(\tau(x, y))), \quad x \in W$$

where x and y stand for the state and decision vectors, respectively, $\tau : W \times D \rightarrow W$ represents the transformation of the process and $p(x)$ denotes the optimal return function with initial state x . Viewing W and D as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving the functional equation

$$p(x) = \sup_{y \in D} \{g(x, y) + G(x, y, p(\tau(x, y)))\}, \quad x \in W \quad (3.7)$$

where $g : W \times D \rightarrow \mathbb{R}$ and $G : W \times D \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded functions. We investigate the existence and uniqueness of solutions for functional equation (3.7). Suppose that the functions ϕ, φ and ψ are given as in Definition 2.5 and the mapping F is defined by

$$F(h(x)) = \sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad h \in B(W), \quad x \in W. \quad (3.8)$$

Suppose that following conditions hold:

(i) that there exists a mapping $\Omega : B(W) \times B(W) \rightarrow \mathbb{R}$ such that

$$\Omega(h, k) \geq 0 \implies \Omega(\sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \sup_{y \in D} \{g(x, y) + G(x, y, k(\tau(x, y)))\}) \geq 0, \quad x \in W$$

where $h, k \in B(W)$;

(ii) there is an $h_0 \in B(W)$ such that

$$\Omega(h_0(x), \sup_{y \in D} \{g(x, y) + G(x, y, h_0(\tau(x, y)))\}) \geq 0 \quad \text{for } x \in W;$$

(iii) if $\{h_n\}$ is a sequence of functions in $B(W)$ such that $\Omega(h_n, h_{n+1}) \geq 0$ for all n and $\{h_n\}$ is pointwise convergent to $h \in B(W)$, then there exists an n_0 such that $\Omega(h_n, h) \geq 0$ for all $n \geq n_0$.

Then we have the following result.

Theorem 3.6. *Suppose that all the conditions (i)-(iii) are satisfied. In addition, for all $h, k \in B(W)$ with $\Omega(h, k) \geq 0$ we have*

$$\psi(d_B(F(h), F(k))) \leq \phi(M(h, k))M(h, k) + \varphi(N(h, k))N(h, k), \quad (3.9)$$

where

$$M(h, k) := \max\{d_B(h, k), d_B(k, F(h))\}$$

and

$$N(h, k) := \min\{d_B(h, k), d_B(h, F(h)), d_B(k, F(k)), d_B(h, F(k)), d_B(k, F(h))\}.$$

Then functional equation (3.7) possesses a solution in $B(W)$.

Proof. Bearing in mind that $(B(W), d_B)$ is a complete metric space, where d_B is the metric induced by the usual metric on $B(W)$. It is easy to see that $B(W)$ is F -invariant, that is, for any $h \in B(W)$ we get $F(h) \in B(W)$. Clearly, F is both hybrid generalized multi-valued α -contractive type mapping and α -admissible with

$$\alpha(h, k) = \begin{cases} 1, & \Omega(h, k) \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

Now, replacing f by the identity mapping and taking $h_{n+1} = F(h_n)$ in the proof of Theorem 2.5 we can find a positive integer n_k such that

$$\phi^{n_k}(d_B(h_{k-1}, h_k)) \leq [1 - \phi(M(h_{k-1}, h_k))]M(h_{k-1}, h_k)$$

and

$$\psi(d_B(h_k, h_{k+1})) \leq \psi(d_B(F(h_{k-1}), F(h_k))) + \phi^{n_k}(d_B(h_{k-1}, h_k)) \quad (3.10)$$

for each $k = 1, 2, \dots$. These inequalities together with the fact that F enjoys condition (3.9) implies that $d_B(h_{k-1}, h_k)$ is a nondecreasing sequence, hence h_k converges to some h in $B(W)$. Now by subadditivity of ψ and using (3.9) we easily conclude that $d_B(h, F(h)) = 0$ which completes the proof. \square

Theorem 3.7. Suppose in addition to conditions of Theorem 3.6, that

$$\Omega(h, k) \geq 0 \implies \left(\frac{1}{d_B} - \left(\frac{\phi}{\psi}\right) \circ d_B\right)(h, k) > 0 \quad (3.11)$$

for $h, k \in B(W)$ such that $h \neq k$. Then functional equation (3.7) has a unique solution in $B(W)$.

Proof. Let $u, v \in B(W)$ be two distinct solutions of functional equation (3.7), that is, $Tu = u$ and $Tv = v$. Then using (3.9) we get

$$\psi(d_B(u, v)) \leq \phi(d_B(u, v))d_B(u, v)$$

which contradicts (3.11) and the conclusion follows. \square

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