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Invariant solutions of Barlett and Whitaker's equations

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Abstract

Lie symmetry group method is applied to study the Barlett and Whitaker's equations. The symmetry group and its optimal system are given,and group invariant solutions associated to the symmetries are obtained. Finally the structure of the Lie algebra symmetries is determined.

Keywords: Lie group analysis, Symmetry group, Optimal system, Invariant solution.

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1 Introduction

Enzymes electrodes are powerful tools for understanding the mechanism and kinetics of fast reactions. Owing to their specicity and sensitivity, enzyme elec-trodes including various amplication, schemes have been developed for many applications such as electrochemical immunoassays, [1, 2] water pollutant detection, [3, 4, 5, 6, 7] and monitoring of biological metabolities [8, 9, 10, 11]. The sensitivity of enzyme electrodes is very often increased by incorporation of a substrate-recycling scheme and several strategies including chemical, enzymatic, or electrochemical recycling have been developed. In the view of numerous application of such bio-sensor with amplied response,we are interested in investigating the concentration s and p in order to improve the metrological characteristics further.

In addition, this theoretical approach is of practical interest since this kind of bio-sensor can be used for the determination of phenolic compounds and catecholamine neurotransmitters in the field of environmental control and clinical analysis [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Such a theoretical and kinetic analysis is a powerful approach to rationalize functions of biosensors. Desprez and Labbe [23] obtained the analytical expression concentration and current for the limiting cases only. The purpose of this communication is to derive a simple accurate polynomial expressions of concentrations generated at a enzyme electrode using Lie Symmetries.

2 Lie Symmetry of the System

We consider the BWEs (Barlett and Whitaker's equations) [24], Desprez and Labbe [23], describing the concentrations of s and p at steady state as follows (with one independent and two dependent):

$$\text{BWEs} \quad : \quad \frac{d^2s}{dx^2} - \frac{\gamma s}{\alpha s + 1} = 0, \qquad \frac{d^2p}{dx^2} + \frac{\gamma s}{\alpha s + 1} = 0, \tag{2.1}$$

where

$$\gamma = \frac{1}{\Lambda^2}, \qquad \alpha = \frac{1}{K_s}, \qquad \Lambda = \sqrt{\frac{mK_s}{K_c E_t}}, \tag{2.2}$$

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x is varible, s and p are functions, and $\gamma, \Lambda, \alpha, K_s, m, K_c$, and E_t are constants. Let

$$\mathbf{v} = \xi(x,s,p)\partial_x + \tau(x,s,p)\partial_s + \varphi(x,s,p)\partial_p, \tag{2.3}$$

be a general vector field on the space of independent and dependent variables. we need the second prolongation:

$$\mathrm{Pr}^{(2)}\mathbf{v} = \mathbf{v} + \tau^x\partial_{s_x} + \varphi^x\partial_{p_x} + \tau^{xx}\partial_{s_{xx}} + \varphi^{xx}\partial_{p_{xx}}, \tag{2.4}$$

of $\mathbf{v}$, with the coefficients

$$\begin{aligned}
\tau^x &= \tau_x + \tau_p p_x + \tau_s s_x - s_x\xi_x - s_x\xi_p p_x - \xi_s s_x^2,\\
\varphi^x &= \varphi_x + \varphi_p p_x + \varphi_s s_x - p_x\xi_x - \xi_p p_x^2 - p_x\xi_s s_x,\\
\tau^{xx} &= 2\tau_{xp}p_x + 2\tau_{xs}s_x - s_x\xi_{xx} - 2\xi_{xs}s_x^2 + \tau_{pp}p_x^2 + p_{xx}\tau_p + \tau_{ss}s_x^2 - \xi_{ss}s_x^3 + s_{xx}\tau_s\\
&\quad -2s_{xx}\xi_x - 2s_x\xi_{xp}p_x + 2p_x\tau_{sp}s_x - s_x\xi_{pp}p_x^2 - 2p_x\xi_{sp}s_x^2 - p_{xx}\xi_p s_x\\
&\quad -3s_{xx}\xi_s s_x - 2s_{xx}\xi_p p_x + \tau_{xx},\\
\varphi^{xx} &= 2\varphi_{xp}p_x + 2\varphi_{xs}s_x - p_x\xi_{xx} - 2\xi_{xp}p_x^2 + \varphi_{pp}p_x^2 - \xi_{pp}p_x^3 + p_{xx}\varphi_p - 2p_{xx}\xi_x + \varphi_{ss}s_x^2\\
&\quad +s_{xx}\varphi_s - 2p_x\xi_{xs}s_x + 2p_x\varphi_{sp}s_x - 2s_x\xi_{sp}p_x^2 - 3p_{xx}\xi_p p_x - 2p_{xx}\xi_s s_x\\
&\quad -p_x\xi_{ss}s_x^2 - s_{xx}\xi_s p_x + \varphi_{xx}.
\end{aligned} \tag{2.5}$$

Applying $\mathrm{Pr}^{(2)}\mathbf{v}$ to equations (2.1), we find the infinitesimal criterion system. determining equations yields:

$$\begin{aligned}
&\varphi_{ss} = \tau_{p,p} = \xi_{ss} = \xi_{p,p} = \xi_{sp} = 0,\\
&\tau_{sp} - \xi_{xp} = \tau_{ss} - 2\xi_{xs} = 2\xi_{xp} - \varphi_{p,p} = \xi_{xs} - \varphi_{sp} = 0,\\
&-2sK_cE_t\xi_p + 2\tau_{xp}mK_s + 2\tau_{xp}ms = 2sK_cE_t\xi_s + 2\varphi_{xs}mK_s + 2\varphi_{xs}ms = 0,\\
&2\tau_{xs}mK_s + 2\tau_{xs}ms - 3sK_cE_t\xi_s - \xi_{xx}mK_s - \xi_{xx}ms + KcE_ts\xi_p = 0,\\
&3sK_cE_t\xi_p - sK_cE_t\xi_s + 2\varphi_{xp}mK_s + 2\varphi_{xp}ms - \xi_{xx}mK_s - \xi_{xx}ms = 0,\\
&-\tau K_cE_tK_s - 2K_cE_ts\xi_xK_s - 2K_cE_ts^2\xi_x - K_cE_ts\tau_pK_s\\
&\qquad -K_cE_ts^2\tau_p + \tau_{xx}mK_s^2 + 2\tau_{xx}mK_ss + \tau_{xx}ms^2 + K_cE_ts\tau_sK_s + K_cE_ts^2\tau_s = 0,\\
&\tau K_cE_tK_s - K_cE_ts\varphi_pK_s - K_cE_ts^2\varphi_p + 2K_cE_ts\xi_xK_s + 2K_cE_ts^2\xi_x\\
&\qquad +K_cE_ts\varphi_sK_s + K_cE_ts^2\varphi_s + \varphi_{xx}mK_s^2 + 2\varphi_{xx}mK_ss + \varphi_{xx}ms^2 = 0.
\end{aligned} \tag{2.6}$$

The solution of the above system gives the following coefficients of the vector field $\mathbf{v}$:

$$\varphi = C_2\,x + C_4\,(s+p) + C_3, \qquad \tau = 0, \qquad \xi = C_1, \tag{2.7}$$

where $C_1, \cdots, C_4$ are arbitrary constants; Thus the Lie algebra $\mathbf{G}$ of the electoenzymatic processes involved in a PPO-rotating-disk-bioelectrode equation is spanned by the four vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x\partial_p, \qquad \mathbf{v}_3 = \partial_p, \qquad \mathbf{v}_4 = (s+p)\partial_p. \tag{2.8}$$

The commutator table of $\mathbf{G}$ is

Table 1. Commutation relations satisfied by infinitesimal generators

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	$\mathbf{v}_3$	0	0
$\mathbf{v}_2$	$-\mathbf{v}_3$	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	0	0	0	$\mathbf{v}_3$
$\mathbf{v}_4$	0	$-\mathbf{v}_2$	$-\mathbf{v}_3$	0

Thus, $\mathbf{G}$ is a solvabel algebra with derived series $\mathbf{G} \geq \mathbf{G}^{(1)} \geq \{0\}$, where $\mathbf{G}^{(1)} = \mathrm{Span}\{\mathbf{v}_2, \mathbf{v}_3\} \cong R^2$, and $\mathbf{G}/\mathbf{G}^{(1)} \cong R^2$ are abelian, thus $\mathbf{G}$ is semidirect product of R^2 by itself.

The one-parameter groups G_i generated by the base of $\mathbf{G}$ are given in the following table.

$$\begin{aligned} G_1 &: (x,s,p) \longmapsto (x+\varepsilon, s, p), \\ G_2 &: (x,s,p) \longmapsto (x, s, x\varepsilon + p), \\ G_3 &: (x,s,p) \longmapsto (x, s, p+\varepsilon), \\ G_4 &: (x,s,p) \longmapsto (x, s, -s + e^{\varepsilon}(s+p)). \end{aligned} \tag{2.9}$$

Since each group G_i is a symmetry group and if $s = S(x), p = P(x)$ are solutions of the equations (2.1), so are the functions

$$\begin{array}{lll} 1) & s = S(x-\varepsilon), & p = P(x-\varepsilon), \\ 2) & s = S(x), & p = P(x) + x\varepsilon, \\ 3) & s = S(x), & p = P(x) + \varepsilon, \\ 4) & s = S(x), & p = e^{\varepsilon}(S(x)+P(x)) - S(x), \end{array} \tag{2.10}$$

where ε is a real number.

3 Optimal system of (2.1)

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. One of the applications of the adjoint representation is classifying group-invariant solutions.

The adjoint action is given by the Lie series

$$\mathbf{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots \tag{3.1}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is a commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 4$. The adjoint table

Table 2. Adjoint relations satisfied by infinitesimal generators

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2 - \epsilon\mathbf{v}_3$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_2$	$\mathbf{v}_1 + \epsilon\mathbf{v}_3$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_3$
$\mathbf{v}_4$	$\mathbf{v}_1$	$e^{\epsilon}\mathbf{v}_2$	$e^{\epsilon}\mathbf{v}_3$	$\mathbf{v}_4$

with (i,j)-th entry indicating $\mathbf{Ad}(\exp(\epsilon\mathbf{v}_i)\mathbf{v}_j)$ and ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4$ of the given vector fields. In particular if g is the action of the symmetry group near the identity, it can be represented in the form $g = \exp(c_1\mathbf{v}_1) \circ \cdots \circ \exp(c_4\mathbf{v}_4)$.

Let $F_i^{\varepsilon} : \mathbf{G} \longrightarrow \mathbf{G}$ defined by $\mathbf{v} \longrightarrow \mathbf{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1, ..., 4$. The matrices M_i^{ε} of F_i^{ε}, $i = 1, \cdots, 4$, with respect to basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\varepsilon & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\varepsilon} & 0 & 0 \\ 0 & 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

respectively, by acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathbf{G}$ is given by

$$1)\ \mathbf{v}_1,\qquad 2)\ \mathbf{v}_3,\qquad 3)\ \mathbf{v}_1+\mathbf{v}_2,\qquad 4)\ \mathbf{v}_1-\mathbf{v}_2,\qquad 5)\ \mathbf{v}_1+a\mathbf{v}_2,\ a\in R. \tag{3.3}$$

4 Conclusion

In this article group classification of (2.1) and the algebraic structure of the symmetry group is considered. Classification of one-dimensional subalgebra is determined by constructing one-dimensional optimal system. The structure of Lie algebra symmetries is analyzed.

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