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On 3-Dissection Property

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Abstract

The purpose of this paper is to derive 3- dissection for $(q^2;q^2)_\infty^{-1}(q^4;q^4)_\infty^{-1}$, $(q^3;q^3)_\infty^{-1}(q^6;q^6)_\infty^{-1}$ and $(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty^{-1}(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty^{-1}$.

Keywords: Partition functions, Generating functions.

2010 MSC: 05A17, 05A15, 11P83.

1 Introduction

$x \sim y$

In 2010, Chan [1] has studied on Ramanujan's cubic continued fraction and defined a function a(n), as

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty \, (q^2;q^2)_\infty} \tag{1.1}$$

In 2011, Zhao and Zhong [2] have studied and investigated the arithmetic properties of a function b(n), as

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 \, (q^2;q^2)_\infty^2} \tag{1.2}$$

Through this paper, we assume

$$(a;q)_\infty = \prod_{n=1}^{\Pi}(1 - aq^{(n-1)}) \quad ; \quad |q| < 1 \tag{1.3}$$

Many properties of $a(n)$ and $b(n)$ are similar with the standard partition function $p(n)$, the function $p(n)$ is defined to be the number of ways of writing n as a sum of positive integers in non-increasing order. Mathematically it is defined as $\sum_{n\geq 0} p(n)q^n = \prod_{n=1}(1-q)^{-1}$. It is convention that, one sets $p(0) = 0$ and $p(n) = 0$ for $n < 0$. Chan[1] obtained the generating function of $a(3n+2)$, as

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3 \, (q^6;q^6)_\infty^3}{(q;q)_\infty^4 \, (q^2;q^2)_\infty^4} \tag{1.4}$$

This identity was prove by Cao[3] by using the 3-dissection for $(q;q)_\infty \, (q^2;q^2)_\infty$. The outline of this paper is as follows. In sections 2, we have recorded some well known results, those are useful to the rest of the paper. In section 3, we state and prove three new theorems, which are not available in the literature.

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2 Preliminaries

Let us recall the definition of cubic theta function $A(q)$,$B(q)$ and $C(q)$due to Borwein et al.[4], as

$$A(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.1}$$

$$B(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}; \ \ \omega = exp\left(\frac{2\Pi\iota}{3}\right) \tag{2.2}$$

$$C(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \tag{2.3}$$

Borwein et al.[4] established the following relations

$$A(q) = A(q^3) + 2qC(q^3) \tag{2.4}$$

$$B(q) = A(q^3) - qC(q^3) \tag{2.5}$$

$$C(q) = \frac{3(q^3;q^3)_\infty^3}{(q;q)_\infty} \tag{2.6}$$

$$A(q)A(q^2) = B(q)B(q^2) + qC(q)C(q^2) \tag{2.7}$$

3 Main results

Now we derive following results by applying 3-dissection

Theorem-I:

$$\frac{1}{(q^2;q^2)_\infty(q^4;q^4)_\infty} = \frac{A(q^{12})C(q^6)}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} + \frac{q^2A(q^6)C(q^{12})}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} + \frac{q^4C(q^6)C(q^{12})}{3(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} \tag{3.1}$$

Theorem-II:

$$\frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{q^3A(q^9)C(q^{18})}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} + \frac{A(q^{18})C(q^9)}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} + \frac{q^6C(q^9)C(q^{18})}{3(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3} \tag{3.2}$$

Theorem-III:

$$\frac{1}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} = \frac{qA(q^2)C(q)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} + \frac{q^{\frac{1}{3}}A(q)C(q^2)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} + \frac{q^{\frac{2}{3}}C(q)C(q^2)}{3(q;q)_\infty^3(q^2;q^2)_\infty^3} \tag{3.3}$$

Proof of Theorem-I: In equation (2.6), by substituting $q = q^2$ and $q = q^4$,we get the values of $C(q^2)$ and $C(q^4)$ respectively. Now by multiplying $C(q^2)$ and $C(q^4)$, and after making suitable arrangement, we get

$$\frac{1}{(q^2;q^2)_\infty(q^4;q^4)_\infty} = \frac{C(q^2)C(q^4)}{9(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty^3} \tag{3.4}$$

In equation (2.4), by substituting $q = q^2$ and $q = q^4$,we get the values of $A(q^2)$ and $A(q^4)$ respectively. Now by multiplying $A(q^2)$ and $A(q^4)$,we get

$$A(q^2)A(q^4) = A(q^6)A(q^{12}) + 2q^2A(q^{12})C(q^6) + 2q^4A(q^6)C(q^{12}) + 4q^6C(q^6)C(q^{12}) \tag{3.5}$$

In equation (2.5), by substituting $q = q^2$ and $q = q^4$,we get the values of $B(q^2)$ and $B(q^4)$ respectively. Now by multiplying $B(q^2)$ and $B(q^4)$,we get

$$B(q^2)B(q^4) = A(q^6)A(q^{12}) - q^2A(q^{12})C(q^6) - q^4A(q^6)C(q^{12}) + q^8C(q^6)C(q^{12}) \tag{3.6}$$

In equation (2.7), by substituting $q = q^2$,we get the values of $C(q^2)C(q^4)$, as

$$q^2C(q^2)C(q^4) = A(q^2)A(q^4) - B(q^2)B(q^4) \tag{3.7}$$

By the equations (3.5),(3.6)and(3.7), we get

$$C(q^2)C(q^4) = 3A(q^{12})C(q^6) + 3q^2A(q^6)C(q^{12}) + 3q^4C(q^6)C(q^{12}) \tag{3.8}$$

By substituting the value of $C(q^2)C(q^4)$ in equation (3.4), from equation (3.8), after simplification, we get the required result as per equation (3.1), and we complete the proof of Theorem-I.

Proof of Theorem-II: In equation (2.6), by substituting $q = q^3$ and $q = q^6$,we get the values of $C(q^3)$ and $C(q^6)$ respectively. Now by multiplying $C(q^3)$ and $C(q^6)$, and after making suitable arrangement, we get

$$\frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{C(q^3)C(q^6)}{9(q^9;q^9)^3_\infty(q^{18};q^{18})^3_\infty} \tag{3.9}$$

In equation (2.4), by substituting $q = q^3$ and $q = q^6$,we get the values of $A(q^3)$ and $A(q^6)$ respectively. Now by multiplying $A(q^3)$ and $A(q^6)$,we get

$$A(q^3)A(q^6) = A(q^9)A(q^{18}) + 2q^6A(q^9)C(q^{18}) + 2q^3A(q^{18})C(q^9) + 4q^9C(q^9)C(q^{18}) \tag{3.10}$$

In equation (2.5), by substituting $q = q^3$ and $q = q^6$,we get the values of $B(q^3)$ and $B(q^6)$ respectively. Now by multiplying $B(q^3)$ and $B(q^6)$,we get

$$B(q^3)B(q^6) = A(q^9)A(q^{18}) - q^6A(q^9)C(q^{18}) - q^3A(q^{18})C(q^9) + q^9C(q^9)C(q^{18}) \tag{3.11}$$

In equation (2.7), by substituting $q = q^3$,we get the values of $C(q^3)C(q^6)$, as

$$q^3C(q^3)C(q^6) = A(q^3)A(q^6) - B(q^3)B(q^6) \tag{3.12}$$

By the equations (3.10),(3.11)and(3.12), we get

$$C(q^3)C(q^6) = 3q^3A(q^9)C(q^{18}) + 3A(q^{18})C(q^9) + 3q^6C(q^9)C(q^{18}) \tag{3.13}$$

By substituting the value of $C(q^3)C(q^6)$ in equation (3.9), from equation (3.13), after simplification, we get the required result as per equation (3.2), and we complete the proof of Theorem-II.

Proof of Theorem-III: In equation (2.6), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $C(q^{\frac{1}{3}})$ and $C(q^{\frac{2}{3}})$ respectively. Now by multiplying $C(q^{\frac{1}{3}})$ and $C(q^{\frac{2}{3}})$, and after making suitable arrangement, we get

$$\frac{1}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty} = \frac{C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})}{9(q;q)^3_\infty(q^2;q^2)^3_\infty} \tag{3.14}$$

In equation (2.4), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $A(q^{\frac{1}{3}})$ and $A(q^{\frac{2}{3}})$ respectively. Now by multiplying $A(q^{\frac{1}{3}})$ and $A(q^{\frac{2}{3}})$,we get

$$A(q^{\frac{1}{3}})A(q^{\frac{2}{3}}) = A(q)A(q^2) + 2q^{\frac{1}{3}}A(q^2)C(q) + 2q^{\frac{2}{3}}A(q)C(q^2) + 4qC(q)C(q^2) \tag{3.15}$$

In equation (2.5), by substituting $q = q^{\frac{1}{3}}$ and $q = q^{\frac{2}{3}}$,we get the values of $B(q^{\frac{1}{3}})$ and $B(q^{\frac{2}{3}})$ respectively. Now by multiplying $B(q^{\frac{1}{3}})$ and $B(q^{\frac{2}{3}})$,we get

$$B(q^{\frac{1}{3}})B(q^{\frac{2}{3}}) = A(q)A(q^2) - q^{\frac{1}{3}}A(q^2)C(q) - q^{\frac{2}{3}}A(q)C(q^2) + qC(q)C(q^2) \tag{3.16}$$

In equation (2.7), by substituting $q = q^{\frac{1}{3}}$,we get the values of $C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})$, as

$$q^{\frac{1}{3}}C(q^{\frac{1}{3}})C(q^{\frac{2}{3}}) = A(q^{\frac{1}{3}})A(q^{\frac{2}{3}}) - B(q^{\frac{1}{3}})B(q^{\frac{2}{3}}) \tag{3.17}$$

By the equations (3.15),(3.16)and(3.17), we get

$$C(q^{\frac{1}{3}})C(q^{\frac{2}{3}}) = 3A(q^2)C(q) + 3(q^{\frac{1}{3}})A(q)C(q^2) + 3q^{\frac{2}{3}}C(q)C(q^2) \tag{3.18}$$

By substituting the value of $C(q^{\frac{1}{3}})C(q^{\frac{2}{3}})$ in equation (3.14), from equation (3.18), after simplification, we get the required result as per equation (3.3), and we complete the proof of Theorem-III.

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