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An existence and uniqueness theorem for fuzzy H-integral equations of fractional order

Mouffak Benchohra[a,b*] and Abderrahmane Boukenkoul[c]

[a]Laboratory of Mathematics, University of Sidi Bel Abbès, Algeria, PO Box 89 Sidi Bel Abbes 22000, Algeria.

[b]Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia.

[c]Département de Foresterie et d'Agronomie, Faculté des sciences de la vie et de l'environnement, Université Abou Bekr Belkaid, Tlemcen, Algérie.

Abstract

We present an existence and uniqueness theorem for H- integral equations of fractional order involving fuzzy set valued mappings of a real variable whose values are normal, convex, upper semi continuous and compactly supported fuzzy sets in $\mathbb{R}^n$. The method of successive approximation is the main tool in our analysis.

Keywords: Fuzzy mapping, fractional orders, Riemann-Liouville H-differentiability, Fuzzy H-integral equation, Hausdorff metric, successive approximation.

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1 Introduction

Dubois and Prade [10] introduced the concept of integration of fuzzy functions. Alternative approaches were later suggested by Goetschel and Voxman [13], Kaleva [15], Nanda [21] and others. While Goetschel and Voxman preferred a Riemann integral type approach, Kaleva chose to define the integral of fuzzy function, using the Lebesgue-type concept of integration. For more information about integration of fuzzy functions and fuzzy integral equations, for instance, see [2, 8, 10, 13, 15, 21, 22, 24, 25] and references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville, see [14] and references therein. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [1, 8, 14, 16, 18, 23, 26].

By means of the fuzzy integral due to Kaleva [15], we investigate the fractional fuzzy integral equation, for the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in $\mathbb{R}^n$. We consider the fuzzy integral equation of Riemann-Liouville fractional order generalized H-differentiability this equation takes the form

$$y(t) = f(t) + \frac{1}{\Gamma(1-q)} \int_0^t \frac{g(s, y(s))}{(t-s)^q}\, ds, \tag{1.1}$$

where $f : [0, T] \to E^n$ and $g : [0, T] \times E^n \to E^n$, and $q \in (0, 1)$. The definition of E^n is given in Section 2.
The paper is organized as follows: in Section 2 auxiliary facts and results are given which will be used later. In Section 3, the Riemann-Liouville H-differentiability is proposed for fuzzy-valued function and the some of important results of it are provided. In Section 4 the main theorem on the existence and uniqueness of solutions of equation (1.1) is given.

*Corresponding author.
E-mail addresses: benchohra@univ-sba.dz (Mouffak Benchohra), ab_koul@yahoo.fr (Abderrahmane Boukenkoul)

2 Auxiliary facts and results

This section is devoted to collect some definitions and results which will be needed further on.

Definition 2.1. Let X be a nonempty set. A *fuzzy set* A in X is characterized by its membership function $A : X \to [0,1]$ and $A(x)$, called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership and values between them are used to represent intermediate degrees of membership. Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. It is clear that $(P_k(\mathbb{R}^n), d)$ is a complete metric space [17].

A fuzzy set $u \in E^n$ is a function $u : \mathbb{R}^n \to [0,1]$ for which

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e., for $x,\ y \in \mathbb{R}^n$ and $\beta \in [0,1]$,

$$u(\beta x + (1-\beta)y) \geq \min(u(x), u(y))$$

(iii) u is upper semi-continuous, and

(iv) the closure of $\{x \in \mathbb{R}^n \ :\ u(x) > 0\}$, denoted by $[u]^0$, is compact.

For $0 < \gamma \leq 1$, the $\alpha-$level set $[u]^\gamma$ is define by $[u]^\gamma = \{x \in \mathbb{R}^n \ :\ u(x) \geq \gamma\}$. Then from $(i)-(iv)$, it follows that $[u]^\gamma \in P_k(\mathbb{R}^n)$ for all $0 \leq \gamma \leq 1$.

We define the supremum metric D on E^n by

$$D(u,\overline{u}) = \sup_{0<\gamma\leq 1} H_d([u]^\gamma, [\overline{u}]^\gamma)$$

for all $u, \overline{u} \in E^n$. (E^n, D) is a complete metric space.

3 Riemann-Liouville Fractional H-differentiability

Now, we define fuzzy Riemann-Liouville fractional derivatives of order $0 \leq r \leq 1$ for fuzzy-valued function f which is a direct extension of strongly generalized H-differentiability in the fractional literature [9].

Definition 3.2. *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, it is denoted by $z = x \ominus y$.*
The sign $\ominus$ always stands for H-difference, also not that $x \ominus y \neq x + (-1)y$.
Also,we define some notations which are used throughout the paper.

- $L_p^F(a,b), 1 \leq p < \infty$ *is the set of all fuzzy-valued measurable and p-integrable functions f on $[a,b]$ where*

$$\|f\|_p = \left(\int_0^1 (d(f(t),0))^p dt\right)^{\frac{1}{p}}.$$

- $C^F[a,b]$ *is a space of fuzzy-valued functions which are continuous on $[a,b]$.*

- $AC^F[a,b]$ *denotes the set of all fuzzy-valued functions which are absolutely continuous.*

Definition 3.3. *Let $f : [a,b] \to E$, $x_0 \in (a,b)$ and $\Phi(x) = \frac{1}{\Gamma(1-q)} \int_a^x \frac{f(t)}{(x-t)^q} dt$. We say that $f(x)$ is fuzzy Riemann-Liouville fractional H-differentiable about order $0 \le q \le 1$ at x_0, if there exists an element $({}^{RL}D^q_{a^+} f)(x_0) \in C^F, 0 \le q \le 1$ such that for all $0 \le r \le 1, h > 0$*

(*i*)

$$({}^{RL}D^q_{a^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}$$

or

(*ii*)

$$({}^{RL}D^q_{a^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h}.$$

For sake of simplicity, we say that a fuzzy-valued function f is ${}^{RL}(1,q)$-differentiable if it is differentiable as in the definition 3.3 case (i), and is ${}^{RL}(2,q)$-differentiable if it is differentiable as in Definition 3.3 case (ii).

Definition 3.4. Let $f \in L^1(a,b)$, $0 \le a < b < \infty$, and let $0 < q < 1$ be a real number. The fractional integral of order q of Riemann-Liouville type is defined by (see; [16, 23]).

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds.$$

Let us consider the $r-$cut representation of fuzzy valued function f as $f(x;r) = [\underline{f}(x;r), \overline{f}(x;r)]$ for $0 \le r \le 1$, then we can indicate the Riemann-Liouville integral of fuzzy-valued function f based on its lower and upper functions as follows:

Theorem 3.1. *Let $f : [a,b] \to E$ be a fuzzy-valued function. The fuzzy Riemann-Liouville integral of f can be expressed as follows:*

$$(I^q f)(x;r) = [(I^q \underline{f})(x;r), (I^q \overline{f})(x;r)], 0 \le r \le 1$$

where

$$(I^q \underline{f})(x;r) = \frac{1}{\Gamma(q)} \int_a^x \frac{\underline{f}(t;r)}{(x-t)^{1-q}} dt$$

$$(I^q \overline{f})(x;r) = \frac{1}{\Gamma(q)} \int_a^x \frac{\overline{f}(t;r)}{(x-t)^{1-q}} dt.$$

Now, we define fuzzy Riemann-Liouville fractional derivatives of order $0 \le r \le 1$ for fuzzy-valued function f which is a direct extension of strongly generalized H-differentiability [9] in the fractional literature. Also, we denote by C^F the space of all fuzzy-valued functions which are continuous on $[a,b]$ and we assume that all fuzzy-valued functions in this work are placed in C^F. We define the fuzzy Riemann-Liouville H-integrals of fuzzy-valued function as follows:

Theorem 3.2. *Let $f : [0,T] \to E^n$, $x_0 \in [0,T]$ and $0 \le q \le 1$ such that for all $0 \le r \le 1$.*

(1) *if $f(x)$ be a ${}^{RL}(1,q)$ differentiable fuzzy-valued function, then*

$$({}^{RL}D^q_0 f)(x_0;r) = [{}^{RL}D^q_0 \underline{f}(x_0;r), {}^{RL}D^q_0 \overline{f}(x_0;r)],$$

(2) *if $f(x)$ be a ${}^{RL}(2,q)$ differentiable fuzzy-valued function, then*

$$({}^{RL}D^q_0 f)(x_0;r) = [{}^{RL}D^q_0 \overline{f}(x_0;r), {}^{RL}D^q_0 \underline{f}(x_0;r)].$$

Where

$$({}^{RL}D^q_0 \underline{f})(x_0) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^q} dt \mid_{x=x_0},$$

and

$$({}^{RL}D^q_0 \overline{f})(x_0) = \ominus \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^q} dt \mid_{x=x_0}.$$

Rewrite Eq.(1.1) in the form

$$y(t) = f(t) +^{RL} I_0^q g(t, y(t)), \quad t \geq 0, \tag{3.2}$$

where ${}^{RL}I_0^q$ is the standard Riemann-Liouville fractional H-integral operator. Notice that, since f is assumed to be integrable and $(x-t)^{q-1}$ is a crisp function, we deduce that $\frac{f(t)}{(x-t)^{1-q}}$ is integrable and then, the existence of integral (1.1) is proved.

Theorem 3.3. *Let $f : [a,b] \to E$, $x_0 \in (a,b)$ and $0 \leq q \leq 1$ for all $0 \leq r \leq 1$, we have*

(1) *if f is ${}^{RL}(1,q)$ H-integrable then*

$${}^{RL}I_0^q(f)(x_0;r) = [{}^{RL}I_0^q \underline{f}(x_0;r), {}^{RL} I_0^q \overline{f}(x_0;r)]$$

(2) *if f is ${}^{RL}(2,q)$ H-integrable then*

$${}^{RL}I_0^q f(x_0;r) = [{}^{RL}I_0^q \overline{f}(x_0;r), {}^{RL} I_0^q \underline{f}(x_0;r)]$$

In this paper, we prove an existence and uniqueness theorem of a solution to the fuzzy integral equation (1.1). The method of successive approximation is the main tool in our analysis.

4 Main Theorem

In this section, we will study Eq(1.1) assuming that the following assumptions are satisfied, Let L and T be positive numbers:

(a_1) $f : [0,T] \to E^n$ is continuous and bounded.

(a_2) $g : [0,T] \times E^n \to E^n$ is continuous and satisfies the Lipschitz condition, i.e.,

$$D\left(g(t, y_2(t)), g(t, y_1(t))\right) \leq L\, D\left(y_2(t), y_1(t)\right),\ t \in [0,T],$$

where $y_i : [0,T] \to E^n$, $i = 1,\ 2$.

(a_3) $g(t, \hat{0})$ is bounded on $[0,T]$.

Now, we are in a position to state and prove our main result in paper

Theorem 4.4. Let the assumptions $(a_1) - (a_3)$ be satisfied. If

$$T < \left(\frac{\Gamma(2-q)}{L}\right)^{\frac{1}{1-q}},$$

then Eq(1.1) has a unique solution y on $[0,T]$ defined as the following:

(1) In the case ${}^{RL}(1;q)$ differentiability, the successive iterations

$$\begin{aligned} y_0(t) &= f(t) \\ y_{n+1}(t) &= f(t) + {}^{RL}I_0{}^q g(t, y_n(t)), \quad n = 0,1,2,\ldots \end{aligned} \tag{4.3}$$

(2) In the case ${}^{RL}(2;q)$ differentiability, the successive iterations

$$\begin{aligned} \hat{y}_0(t) &= f(t) \\ \hat{y}_{n+1}(t) &= f(t) \ominus {}^{RL}I_0{}^q g(t, \hat{y}_n(t)), \quad n = 0,1,2,\ldots \end{aligned} \tag{4.4}$$

are uniformly convergent to y on $[0,T]$.

Proof. (1) Case (1): If f is ${}^{RL}(1;q)$ differentiable
First we prove that y_n are bounded on $[0,T]$. We have $y_0 = f(t)$ is bounded, thanks (a_1). Assume that y_{n-1} is bounded. From (4.3) we have

$$\begin{aligned}
D(y_n(t),\hat{0}) &= D\left(f(t) + {}^{RL}I_0^q g(t,y_{n-1}(t)),\ \hat{0}\right)\\
&\leq D\left(f(t),\ \hat{0}\right) + D\left({}^{RL}I_0^q g(t,y_{n-1}(t)),\ \hat{0}\right)\\
&\leq D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(1-q)}\int_0^t D\left(\frac{g(s,y_{n-1}(s))}{(t-s)^q}\ ,\ \hat{0}\right)\,ds\\
&\leq D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\int_0^t \frac{ds}{(t-s)^q}.
\end{aligned}$$

But

$$\begin{aligned}
D(g(t,y_{n-1}(t)),\ \hat{0}) &\leq D(g(t,y_{n-1}(t)),\ g(t,\ \hat{0})) + D(g(t,\ \hat{0}),\ \hat{0})\\
&\leq L\, D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0}).
\end{aligned}$$

So

$$\begin{aligned}
D(y_n(t),\hat{0}) &\leq D\left(f(t),\ \hat{0}\right) + \frac{T^{1-q}}{\Gamma(2-q)}\sup_{0\leq t\leq T}[L\, D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0})]\\
&\leq D\left(f(t),\ \hat{0}\right) + \sup_{0\leq t\leq T} D(y_{n-1}(t),\ \hat{0}) + \frac{T^{1-q}}{\Gamma(2-q)}\sup_{0\leq t\leq T} D(g(t,\ \hat{0}),\ \hat{0}).
\end{aligned}$$

This proves that y_n is bounded. Therefore, $\{y_n\}$ is a sequence of bounded functions on $[0,T]$. Second we prove that y_n are continuous on $[0,T]$. For $0 \leq t \leq \tau \leq T$, we have

$$\begin{aligned}
D(y_n(t),y_n(\tau)) &\leq D(f(t),f(\tau)) + \frac{1}{\Gamma(1-q)}\,D\left(\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds,\int_0^\tau \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds\right)\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}D\left(\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds,\int_0^t \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds\right)\\
&\quad+\frac{1}{\Gamma(1-q)}D\left(\int_t^\tau \frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\,ds,\ \hat{0}\right)\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}\int_0^t D\left(\frac{g(s,y_{n-1}(s))}{(t-s)^q},\frac{g(s,y_{n-1}(s))}{(\tau-s)^q}\right)\,ds\\
&\quad+\frac{1}{\Gamma(1-q)}\int_t^\tau D\left(\frac{g(s,y_{n-1}(s))}{(\tau-s)^q},\ \hat{0}\right)\,ds\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
&\quad\int_0^t |(t-s)^{-q} - (\tau-s)^{-q}|\,ds\\
&\quad+\frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\int_t^\tau \frac{ds}{(\tau-s)^q}\,ds\\
&\leq D\left(f(t),f(\tau)\right) + \frac{1}{\Gamma(1-q)}[|t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|]\\
&\quad\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
&\quad+\frac{1}{\Gamma(2-q)}\ |t-\tau|^{\alpha}\ \sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})
\end{aligned}$$

$$
\begin{aligned}
&\leq D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)}[2\,|t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|] \\
&\qquad \sup_{0\leq t\leq T} D(g(t, y_{n-1}(t)),\, \hat{0}) \\
&\leq D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)}[2\,|t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|] \\
&\qquad \sup_{0\leq t\leq T} [L\, D(g(y_{n-1}(t)),\, \hat{0}) + D(g(t,\hat{0}),\, \hat{0})].
\end{aligned}
$$

The last inequality, by symmetry, is valid for all $t,\ \tau \in [0, T]$ regardless whether or not $t \leq \tau$. Thus, $D(y_n(t), y_n(\tau)) \to 0$ as $t \to \tau$. Therefore, the sequence $\{y_n\}$ is continuous on $[0, T]$. For $n \geq 1$, we have

$$
\begin{aligned}
D(y_{n+1}(t), y_n(t)) &= \frac{1}{\Gamma(1-q)}\, D\left(\int_0^t \frac{g(s, y_n(s))}{(t-s)^q}\, ds, \int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^q}\, ds\right) \\
&\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s, y_n(s))}{(t-s)^q}, \int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^q}\right)\, ds \\
&\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(g(s, y_n(s)), g(s, y_{n-1}(s))\right) \frac{ds}{(t-s)^q} \\
&\leq \frac{1}{\Gamma(1-q)} \sup_{0\leq t\leq T} D(g(t, y_n(t)), g(t, y_{n-1}(t))) \int_0^t \frac{ds}{(t-s)^q} \\
&\leq \frac{L\, T^{(1-q)}}{\Gamma(2-q)} \sup_{0\leq t\leq T} D(y_n(t), y_{n-1}(t)) \\
&\leq \left(\frac{L\, T^{(1-q)}}{\Gamma(2-q)}\right)^2 \sup_{0\leq t\leq T} D(y_{n-1}(t), y_{n-2}(t)) \\
&\ \ \vdots \\
&\leq \left(\frac{L\, T^{(1-q)}}{\Gamma(2-q)}\right)^n \sup_{0\leq t\leq T} D(y_1(t), y_0(t)).
\end{aligned} \tag{4.5}
$$

But

$$
\begin{aligned}
D(y_1(t), y_0(t)) &= \frac{1}{\Gamma((1-q))}\, D\left(\int_0^t \frac{g(s, f(s))}{(t-s)^q}\, ds, \hat{0}\right) \\
&\leq \frac{1}{\Gamma((1-q))} \int_0^t D\left(\frac{g(s, f(s))}{(t-s)^q}, \hat{0}\right)\, ds \\
&\leq \frac{1}{\Gamma((1-q))} \sup_{0\leq t\leq T} D(g(t, f(t)), \hat{0}) \int_0^t \frac{ds}{(t-s)^q}.
\end{aligned}
$$

Thus

$$
\sup_{0\leq t\leq T} D(y_1(t), y_0(t)) \leq \frac{T^{(1-q)}}{\Gamma(2-q)}[LM + N] := R,
$$

where

$$
M = \sup_{0\leq t\leq T} D(f(t), \hat{0}) \ \text{ and } \ N = \sup_{0\leq t\leq T} D(g(t, \hat{0}), \hat{0}).
$$

Therefore (4.5) takes the form

$$
D(y_{n+1}(t), y_n(t)) \leq R\, \left(\frac{L\, T^{(1-q)}}{\Gamma(2-q)}\right)^n. \tag{4.6}
$$

Next, we show that for each $t \in [0, T]$ the sequence $\{y_n(t)\}$ is a Cauchy sequence in E^n. Let $m_1,\ m_2$ be such that $m_2 > m_1$ and $t \in [0, T]$. Then, by using (4.6), we have

$$\begin{aligned}
D(y_{m_1}(t),y_{m_2}(t)) &\leq D(y_{m_2}(t),y_{m_2-1}(t))+D(y_{m_2-1}(t),y_{m_2-2}(t))\\
&\quad +\ldots+D(y_{m_1+1}(t),y_{m_1}(t))\\
&\leq R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}+R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-2}\\
&\quad +\ldots+R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_1}\\
&= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[1+\frac{\Gamma(2-q)}{L\,T^{1-q}}+\left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^2\right.\\
&\qquad\qquad \left.+\ldots+\left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1-1}\right]\\
&= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[\frac{1-\left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1}}{1-\frac{\Gamma(2-q)}{L\,T^{1-q}}}\right].
\end{aligned}$$

The right hand side of the last inequality tends to zero as $m_1,\ m_2\to\infty$. This implies that $\{y_n(t)\}$ is a Cauchy sequence. Consequently, the sequence $\{y_n(t)\}$ is convergent, thanks to the completeness of the metric space (E^n,D). If we denote $y(t)=\lim_{n\to\infty}y_n(t)$, then $y(t)$ satisfies (1.1). It is continuous and bounded on $[0,T]$. To prove the uniqueness, let $x(t)$ be a continuous solution of (1.1) on $[0,T]$. Then

$$x(t)=f(t)+\ {}^{RL}I^q g(t,x(t)),\ \ t\geq 0.$$

Now, for $n\geq 1$, we have

$$\begin{aligned}
D(x(t),y_n(t)) &= D\left({}^{RL}I^{1-q}g(t,x(t)),{}^{RL}I^{1-q}g(t,y_n(t))\right)\\
&\leq \frac{1}{\Gamma(1-q)}\int_0^t D\left(\frac{g(s,x(s))}{(t-s)^q},\int_0^t\frac{g(s,y_n(s))}{(t-s)^q}\right)ds\\
&\leq \frac{1}{\Gamma(1-q)}\int_0^t D\left(g(s,x(s)),g(s,y_n(s))\right)\frac{ds}{(t-s)^q}\\
&\leq \frac{1}{\Gamma(1-q)}\sup_{0\leq t\leq T}D(g(t,x(t)),g(t,y_n(t)))\int_0^t\frac{ds}{(t-s)^q}\\
&\leq \frac{L\,T^{1-q}}{\Gamma(2-q)}\sup_{0\leq t\leq T}D(x(t),y_n(t))\\
&\ \ \vdots\\
&\leq \left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^n\sup_{0\leq t\leq T}D(x(t),y_0(t)).
\end{aligned}$$

Since $\frac{L\,T^{1-q}}{\Gamma(2-q)}<1$

$$\lim_{n\to\infty}y_n(t)=x(t)=y(t),\ \ t\in[0,T].$$

This completes the proof.

(2) Case (2): If f is ${}^{RL}(2;q)$ differentiable, with the same argument as above, we can prove that the solution is (4.4) with

$$\lim_{n\to\infty}\hat{y}_n(t)=\hat{x}(t)=\hat{y}(t),\ \ t\in[0,T].$$

□

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[2] R.P. Agarwal, M. Benchohra, D. O'Regan and A. Ouahab, Fuzzy solutions for multi-point boundary value problems, *Mem. Differential Equations Math. Phys.*, 35(2005), 1-14.

[3] R. P. Agarwal, V. Lakshmikantham and J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.,* 72(2010), 2859-2862.

[4] R.P. Agarwal, D. O'Regan, and V. Lakshmikantham, Fuzzy Volterra integral equations: a stacking theorem approach, *Appl. Anal.*, 83(5)(2004), 521-532.

[5] A. Arara and M. Benchohra, Fuzzy solutions for boundary value problems with integral boundary conditions, *Acta Math. Univ. Comenianae,* LXXV(1)(2006), 119-126.

[6] S. Arshad and V. Lupulescu, *On the fractional differential equations with uncertainty*, *Nonlinear Anal.,* 74(2011), 3685-3693.

[7] M. Benchohra and M.A. Darwish, Existence and uniqueness theorem for fuzzy integral equation of fractional order, *Commun. Appl. Anal.,* 12(1)(2008), 13-22.

[8] M.A. Darwish, On quadratic integral equation of fractional orders, *J. Math. Anal. Appl.,* 311(2005), 112-119.

[9] M.A. Darwish and A.A. El-Bary, Existence of fractional integral equation with hysteresis, *Appl. Math. Comput.,* 176(2006), 684-687.

[10] D. Dubois and H. Prade, Towards fuzzy differential calculus, Part 1. Integraation of fuzzy mappings, *Fuzzy Sets Systems* 8(1982), 1-17.

[11] D. Dubois and H. Prade, Towards fuzzy differential calculus, Part 2. Integraation of fuzzy mappings, *Fuzzy Sets Systems* 8(1982), 105-116.

[12] M. Friedman, Ma Ming and A. Kandel, Solutions to fuzzy interal equations with arbitrary kernels, *Internat. J. Approx. Reason,* 20(3)(1999), 249-262.

[13] R. Goetschel and W. Voxman, Elementary Calculus, *Fuzzy Sets Systems,* 18(1986), 31-43.

[14] R. Hilfer, *Applications of Fractional calculus in Physics*, World Scientific, Singapore, 2000.

[15] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets Systems **24** (1987), 301-317.

[16] A. A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.

[17] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[18] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[19] V. Lakshmikantham and R. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, New York, 2003.

[20] J. Mordeson and W. Newman, Fuzzy integral equations, *Information Sciences,* 87(1995), 215-229.

[21] S. Nanda, On integration of fuzzy mappings, *Fuzzy Sets Systems* 32(1989), 95-101.

[22] J. Y. Park and J.U. Jeong, A note on fuzzy integral equations, *Fuzzy Sets Systems,* 108(1999), 193-200.

[23] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York and London, 1999.

[24] M. L. Puri and D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.,* 114(1986), 409-422.

[25] P. V. Subrahmanyam and S.K. Sudarsanam, A note on fuzzy Volterra integral equations, *Fuzzy Sets Systems,* 81(1996), 237-240.

[26] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

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