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Ulam - Hyers stability of a 2- variable AC - mixed type functional equation in quasi - beta normed spaces: direct and fixed point methods

John M. Rassias,[a] M. Arunkumar,[b] S. Ramamoorthi[c] and S. Hemalatha[d,*]

[a]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, Athens 15342, Greece.

[b]Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

[c]Department of Mathematics, Arunai Engineering College, Tiruvannamalai - 606 604, TamilNadu, India.

[d]Department of Mathematics, Annai Veilankanni's College of Arts and Science, Saidapet, Chennai - 600 015.

Abstract

In this paper, we obtain the generalized Ulam - Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w)$$

in Quasi - Beta normed space using direct and fixed point methods.

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1 Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem was raised by S.M. Ulam [24] during his talk at the University of Wisconsin in 1940. In 1941, D.H. Hyers [8] gave an first affirmative answer to Ulam problem for Banach spaces. It was further generalized and excellent results were obtained by a number of authors.

Over the last seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations including mixed type additive and cubic functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 4, 6, 7, 9, 10, 11, 16, 17, 19, 21, 23, 25].

Very recently, M. Arunkumar et.al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \tag{1.1}$$

having solutions

$$f(x,y)=ax+by \tag{1.2}$$

*Corresponding author.
E-mail addresses: jrassias@primedu.uoa.gr (John M. Rassias), annarun2002@yahoo.co.in (M. Arunkumar), ramsdmaths@yahoo.com (S. Ramamoorthi), hemsjes@yahoo.co.in(S. Hemalatha).

and

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{1.3}$$

in Banach space using direct and fixed point approach.

The solution of the AC functional equation (1.1) is given in the following lemmas.

Lemma 1.1. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $g : U^2 \to V$ be a mapping given by*

$$g(x,x) = f(2x,2x) - 8f(x,x) \tag{1.4}$$

for all $x \in U$ then

$$g(2x,2x) = 2g(x,x) \tag{1.5}$$

for all $x \in U$ such that g is additive.

Lemma 1.2. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $h : U^2 \to V$ be a mapping given by*

$$h(x,x) = f(2x,2x) - 2f(x,x) \tag{1.6}$$

for all $x \in U$ then

$$h(2x,2x) = 8h(x,x) \tag{1.7}$$

for all $x \in U$ such that h is cubic.

Remark 1.1. *[3] If $f : U^2 \to V$ be a mapping satisfying (1.1) and let $g, h : U^2 \to V$ be a mapping defined in (1.4) and (1.6) then*

$$f(x,x) = \frac{1}{6}(h(x,x) - g(x,x)) \tag{1.8}$$

for all $x \in U$.

In this paper, the authors established the generalized Ulam-Hyers stability of the 2-variable AC functional equation (1.1) in Quasi-Beta Normed spaces using direct and fixed point methods are discussed in Section 3 and Section 4, respectively.

2 Preliminary results on quasi-beta normed spaces

In this section, we present some preliminary results concerning to quasi-β-normed spaces.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathcal{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let X be a linear space over $\mathcal{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following:*

(i) $\| x \| \geq 0$ *for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.*

(ii) $\| \lambda x \| = | \lambda |^{\beta} . \| x \|$ *for all $\lambda \in \mathcal{K}$ and all $x \in X$.*

(iii) *There is a constant $K \geq 1$ such that $\| x + y \| \leq K (\| x \| + \| y \|)$ for all $x, y \in X$.*

The pair $(X, \| \cdot \|)$ is called quasi-β-normed space if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.

Definition 2.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 2.3. *A quasi-β-norm $\| \cdot \|$ is called a (β, p)-norm $(0 < p \leq 1)$ if*

$$\| x + y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

More details, one can refer [7, 25] for the concepts of quasi-normed spaces and p-Banach space.

3 Stability results: Direct method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) using direct method.

Through out this section, let us take U is a linear space over $\mathcal{K}$ and V is a (β, p) Banach space with $p-$norm $\|.\|_V$. Let K be the modulus of concavity of $\|.\|_V$. Define a mapping $F : U^2 \to V$ by

$$\begin{aligned} F(x, y, z, w) = f(2x + y, 2z + w) - f(2x - y, 2z - w) - 4f(x + y, z + w) \\ + 4f(x - y, z - w) + 6f(y, w) \end{aligned}$$

for all $x, y, z, w \in U$.

Theorem 3.1. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{3.1}$$

such that the functional inequality

$$\|F(x, y, z, w)\|_V \le \alpha(x, y, z, w) \tag{3.2}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(2x, 2x) - 8f(x, x) - A(x, x)\|_V^p \le \frac{K^{np}}{2^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{2^{kjp}} \tag{3.3}$$

where $\delta(2^{kj}x)$ and $A(x, x)$ are defined by

$$\delta(2^{kj}x) = 4^{\beta}\, \alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{3.4}$$

$$A(x, x) = \lim_{n\to\infty} \frac{1}{2^{nj}} (f(2^{(n+1)j}x, 2^{(n+1)j}x) - 8f(2^{nj}x, 2^{nj}x)) \tag{3.5}$$

for all $x \in U$.

Proof. Assume $j = 1$. Letting (x, y, z, w) by (x, x, x, x) in (3.2), we obtain

$$\|f(3x, 3x) - 4f(2x, 2x) + 5f(x, x)\|_V \le \alpha(x, x, x, x) \tag{3.6}$$

for all $x \in U$. Replacing (x, y, z, w) by $(x, 2x, x, 2x)$ in (3.2), we get

$$\|f(4x, 4x) - 4f(3x, 3x) + 6f(2x, 2x) - 4f(x, x)\|_V \le \alpha(x, 2x, x, 2x) \tag{3.7}$$

for all $x \in U$. Now, from (3.6) and (3.7), we have

$$\begin{aligned} &\|f(4x, 4x) - 10f(2x, 2x) + 16f(x, x)\|_V \\ &\le K \left(4^{\beta} \|f(3x, 3x) - 4f(2x, 2x) + 5f(x, x)\|_V + \|f(4x, 4x) - 4f(3x, 3x) + 6f(2x, 2x) - 4f(x, x)\|_V\right) \\ &\le K(4^{\beta}\alpha(x, x, x, x) + \alpha(x, 2x, x, 2x)) \end{aligned} \tag{3.8}$$

for all $x \in U$. From (3.8), we arrive

$$\|f(4x, 4x) - 10f(2x, 2x) + 16f(x, x)\|_V \le K\delta(x) \tag{3.9}$$

where

$$\delta(x) = 4^{\beta}\alpha(x, x, x, x) + \alpha(x, 2x, x, 2x) \tag{3.10}$$

for all $x \in U$. It is easy to see from (3.9) that

$$\|f(4x, 4x) - 8f(2x, 2x) - 2(f(2x, 2x) - 8f(x, x))\|_V \le K\delta(x) \tag{3.11}$$

for all $x \in U$. Using (1.4) in (3.11), we obtain

$$\|g(2x, 2x) - 2g(x, x)\|_V \le K\delta(x) \tag{3.12}$$

for all $x \in U$. From (3.12), we arrive

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V \leq K \frac{\delta(x)}{2^\beta} \tag{3.13}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 2 in (3.13), we get

$$\left\| \frac{g(2^2x,2^2x)}{2^2} - \frac{g(2x,2x)}{2} \right\|_V \leq K \frac{\delta(2x)}{2^{\beta+1}} \tag{3.14}$$

for all $x \in U$. From (3.13) and (3.14), we obtain

$$\begin{aligned} \left\| \frac{g(2^2x,2^2x)}{2^2} - g(x,x) \right\|_V &\leq K \left(\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V + \left\| \frac{g(2^2x,2^2x)}{2^2} - \frac{g(2x,2x)}{2} \right\|_V \right) \\ &\leq \frac{K^2}{2^\beta} \left[\delta(x) + \frac{\delta(2x)}{2} \right] \end{aligned} \tag{3.15}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$\left\| \frac{g(2^nx,2^nx)}{2^n} - g(x,x) \right\|_V \leq \frac{K^n}{2^\beta} \sum_{k=0}^{n-1} \frac{\delta(2^kx)}{2^k} \leq \frac{K^n}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^kx)}{2^k} \tag{3.16}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{ \frac{g(2^nx,2^nx)}{2^n} \right\}$, replacing x by 2^mx and dividing by 2^m in (3.16), for any $m,n > 0$, we deduce

$$\begin{aligned} \left\| \frac{g(2^{n+m}x,2^{n+m}x)}{2^{(n+m)}} - \frac{g(2^mx,2^mx)}{2^m} \right\|_V &= \frac{1}{2^{m\beta}} \left\| \frac{g(2^n \cdot 2^mx, 2^n \cdot 2^mx)}{2^n} - g(2^mx,2^mx) \right\|_V \\ &\leq \frac{K^n}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^{k+m}x)}{2^{k+m\beta}} \to 0 \quad as \;\; m \to \infty \end{aligned}$$

for all $x \in U$. This shows that the sequence $\left\{ \frac{g(2^nx,2^nx)}{2^n} \right\}$ is a Cauchy sequence. Since V is complete, there exists a mapping $A(x,x) : U^2 \to V$ such that

$$A(x,x) = \lim_{n\to\infty} \frac{g(2^nx,2^nx)}{2^n}, \quad \forall \;\; x \in U.$$

Letting $n \to \infty$ in (3.16) and using (1.4), we see that (3.3) holds for all $x \in U$. To show that A satisfies (1.1), replacing (x,y,z,w) by $(2^nx,2^ny,2^nz,2^nw)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n} \left\| F(2^nx,2^ny,2^nz,2^nw) \right\|_V \leq \frac{1}{2^n} \alpha(2^nx,2^ny,2^nz,2^nw)$$

for all $x,y,z,w \in U$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x,x)$, we see that A satisfies (1.1) for all $x,y,z,w \in U$. To prove A is a unique 2-variable additive function satisfying (1.1), we let $B(x,x)$ be another 2-variable additive mapping satisfying (1.1) and (3.3), then

$$\begin{aligned} \|A(x,x) - B(x,x)\|_V \leq \frac{K}{2^{n\beta}} \Big\{ &\left\| A(2^nx,2^nx) - f(2^{n+1}x,2^{n+1}x) + 8f(2^nx,2^nx) \right\|_V \\ &+ \left\| f(2^{n+1}x,2^{n+1}x) - 8f(2^nx,2^nx) - B(2^nx,2^nx) \right\|_V \Big\} \\ \leq \frac{2K^{n+1}}{2^\beta} \sum_{k=0}^{\infty} \frac{\delta(2^{k+n}x)}{2^{(k+n\beta)}} &\to 0 \quad as \;\; n \to \infty \end{aligned}$$

for all $x \in U$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.1. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, & \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ \lambda \left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \end{cases} \tag{3.17}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V^p \leq \begin{cases} \left(\dfrac{K^n 2\lambda(4^\beta + 1)}{2^\beta}\right)^p, \\ \left(\dfrac{K^n 2\lambda(4^{\beta+1} + 2^{\beta s+1} + 2)\lambda||x||^s}{2^\beta|2 - 2^{\beta s}|}\right)^p, \\ \left(\dfrac{K^n 2\lambda(4^\beta + 2^{2\beta s})\lambda||x||^{4s}}{2^\beta|2 - 2^{\beta 4s}|}\right)^p \\ \left(\dfrac{K^n 2\lambda(5 \cdot 4^\beta + 2^{2\beta s} + 2^{4\beta s+1} + 2)\lambda||x||^{4s}}{2^\beta|2 - 2^{\beta 4s}|}\right)^p \end{cases} \tag{3.18}$$

for all $x \in U$.

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x, & if\, |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathcal{K}^2 \to \mathcal{K}$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq 32\, \mu\, (|x| + |y| + |z| + |w|) \tag{3.19}$$

for all $x, y, z, w \in \mathcal{K}$. Then there do not exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ such that

$$|f(2x,2x) - 8f(x,x) - A(x,x)|_V \leq \rho|x| \qquad for\ all \quad x \in \mathcal{K}. \tag{3.20}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\,\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.19).

If $x = y = z = w = 0$ then (3.19) is trivial. If $|x| + |y| + |z| + |w| \geq \frac{1}{2}$ then the left hand side of (3.19) is less than 32μ. Now suppose that $0 < |x| + |y| + |z| + |w| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x| + |y| + |z| + |w| < \frac{1}{2^{k-1}}, \tag{3.21}$$

so that $2^{k-1}x < \frac{1}{2}$, $2^{k-1}y < \frac{1}{2}$, $2^{k-1}z < \frac{1}{2}$, $2^{k-1}w < \frac{1}{2}$ and consequently

$$2^{k-1}(y,w), 2^{k-1}(x+y, z+w), 2^{k-1}(x-y, z-w),$$
$$2^{k-1}(2x+y, 2z+w), 2^{k-1}(2x-y, 2z-w), \in (-1,1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(y,w), 2^n(x+y, z+w), 2^n(x-y, z-w), \\ 2^n(2x+y, 2z+w), 2^n(2x-y, 2z-w), \in (-1,1)$$

and

$$\alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (3.21), we obtain that

$$\begin{aligned} &\Big| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w) \Big|_V \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ &\qquad + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) \Big|_V \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ &\qquad + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) \Big|_V \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n} 16\mu = 16\,\mu \times \frac{2}{2^k} = 32\,\mu\,(|x|+|y|+|z|+|w|)\,. \end{aligned}$$

Thus f satisfies (3.19) for all $x, y, z, w \in \mathcal{K}$ with $0 < |x|+|y|+|z|+|w| < \frac{1}{2}$.

We claim that the additive functional equation (1.1) is not stable for $s = 1$ in condition (ii) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ satisfying (3.20). Since f is bounded and continuous for all $x \in \mathcal{K}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x,x) = cx$ for any x in $\mathcal{K}$. Thus we obtain that

$$|f(2x,2x) - 8f(x,x)|_V \le (\rho + |c|)\,|x|. \tag{3.22}$$

But we can choose a positive integer m with $m\mu > \rho + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(2x,2x) - 8f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\rho + |c|)\,x$$

which contradicts (3.22). Therefore the additive functional equation (1.1) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (3.17). □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.1 is given in the following example.

Example 3.2. *Let s be such that $0 < s < \frac{1}{4}$. Then there is a function $F : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\lambda > 0$ satisfying*

$$|F(x,y,z,w)|_V \le \lambda |x|^{\frac{s}{4}}\ |y|^{\frac{s}{4}}\ |z|^{\frac{s}{4}}\ |w|^{\frac{1-3s}{4}} \tag{3.23}$$

for all $x, y, z, w \in \mathcal{K}$ and

$$\sup_{x \ne 0} \frac{|f(2x,2x) - 8f(x,x) - A(x,x)|_V}{|x|} = +\infty \tag{3.24}$$

for every additive mapping $A(x,x) : \mathcal{K}^2 \to \mathcal{K}$.

Proof. If we take

$$f(x,x)=\begin{cases}(x,x)\ln|x,x| & if \quad x\neq 0,\\ 0, & if \quad x=0.\end{cases}$$

Then from the relation (3.24), it follows that

$$\begin{aligned}\sup_{x\neq 0}\frac{|f(2x,2x)-8f(x,x)-A(x,x)|_V}{|x|} &\geq \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|f(2n,2n)-8f(n,n)-A(n,n)|_V}{|n|}\\ &=\sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|n(2,2)\ln|2n,2n|-8n(1,1)\ln|n,n|-n\,A(1,1)|_V}{|n|}\\ &=\sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}|(2,2)\ln|2n,2n|-8(1,1)\ln|n,n|-A(1,1)|_V=\infty.\end{aligned}$$

We have to prove (3.23) is true.

Case (i): If $x,y,z,w>0$ in (3.23) then,

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\ &=|(2x+y,2z+w)\ln|2x+y,2z+w|-(2x-y,2z-w)\ln|2x-y,2z-w|\\ &\quad -4(x+y,z+w)\ln|x+y,z+w|+4(x-y,z-w)\ln|x-y,z-w|+6(y,w)\ln|y,w||_V.\end{aligned}$$

Set $x=v_1, y=v_2, z=v_3, w=v_4$ it follows that

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\ &=|(2v_1+v_2,2v_3+v_4)\ln|2v_1+v_2,2v_3+v_4|-|2v_1-v_2,2v_3-v_4|\ln|2v_1-v_2,2v_3-v_4|\\ &\quad -4(v_1+v_2,v_3+v_4)\ln|v_1+v_2,v_3+v_4|+4|v_1-v_2,v_3-v_4|\ln|v_1-v_2,v_3-v_4|\\ &\qquad +6(v_2,v_4)\ln|v_2,v_4||_V.\\ &=|f(2v_1+v_2,2v_3+v_4)-f(2v_1-v_2,2v_3-v_4)-4f(v_1+v_2,v_3+v_4)\\ &\qquad +4f(v_1-v_2,v_3-v_4)+6f(v_2,v_4)|_V\\ &\leq\lambda|v_1|^{\frac{s}{4}}\,|v_2|^{\frac{s}{4}}\,|v_3|^{\frac{s}{4}}\,|v_4|^{\frac{1-3s}{4}}\\ &=\lambda|x|^{\frac{s}{4}}\,|y|^{\frac{s}{4}}\,|z|^{\frac{s}{4}}\,|w|^{\frac{1-3s}{4}}.\end{aligned}$$

Case (ii): If $x,y,z,w<0$ in (3.23) then,

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\ &=|(2x+y,2z+w)\ln|2x+y,2z+w|-(2x-y,2z-w)\ln|2x-y,2z-w|\\ &\quad -4(x+y,z+w)\ln|x+y,z+w|+4(x-y,z-w)\ln|x-y,z-w|+6(y,w)\ln|y,w||_V.\end{aligned}$$

Set $-x=v_1, -y=v_2, -z=v_3, -w=v_4$ it follows that

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\ &=|(-2v_1-v_2,-2v_3-v_4)\ln|-2v_1-v_2,-2v_3-v_4|\\ &\qquad -(-2v_1+v_2,-2v_3+v_4)\ln|-2v_1+v_2,-2v_3+v_4|\\ &\qquad -4(-v_1-v_2,-v_3-v_4)\ln|-v_1-v_2,-v_3-v_4|\\ &\qquad +4(-v_1+v_2,-v_3+v_4)\ln|-v_1+v_2,-v_3+v_4|\\ &\qquad +6(-v_2,-v_4)\ln|-v_2,-v_4||_V.\\ &=|f(-2v_1-v_2,-2v_3-v_4)-f(-2v_1+v_2,-2v_3+v_4)-4f(-v_1-v_2,-v_3-v_4)\\ &\qquad +4f(-v_1+v_2,-v_3+v_4)+6f(-v_2,-v_4)|_V\\ &\leq\lambda|-v_1|^{\frac{s}{4}}\,|-v_2|^{\frac{s}{4}}\,|-v_3|^{\frac{s}{4}}\,|-v_4|^{\frac{1-3s}{4}}\\ &=\lambda|x|^{\frac{s}{4}}\,|y|^{\frac{s}{4}}\,|z|^{\frac{s}{4}}\,|w|^{\frac{1-3s}{4}}.\end{aligned}$$

Case (iii): If $x, z > 0, y, w < 0$ then $2x+y, 2z+w, x+y, z+w > 0$, $2x-y, 2z-w, x-y, z-w < 0$ in (3.23) then,

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\&=|(2x+y,2z+w)\ln|2x+y,2z+w|-(2x-y,2z-w)\ln|2x-y,2z-w|\\&\quad-4(x+y,z+w)\ln|x+y,z+w|+4(x-y,z-w)\ln|x-y,z-w|+6(y,w)\ln|y,w||_V.\end{aligned}$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\&=|(2v_1-v_2,2v_3-v_4)\ln(2v_1-v_2,2v_3-v_4)\\&\qquad-(2v_1+v_2,2v_3+v_4)\ln|-(2v_1+v_2,2v_3+v_4)|\\&\qquad-4(v_1-v_2,v_3-v_4)\ln|v_1-v_2,v_3-v_4|\\&\qquad+4(v_1+v_2,v_3+v_4)\ln|-(v_1+v_2,v_3+v_4)|\\&\qquad+6(-v_2,-v_4)\ln(-v_2,-v_4)|_V.\\&=|f(2v_1-v_2,2v_3-v_4)-f(2v_1+v_2,2v_3+v_4)-4f(v_1-v_2,v_3-v_4)\\&\qquad+4f(v_1+v_2,v_3+v_4)+6f(-v_2,-v_4)|_V\\&\le\lambda|v_1|^{\frac{s}{4}}\,|-v_2|^{\frac{s}{4}}\,|v_3|^{\frac{s}{4}}\,|-v_4|^{\frac{1-3s}{4}}\\&=\lambda|x|^{\frac{s}{4}}\,|y|^{\frac{s}{4}}\,|z|^{\frac{s}{4}}\,|w|^{\frac{1-3s}{4}}.\end{aligned}$$

Case (iv): If $x, z > 0, y, w < 0$ then $2x+y, 2z+w, x+y, z+w < 0$, $2x-y, 2z-w, x-y, z-w > 0$ in (3.23) then,

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\&=|(2x+y,2z+w)\ln|2x+y,2z+w|-(2x-y,2z-w)\ln|2x-y,2z-w|\\&\quad-4(x+y,z+w)\ln|x+y,z+w|+4(x-y,z-w)\ln|x-y,z-w|+6(y,w)\ln|y,w||_V.\end{aligned}$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$\begin{aligned}&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)+4f(x-y,z-w)+6f(y,w)|_V\\&=|(2v_1-v_2,2v_3-v_4)\ln|-(2v_1-v_2,2v_3-v_4)|\\&\qquad-(2v_1+v_2,2v_3+v_4)\ln|2v_1+v_2,2v_3+v_4|\\&\qquad-4(v_1-v_2,v_3-v_4)\ln|-(v_1-v_2,v_3-v_4)|\\&\qquad+4(v_1+v_2,v_3+v_4)\ln|v_1+v_2,v_3+v_4|\\&\qquad+6(-v_2,-v_4)\ln(-v_2,-v_4)|_V.\\&=|f(2v_1-v_2,2v_3-v_4)-f(2v_1+v_2,2v_3+v_4)-4f(v_1-v_2,v_3-v_4)\\&\qquad+4f(v_1+v_2,v_3+v_4)+6f(-v_2,-v_4)|_V\\&\le\lambda|v_1|^{\frac{s}{4}}\,|-v_2|^{\frac{s}{4}}\,|v_3|^{\frac{s}{4}}\,|-v_4|^{\frac{1-3s}{4}}\\&=\lambda|x|^{\frac{s}{4}}\,|y|^{\frac{s}{4}}\,|z|^{\frac{s}{4}}\,|w|^{\frac{1-3s}{4}}.\end{aligned}$$

Case (v): If $x = y = z = w = 0$ in (3.23) then it is trivial. □

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = \frac{1}{4}$ in condition (iv) of Corollary 3.1.

Example 3.3. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{4} \\ \frac{\mu}{4}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathcal{K}^2 \to \mathcal{K}$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq 8\,\mu \left(|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} \right) \tag{3.25}$$

for all $x, y, z, w \in \mathcal{K}$. Then there do not exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ such that

$$|f(2x,2x) - 8f(x,x) - A(x,x)|_V \leq \rho|x| \qquad for\ all \quad x \in \mathcal{K}. \tag{3.26}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{1}{2^n} \times \frac{\mu}{4} = \frac{\mu}{2}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.25).

If $x = y = z = w = 0$ then (3.25) is trivial.

If $|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} \geq \frac{1}{2}$ then the left hand side of (3.25) is less than 8μ. Now suppose that $0 < |x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} + \{|x| + |y| + |w| + |z|\} < \frac{1}{2^{k-1}}, \tag{3.27}$$

so that $2^{k-1}|x|^{\frac{1}{4}}\ |y|^{\frac{1}{4}}\ |z|^{\frac{1}{4}}\ |w|^{\frac{1}{4}} < \frac{1}{2}$, $2^{k-1}|x| < \frac{1}{2}$, $2^{k-1}|y| < \frac{1}{2}$, $2^{k-1}|z| < \frac{1}{2}$, $2^{k-1}|w| < \frac{1}{2}$ and consequently

$$\begin{aligned} &2^{k-1}(y,w), 2^{k-1}(x+y,z+w), 2^{k-1}(x-y,z-w), \\ &\qquad 2^{k-1}(2x+y,2z+w), 2^{k-1}(2x-y,2z-w), \in \left(-\frac{1}{4}, \frac{1}{4}\right). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^{n}(y,w), 2^{n}(x+y,z+w), 2^{n}(x-y,z-w), \\ &\qquad 2^{n}(2x+y,2z+w), 2^{n}(2x-y,2z-w), \in \left(-\frac{1}{4}, \frac{1}{4}\right) \end{aligned}$$

and

$$\begin{aligned} &\alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ &\qquad + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (3.27), we obtain that

$$\begin{aligned} &\Big| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) + 4f(x-y,z-w) + 6f(y,w) \Big|_V \\ &\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ &\qquad + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) \Big|_V \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \alpha(2^n(2x+y,2z+w)) - \alpha(2^n(2x-y,2z-w)) - 4\alpha(2^n(x+y,z+w)) \\ &\qquad + 4\alpha(2^n(x-y,z-w)) + 6\alpha(2^n(y,w)) \Big|_V \end{aligned}$$

$$\leq \sum_{n=k}^{\infty} \frac{16\mu}{4} \times \frac{1}{2^n} = \frac{16\mu}{4} \times \frac{2}{2^k} = 8\,\mu\left(|x|^{\frac{1}{4}}\,|y|^{\frac{1}{4}}\,|z|^{\frac{1}{4}}\,|w|^{\frac{1}{4}} + \{|x|+|y|+|w|+|z|\}\right).$$

Thus f satisfies (3.25) for all $x, y, z, w \in \mathcal{K}$ with
$0 < |x|^{\frac{1}{4}}\,|y|^{\frac{1}{4}}\,|z|^{\frac{1}{4}}\,|w|^{\frac{1}{4}} + \{|x|+|y|+|w|+|z|\} < \frac{1}{2}$.

We claim that the additive functional equation (1.1) is not stable for $s = \frac{1}{4}$ in condition (iv) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\rho > 0$ satisfying (3.26). Since f is bounded and continuous for all $x \in \mathcal{K}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x, x) = cx$ for any x in $\mathcal{K}$. Thus we obtain that

$$|f(2x, 2x) - 8f(x, x)|_V \leq (\rho + |c|)\,|x|. \tag{3.28}$$

But we can choose a positive integer m with $m\mu > \rho + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(2x, 2x) - 8f(x, x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\rho + |c|)\,x$$

which contradicts (3.28). Therefore the additive functional equation (1.1) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{4}$, assumed in the inequality condition (iv) of (3.17).

□

Theorem 3.2. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{8^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{3.29}$$

such that the functional inequality

$$\|F(x, y, z, w)\|_V \leq \alpha(x, y, z, w) \tag{3.30}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(2x, 2x) - 2f(x, x) - C(x, x)\|_V^p \leq \frac{K^{np}}{8^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{8^{kjp}} \tag{3.31}$$

where $\delta(2^{kj}x)$ and $C(x, x)$ are defined by

$$\delta(2^{kj}x) = 4^{\beta}\alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{3.32}$$

$$C(x, x) = \lim_{n\to\infty} \frac{1}{8^{nj}} (f(2^{(n+1)j}x, 2^{(n+1)j}x) - 2f(2^{nj}x, 2^{nj}x)) \tag{3.33}$$

for all $x \in U$.

Proof. It is easy to see from (3.9) that

$$\|f(4x, 4x) - 2f(2x, 2x) - 8(f(2x, 2x) - 2f(x, x))\|_V \leq K\delta(x) \tag{3.34}$$

for all $x \in U$. Using (1.6) in (3.34), we obtain

$$\|h(2x, 2x) - 8h(x, x)\|_V \leq K\delta(x) \tag{3.35}$$

for all $x \in U$. From (3.35), we arrive

$$\left\|\frac{h(2x, 2x)}{8} - h(x, x)\right\|_V \leq K\frac{\delta(x)}{8^{\beta}} \tag{3.36}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1 □

The following Corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.1).

Corollary 3.2. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s<3 \quad or \quad s>3; \\ \lambda\, ||x||^s||y||^s||z||^s||w||^s, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \end{cases} \tag{3.37}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$\|f(2x,2x) - 2f(x,x) - C(x,x)\|_V^p \leq \begin{cases} \left(\dfrac{K^n 8\lambda(4^\beta+1)}{7\cdot 8^\beta}\right)^p, \\ \left(\dfrac{K^n 8\lambda(4^{\beta+1}+2^{\beta s+1}+2)\lambda||x||^s}{8^\beta|8-2^{\beta s}|}\right)^p, \\ \left(\dfrac{K^n 8\lambda(4^\beta+2^{2\beta s})\lambda||x||^{4s}}{8^\beta|8-2^{\beta 4s}|}\right)^p \\ \left(\dfrac{K^n 8\lambda(5\cdot 4^\beta+2^{2\beta s}+2^{4\beta s+1}+2)\lambda||x||^{4s}}{8^\beta|8-2^{\beta 4s}|}\right)^p \end{cases} \tag{3.38}$$

for all $x \in U$.

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = 3$ in condition (ii) of Corollary 3.2.

Example 3.4. *Let $\alpha : \mathcal{K} \to \mathcal{K}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^3, & if\ |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathcal{K}^2 \to \mathcal{K}$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \leq \frac{16\ \mu \times 8^3}{7}\left(|x|^3 + |y|^3 + |z|^3 + |w|^3\right) \tag{3.39}$$

for all $x, y, z, w \in \mathcal{K}$. Then there do not exist a cubic mapping $C : \mathcal{K}^2 \to \mathcal{K}$ and a constant $\beta > 0$ such that

$$|f(2x,2x) - 2f(x,x) - C(x,x)|_V \leq \beta|x|^3 \qquad for\ all \quad x \in \mathcal{K}. \tag{3.40}$$

Proof. Now

$$|f(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{\mu}{8^n} = \frac{8\ \mu}{7}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.39).

If $x = y = z = w = 0$ then (3.39) is trivial. If $|x|^3 + |y|^3 + |z|^3 + |w|^3 \geq \frac{1}{8}$ then the left hand side of (3.39) is less than $\frac{16 \times 8\mu}{7}$. Now suppose that $0 < |x|^3 + |y|^3 + |z|^3 + |w|^3 < \frac{1}{8}$. Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^3 + |y|^3 + |z|^3 + |w|^3 < \frac{1}{8^{k+1}}, \tag{3.41}$$

the rest of the proof is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.2 is given in the following example.

Example 3.5. *Let s be such that* $0 < s < \frac{3}{4}$. *Then there is a function* $F : \mathcal{K}^2 \to \mathcal{K}$ *and a constant* $\lambda > 0$ *satisfying*

$$|F(x,y,z,w)|_V \le \lambda |x|^{\frac{s}{4}} \, |y|^{\frac{s}{4}} \, |z|^{\frac{s}{4}} \, |w|^{\frac{3-3s}{4}} \tag{3.42}$$

for all $x, y, z, w \in \mathcal{K}$ *and*

$$\sup_{x \neq 0} \frac{|f(2x,2x) - 2f(x,x) - C(x,x)|_V}{|x|^3} = +\infty \tag{3.43}$$

for every cubic mapping $C : \mathcal{K}^2 \to \mathcal{K}$.

Proof. If we take

$$f(x,x) = \begin{cases} (x,x)^3 \ln |x,x| & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.43), it follows that

$$\begin{aligned} &\sup_{x \neq 0} \frac{|f(2x,2x) - 2f(x,x) - C(x,x)|_V}{|x|^3} \\ &\qquad \ge \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f(2n,2n) - 2f(n,n) - C(n,n)|_V}{|n|^3} \\ &\qquad = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^3 (2,2)^3 \ln |n,n| - 2n^3 (1,1)^3 \ln |n,n| - n^3 \, C\,(1,1)\right|_V}{|n|^3} \\ &\qquad = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \left|(2,2)^3 \ln |n,n| - 2(1,1)^3 \ln |n,n| - C\,(1,1)\right|_V = \infty. \end{aligned}$$

Rest of the proof is similar to that of Example 3.2. □

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = \frac{3}{4}$ in condition (iv) of Corollary 3.2.

Example 3.6. *Let* $\alpha : \mathcal{K} \to \mathcal{K}$ *be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^3, & if \, |x| < \dfrac{3}{4} \\ \dfrac{3\mu}{4}, & otherwise \end{cases}$$

where $\mu > 0$ *is a constant, and define a function* $f : \mathcal{K}^2 \to \mathcal{K}$ *by*

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for\ all \quad x \in \mathcal{K}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)|_V \le \frac{96\mu \times 8^2}{7} \left(|x|^{\frac{3}{4}} \, |y|^{\frac{3}{4}} \, |z|^{\frac{3}{4}} \, |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |w|^3 + |z|^3 \right\} \right) \tag{3.44}$$

for all $x, y, z, w \in \mathcal{K}$. *Then there do not exist a cubic mapping* $C : \mathcal{K}^2 \to \mathcal{K}$ *and a constant* $\rho > 0$ *such that*

$$|f(2x,2x) - 2f(x,x) - C(x,x)|_V \le \rho |x| \qquad for\ all \quad x \in \mathcal{K}. \tag{3.45}$$

Proof. Now

$$|f(x,x)| \le \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{1}{8^n} \times \frac{3\mu}{4} = \frac{6\,\mu}{7}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.44).

If $x = y = z = w = 0$ then (3.44) is trivial. If $|x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}}$ $+ \left\{|x|^3 + |y|^3 + |w|^3 + |z|^3\right\} \geq \frac{1}{8}$ then the left hand side of (3.44) is less than $\frac{96\ \mu}{7}$. Now suppose that $0 < |x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}} + \left\{|x|^3 + |y|^3 + |w|^3 + |z|^3\right\} < \frac{1}{8}$. Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^{\frac{3}{4}}\ |y|^{\frac{3}{4}}\ |z|^{\frac{3}{4}}\ |w|^{\frac{3}{4}} + \left\{|x|^3 + |y|^3 + |w|^3 + |z|^3\right\} < \frac{1}{8^{k+1}}, \tag{3.46}$$

the rest of the proof is similar to that of Example 3.3. □

Now, we are ready to prove our main stability results.

Theorem 3.3. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the conditions given in (3.1) and (3.29) respectively, such that the functional inequality*

$$\|F(x, y, z, w)\|_V \leq \alpha(x, y, z, w) \tag{3.47}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\|f(x, x) - A(x, x) - C(x, x)\|_V^p \leq \frac{K^p}{6^p}\left\{\frac{K^{np}}{2^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{2^{kjp}} + \frac{K^{np}}{8^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{8^{kjp}}\right\} \tag{3.48}$$

where $\delta(2^{kj}x)$, $A(x, x)$ and $C(x, x)$ are respectively defined in (3.4), (3.5) and (3.33) for all $x \in U$.

Proof. By Theorems 3.1 and 3.2, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$\|f(2x, 2x) - 8f(x, x) - A_1(x, x)\|_V^p \leq \frac{K^{np}}{2^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{2^{kjp}} \tag{3.49}$$

$$\|f(2x, 2x) - 2f(x, x) - C_1(x, x)\|_V^p \leq \frac{K^{np}}{8^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{8^{kjp}} \tag{3.50}$$

for all $x \in U$. Now from (3.49) and (3.50), one can see that

$$\begin{aligned}
&\left\|f(x, x) + \frac{1}{6}A_1(x, x) - \frac{1}{6}C_1(x, x)\right\|_V^p \\
&= \left\|\left\{-\frac{f(2x, 2x)}{6} + \frac{8f(x, x)}{6} + \frac{A_1(x, x)}{6}\right\} + \left\{\frac{f(2x, 2x)}{6} - \frac{2f(x, x)}{6} - \frac{C_1(x, x)}{6}\right\}\right\|_V^p \\
&\leq \frac{K^p}{6^p}\left\{\|f(2x, 2x) - 8f(x, x) - A_1(x, x)\|_V^p + \|f(2x, 2x) - 2f(x, x) - C_1(x, x)\|_V^p\right\} \\
&\leq \frac{K^p}{6^p}\left\{\frac{K^{np}}{2^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{2^{kjp}} + \frac{K^{np}}{8^{\beta p}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\delta(2^{kj}x)^p}{8^{kjp}}\right\}
\end{aligned}$$

for all $x \in U$. Thus we obtain (3.50) by defining $A(x, x) = \frac{-1}{6}A_1(x, x)$ and $C(x, x) = \frac{1}{6}C_1(x, x)$, $\delta(2^{kj}x)$, $A(x, x)$ and $C(x, x)$ are respectively defined in (3.4), (3.5) and (3.33) for all $x \in U$. □

The following corollary is the immediate consequence of Theorem 3.3, using Corollaries 3.1 and 3.2 concerning the stability of (1.1).

Corollary 3.3. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\begin{aligned}
&\|F(x, y, z, w))\|_V \\
&\leq \begin{cases}
\lambda, & \\
\lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s \neq 1, 3; \\
\lambda\ ||x||^s||y||^s||z||^s||w||^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\
\lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s \neq \frac{1}{4}, \frac{3}{4};
\end{cases}
\end{aligned} \tag{3.51}$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\|f(x,x) - A(x,x) - C(x,x)\|_V^p \leq \begin{cases} \left(\frac{K^{n+1}\lambda(4^\beta+1)}{6}\left[\frac{2}{2^\beta}+\frac{8}{7\cdot 8^\beta}\right]\right)^p, \\ \left(\frac{K^{n+1}\lambda(4^{\beta+1}+2^{\beta s+1}+2)||x||^s}{6}\left[\frac{2}{2^\beta|2-2^{\beta s}|}+\frac{8}{8^\beta|8-2^{\beta s}|}\right]\right)^p, \\ \left(\frac{K^{n+1}\lambda(4^\beta+2^{2\beta s})||x||^{4s}}{6}\left[\frac{2}{2^\beta|2-2^{\beta 4s}|}+\frac{8}{8^\beta|8-2^{\beta 4s}|}\right]\right)^p, \\ \left(\frac{K^{n+1}8\lambda(5\cdot 4^\beta+2^{2\beta s}+2^{4\beta s+1}+2)\lambda||x||^{4s}}{6}\left[\frac{2}{2^\beta|2-2^{\beta 4s}|}+\frac{8}{8^\beta|8-2^{\beta 4s}|}\right]\right)^p, \end{cases} \tag{3.52}$$

for all $x \in U$.

4 Stability results: Fixed point method

In this section, we apply a fixed point method for achieving stability of the 2-variable AC functional equation (1.1).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [12] for fixed point Theory.

Theorem 4.1. *[12] Suppose that for a complete generalized metric space (Ω, β) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that
(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(ii) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(iii) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1.1).

Through out this section let U be a normed space and V is a (β, p) Banach space with $p-$norm $\|.\|_V$. Define a mapping $F : U^2 \to V$ by

$$F(x,y,z,w) = f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w) + 6f(y,w)$$

for all $x, y, z, w \in U$.

Theorem 4.2. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^n} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \tag{4.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x,y,z,w)\|_V \leq \alpha(x,y,z,w) \tag{4.2}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the property

$$\gamma(x) \leq L\, \mu_i\, \gamma(\mu_i x). \tag{4.3}$$

for all $x \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1.1) and

$$\| f(2x,2x) - 8f(x,x) - A(x,x) \|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.4}$$

for all $x \in U$.

Proof. Consider the set

$$\Omega = \{q_1/q_1 : U^2 \to V,\ q_1(0,0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(q_1,q_2) = d_\gamma(q_1,q_2) = \inf\{M \in (0,\infty) : \| q_1(x,x) - q_2(x,x) \| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega^2 \to \Omega$ by

$$Tq_1(x,x) = \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x),$$

for all $x \in U$. Now $q_1, q_2 \in \Omega$,

$$\begin{aligned} d(q_1,q_2) \leq M &\Rightarrow \| q_1(x,x) - q_2(x,x) \| \leq M\gamma(x), x \in U. \\ &\Rightarrow \left\| \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q_2(\mu_i x, \mu_i x) \right\| \leq \frac{1}{\mu_i} M\gamma(\mu_i x), x \in U, \\ &\Rightarrow \left\| \frac{1}{\mu_i} q_1(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q_2(\mu_i x, \mu_i x) \right\| \leq LM\gamma(x), x \in U, \\ &\Rightarrow \| Tq_1(x,x) - Tq_2(x,x) \| \leq LM\gamma(x), x \in U, \\ &\Rightarrow d_\gamma(q_1,q_2) \leq LM. \end{aligned}$$

This implies $d(Tq_1, Tq_2) \leq Ld(q_1,q_2)$, for all $q_1, q_2 \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
From (3.12), we arrive

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\|_V \leq K \frac{\delta(x)}{2^\beta} \tag{4.5}$$

for all $x \in U$. Using (4.3) for the case $i = 0$ it reduces to

$$\left\| \frac{g(2x,2x)}{2} - g(x,x) \right\| \leq L\gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d(g,Tg) \leq L = \frac{1}{2^\beta} \Rightarrow d(g,Tg) \leq L = L^1 < \infty. \tag{4.6}$$

Again replacing $x = \frac{x}{2}$ in (4.5), we get,

$$\left\| g(x,x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_V \leq K\delta\left(\frac{x}{2}\right) \tag{4.7}$$

Using (4.3) for the case $i = 1$ it reduces to

$$\left\| g(x,x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_V \leq \gamma(x)$$

for all $x \in U$,

$$\text{i.e.,} \quad d(g,Tg) \leq 1 \Rightarrow d(g,Tg) \leq 1 = L^0 < \infty. \tag{4.8}$$

From (4.6) and (4.8), we have

$$d(Tg,g) \leq L^{1-i}. \tag{4.9}$$

Now from from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x,x) = \lim_{n\to\infty} \frac{1}{\mu_i^n} (f(\mu_i^{(n+1)} x, \mu_i^{(n+1)} x) - 8f(\mu_i^n x, \mu_i^n x)) \tag{4.10}$$

for all $x \in U$.

To prove $A : U^2 \to V$ is additive. Replacing (x, y, z, w) by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w\right)$ in (4.2) and dividing by μ_i^n, it follows from (4.1) that

$$\|A(x,y,z,w)\|_V = \lim_{n\to\infty} \frac{\left\|F(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)\right\|_V}{\mu_i^n} \le \lim_{n\to\infty} \frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = 0$$

for all $x, y, z, w \in U$ i.e., A satisfies the functional equation (1.1).

According to the fixed point alternative, since A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f, A) < \infty\}$, A is the unique function such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V \le M\gamma(x)$$

for all $x \in U$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f, A) \le \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, A) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2x,2x) - 8f(x,x) - A(x,x) \|_V^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p$$

this completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1.1).

Corollary 4.4. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \le \begin{cases} \lambda, & \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ \lambda \left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \end{cases} \tag{4.11}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|_V^p \le \begin{cases} (K\lambda(4^\beta + 1))^p, \\ \left(\dfrac{(2 + 2^{s+1} + 4^{\beta+1})K\lambda||x||^s}{|2 - 2^{\beta s}|}\right)^p, \\ \left(\dfrac{(4^\beta + 2^{2s})K\lambda||x||^{4s}}{|2 - 2^{\beta 4s}|}\right)^p \\ \left(\dfrac{(5 \cdot 4^\beta + 2^{2s} + 2^{4s+1} + 2)K\lambda||x||^{4s}}{|2 - 2^{\beta 4s}|}\right)^p \end{cases} \tag{4.12}$$

for all $x \in U$.

Proof. Setting

$$\alpha(x,y,z,w) = \begin{cases} \lambda, \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, \\ \lambda \;\; ||x||^s \; ||y||^s \; ||z||^s \; ||w||^s \\ \lambda \left\{||x||^s ||y||^s ||z||^s ||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\} \end{cases}$$

for all $x, y, z, w \in U$. Now

$$\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = \begin{cases} \frac{\lambda}{\mu_i^n}, \\ \frac{\lambda}{\mu_i^n}\{||\mu_i^n x||^s + ||\mu_i^n y||^s + ||\mu_i^n z||^s + ||\mu_i^n w||^s\}, \\ \frac{\lambda}{\mu_i^n}\ ||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n w||^s \\ \frac{\lambda}{\mu_i^n}\ \Big\{||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n w||^s \\ \Big\{||\mu_i^n x||^{4s} + ||\mu_i^n y||^{4s} + ||\mu_i^n z||^{4s} + ||\mu_i^n w||^{4s}\Big\}\Big\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (4.1) is holds.

But we have $\gamma(x) = K\delta\left(\frac{x}{2}\right)$ has the property $\gamma(x) \leq L \cdot \mu_i\ \gamma(\mu_i x)$ for all $x \in U$. Hence

$$\gamma(x) = K\delta\left(\frac{x}{2}\right) = K\left(4\alpha\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \alpha\left(\frac{x}{2}, x, \frac{x}{2}, x\right)\right) = \begin{cases} K\lambda\left(4^\beta + 1\right), \\ \frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s, \\ \frac{K\lambda}{2^{4s}}\left(2^{2s} + 4^\beta\right)||x||^{4s}, \\ \frac{K\lambda}{2^{4s}}\left(2^{2s} + 2^{s+1} + 2^{4s+1} + 5\cdot 4^\beta\right)||x||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma(\mu_i x) = \begin{cases} \mu_i^{-1}K\lambda\left(4^\beta + 1\right), \\ \mu_i^{s-1}K\frac{\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s, \\ \mu_i^{4s-1}K\frac{\lambda}{2^{4s}}\left(2^{2s} + 4^\beta\right)||x||^{4s}, \\ \mu_i^{4s-1}K\frac{\lambda}{2^{4s}}\left(2^{2s} + 2^{s+1} + 2^{4s+1} + 5\cdot 4^\beta\right)||x||^{4s} \end{cases} = \begin{cases} \mu_i^{-1}\gamma(x), \\ \mu_i^{s-1}\gamma(x), \\ \mu_i^{4s-1}\gamma(x), \\ \mu_i^{4s-1}\gamma(x). \end{cases}$$

Hence the inequality (4.3) holds either, $L = 2^{s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 2$ if $i = 1$. Now from (4.4), we prove the following cases for condition (ii).

Case:1 $L = 2^{s-1}$ for $s < 1$ if $i = 0$

$$\begin{aligned} \|f(2x,2x) - 8f(x,x) - A(x,x)\| &\leq \frac{\left(2^{(s-1)}\right)^{1-0}}{1 - 2^{(s-1)}}\frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s \\ &= \frac{K\lambda\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s}{2 - 2^s} \end{aligned}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned} \|f(2x,2x) - 8f(x,x) - A(x,x)\| &\leq \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-1)}}}\frac{K\lambda}{2^s}\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s \\ &= \frac{K\lambda\left(2 + 2^{s+1} + 4^{\delta+1}\right)||x||^s}{2^s - 2} \end{aligned}$$

Similarly, the inequality (4.3) holds either, $L = 2^{-1}$ for $s = 0$ if $i = 0$ and $L = \frac{1}{2^{-1}}$ for $s = 0$ if $i = 1$ for condition (i), the inequality (4.3) holds either, $L = 2^{4s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > 2$ if $i = 1$ for condition (iii) and the inequality (4.3) holds either, $L = 2^{4s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > 2$ if $i = 1$ for condition (iv).

Hence the proof is complete □

The proof of the following Theorem and Corollary is similar to that of Theorem 4.2 and Corollary 4.4. Hence the details of the proof is omitted.

Theorem 4.3. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{3n}} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \tag{4.13}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x,y,z,w)\|_V \le \alpha(x,y,z,w) \tag{4.14}$$

for all $x,y,z,w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the property

$$\gamma(x) \le L\, \mu_i^3\, \gamma(\mu_i x). \tag{4.15}$$

Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\| f(2x,2x) - 2f(x,x) - C(x,x) \|_V^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p \tag{4.16}$$

for all $x \in U$.

Corollary 4.5. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w))\|_V \le \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s<1 \quad or \quad s>1; \\ \lambda\, ||x||^s||y||^s||z||^s||w||^s, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \end{cases} \tag{4.17}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$\|f(2x,2x) - 2f(x,x) - C(x,x)\|_V^p \le \begin{cases} \left(\dfrac{K\lambda(4^\beta+1)}{7}\right)^p, \\ \left(\dfrac{(2+2^{s+1}+4^{\beta+1})K\lambda||x||^s}{|8-2^{\beta s}|}\right)^p, \\ \left(\dfrac{(4^\beta+2^{2s})K\lambda||x||^{4s}}{|8-2^{\beta 4s}|}\right)^p \\ \left(\dfrac{(5\cdot 4^\beta+2^{2s}+2^{4s+1}+2)K\lambda||x||^{4s}}{|8-2^{\beta 4s}|}\right)^p \end{cases} \tag{4.18}$$

for all $x \in U$.

Now, we are ready to prove the main fixed point stability results.

Theorem 4.4. *Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0,\infty)$ with the conditions (4.1) and (4.13) where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality*

$$\|F(x,y,z,w)\|_V \le \alpha(x,y,z,w) \tag{4.19}$$

for all $x,y,z,w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = K\delta\left(\frac{x}{2}\right),$$

has the properties (4.3) and (4.15) Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1.1) and

$$\| f(x,x) - A(x,x) - C(x,x) \|_V^p \leq \frac{2\,K^p}{6^{\beta\,p}} \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.20}$$

for all $x \in U$.

Proof. By Theorems 4.2 and 4.3, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.21}$$

and

$$\|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \tag{4.22}$$

for all $x \in U$. Now from (4.21) and (4.22), one can see that

$$\begin{aligned} &\left\| f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x) \right\|_V^p \\ &= \left\| \left\{ -\frac{f(2x,2x)}{6} + \frac{8f(x,x)}{6} + \frac{A_1(x,x)}{6} \right\} + \left\{ \frac{f(2x,2x)}{6} - \frac{2f(x,x)}{6} - \frac{C_1(x,x)}{6} \right\} \right\|_V^p \\ &\leq \frac{K^p}{6^{\beta p}} \left\{ \|f(2x,2x) - 8f(x,x) - A_1(x,x)\|_V^p + \|f(2x,2x) - 2f(x,x) - C_1(x,x)\|_V^p \right\} \\ &\leq \frac{K^p}{6^{\beta p}} \left\{ \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p + \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p \right\} \end{aligned}$$

for all $x \in U$. Thus we obtain (4.20) by defining $A(x,x) = \frac{-1}{6}A_1(x,x)$ and $C(x,x) = \frac{1}{6}C_1(x,x)$, for all $x \in U$. □

The following Corollary is an immediate consequence of Theorem 4.4, using Corollaries 4.4 and 4.5 concerning the stability of (1.1).

Corollary 4.6. *Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\begin{aligned} &\|F(x,y,z,w))\|_V \\ &\leq \begin{cases} \lambda, & \\ \lambda \left\{ ||x||^s + ||y||^s + ||z||^s + ||w||^s \right\}, & s \neq 1,3; \\ \lambda\, ||x||^s ||y||^s ||z||^s ||w||^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda \left\{ ||x||^s ||y||^s ||z||^s ||w||^s + \left\{ ||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s} \right\} \right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \end{aligned} \tag{4.23}$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\begin{aligned} &\|f(x,x) - A(x,x) - C(x,x)\|_V \\ &\leq \begin{cases} \left(\dfrac{K^2\lambda(4^\beta + 1)}{6} \left[1 + \dfrac{1}{7} \right] \right)^p, \\ \left(\dfrac{(2 + 2^{s+1} + 4^{\beta+1})K^2\lambda||x||^s}{6} \left[\dfrac{1}{|2 - 2^{\beta s}|} + \dfrac{1}{|8 - 2^{\beta s}|} \right] \right)^p, \\ \left(\dfrac{(4^\beta + 2^{2s})K^2\lambda||x||^{4s}}{6} \left[\dfrac{1}{|2 - 2^{\beta 4s}|} + \dfrac{1}{|8 - 2^{\beta 4s}|} \right] \right)^p \\ \left(\dfrac{(5 \cdot 4^\beta + 2^{2s} + 2^{4s+1} + 2)K^2\lambda||x||^{4s}}{6} \left[\dfrac{1}{|2 - 2^{\beta 4s}|} + \dfrac{1}{|8 - 2^{\beta 4s}|} \right] \right)^p \end{cases} \end{aligned} \tag{4.24}$$

for all $x \in U$.

References

[1] **J. Aczel and J. Dhombres**, *Functional Equations in Several Variables,* Cambridge University Press, 1989.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, 2(1950), 64-66.

[3] M. Arunkumar, Matina J. Rassias, Yanhui Zhang, Ulam - Hyers stability of a 2- variable AC - mixed type functional equation: direct and fixed point methods, *Journal of Modern Mathematics Frontier,* 1(3)(2012), 10-26.

[4] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[5] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings , *J. Math. Anal. Appl.*, 184(1994), 431-436.

[6] M. Eshaghi Gordji, H. Khodaie, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.

[7] M. Eshaghi Gordji, H. Khodaei, J.M. Rassias, Fixed point methods for the stability of general quadratic functional equation, *Fixed Point Theory,* 12(1)(2011), 71-82.

[8] D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.,* 27(1941), 222-224.

[9] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*,Birkhauser, Basel, 1998.

[10] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[11] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[12] B.Margoils, J.B.Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.,* 126(74) (1968), 305-309.

[13] M.M. Pourpasha, J. M. Rassias, R. Saadati, S.M. Vaezpour, A fixed point approach to the stability of Pexider quadratic functional equation with involution, *J. Inequal. Appl.,* 2010, Art. ID 839639, 18 pp.

[14] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal.,* 46(1982), 126-130.

[15] J.M. Rassias, H.M. Kim, Generalized Hyers-Ulam stability for general additive functional equations in quasi-δ-normed spaces, *J. Math. Anal. Appl.,* 356(1)(2009), 302-309.

[16] J.M. Rassias, K.Ravi, M.Arunkumar and B.V.Senthil Kumar, *Ulam Stability of Mixed type Cubic and Additive functional equation*, Functional Ulam Notions (F.U.N) Nova Science Publishers, 2010, Chapter 13, 149 - 175.

[17] J.M. Rassias, E. Son, H.M. Kim, On the Hyers-Ulam stability of 3D and 4D mixed type mappings, *Far East J. Math. Sci.,* 48(1)(2011), 83-102.

[18] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, 72 (1978), 297-300.

[19] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[20] K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematical Sciences*, Autumn 2008 Vol.3, No. 08, 36-47.

[21] K. Ravi, J.M. Rassias, M. Arunkumar, R. Kodandan, Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation, *J. Inequal. Pure Appl. Math.*, 10(4)(2009), Article 114, 29 pp.

[22] K. Ravi, J.M. Rassias, R. Kodandan, Generalized Ulam-Hyers stability of an AQ-functional equation in quasi-β-normed spaces, *Math. AEterna*, 1(3-4)(2011), 217-236.

[23] S.M. Jung, J.M. Rassias, A fixed point approach to the stability of a functional equation of the spiral of Theodorus, *Fixed Point Theory Appl.*, 2008, Art. ID 945010, 7 pp.

[24] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, 1964.

[25] T.Z. Xu, J.M. Rassias, M.J. Rassias, W.X. Xu, A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces, *J. Inequal. Appl.*, 2010, Art. ID 423231, 23 pp.

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