Exact solution of the(2+1)-dimensional hyperbolic nonlinear Schrödinger equation by Adomian decomposition method

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Abstract

This paper studies the exact solution of the the(2+1)-dimensional hyperbolic nonlinear Schrödinger equation by the aid of Adomian decomposition method.

Keywords: Exact solution, Hyperbolic Schrödinger equation, Adomian decomposition method.

1 Introduction

Nonlinear equations describe fundamental physical phenomena in nature ranging from chaotic behaviour in biological systems, plasma containment in tokamaks and stellarators for energy generation, to solitonic fibre optical communication devices. The construction of the exact solutions of nonlinear partial differential equations (PDEs) is one of the most important and essential tasks which help us for better understanding of nonlinear complex physical phenomena. In the past couple of decades, there are various mathematical techniques have been developed to carry out the integration of these equations. Some of these commonly studied techniques are Inverse Scattering Transform [5], bilinear transformation[4], the tanh-sech method[6, 7], adomian decomposition method [3], the tanh-coth method[8], homogeneous balance method[9], Exp-function method [10], and many others.

The Adomian decomposition method was introduced and developed by George Adomian in [11, 12] and is well addressed in the literature. A reliable modification of the Adomian decomposition method developed by Wazwaz and presented in [3]. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear equations for detail see [13, 14, 15, 16, 17, 18] and the references therein.

In this paper the Adomian decomposition method will determine exact solution to (2+1)-dimensional hyperbolic nonlinear Schrödinger equation. In Section 2, we described this method for finding exact solutions for nonlinear PDEs. In Section 3, we illustrated this method in detail with the hyperbolic Schrödinger equation. In Section 4, we gave some conclusions.

2 Adomian decomposition method for nonlinear PDEs

We first consider the nonlinear partial differential equation given in an operator form

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y)) + F(u(x,y)) = g(x,y), \qquad (2.1)$$

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where L_x is the highest order differential in x, L_y is the highest order differential in y, R contains the remaining linear terms of lower derivatives, $F(u(x, y))$ is an analytic nonlinear term, and $g(x, y)$ is an inhomogeneous or forcing term. the decision as to which operator L_x or L_y should be used to solve the problem depends mainly on two bases: (i) The operator of lowest order should be selected to minimize the size of computational work. (ii) The selected operator of lowest order should be of best known conditions to accelerate the evaluation of the components of the solution.For more detail see[3]. Assume that L_y meet these two conditions, therefore we set

$$L_y u(x, y) = g(x, y) - L_x u(x, y) - R(u(x, y)) - F(u(x, y)). \tag{2.2}$$

Applying L_y^{-1} to both sides of (2.2) gives

$$u(x, y) = \Phi_0 - L_y^{-1} g(x, y) - L_y^{-1} L_x u(x, y) - L_y^{-1} R(u(x, y)) - L_y^{-1} F(u(x, y)), \tag{2.3}$$

where

$$\Phi_0 = \begin{cases} u(x, 0) & L = \frac{\partial}{\partial y}, \\ u(x, 0) + y u_y(x, 0) & L = \frac{\partial^2}{\partial y^2}, \\ u(x, 0) + y u_y(x, 0) + \frac{1}{2!} y^2 u_{yy}(x, 0) & L = \frac{\partial^3}{\partial y^3}, \\ u(x, 0) + y u_y(x, 0) + \frac{1}{2!} y^2 u_{yy}(x, 0) + \frac{1}{3!} y^3 u_{yyy}(x, 0) & L = \frac{\partial^4}{\partial y^4}, \end{cases}$$

Take the solution $u(x, y)$ in a series form

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \tag{2.4}$$

and the nonlinear term $F(u(x, y))$ by

$$F(u(x, y)) = \sum_{n=0}^{\infty} A_n, \tag{2.5}$$

where A_n are Adomian polynomials that can be generated for all forms of nonlinearity and can be evaluated by using the following expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2 \tag{2.6}$$

Based on these assumptions, Eq. (2.3) become

$$\sum_{n=0}^{\infty} u_n(x, y) = \Phi_0 - L_y^{-1} g(x, y) - L_y^{-1} L_x \left(\sum_{n=0}^{\infty} u_n(x, y) \right)$$
$$- L_y^{-1} R \left(\sum_{n=0}^{\infty} u_n(x, y) \right) - L_y^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \tag{2.7}$$

The components $u_n(x, y)$, $n \geq 0$ of the solution $u(x, y)$ can be recursively determined by using the relation

$$u_0(x, y) = \Phi_0 - L_y^{-1} g(x, y),$$
$$u_{k+1}(x, y) = -L_y^{-1} L_x u_k - L_y^{-1} R(u_k) - L_y^{-1}(A_k), \quad k \geq 0. \tag{2.8}$$

Next find the components of $\sum_{n=0}^{\infty} u_n(x, y)$ by

$$u_0(x, y) = \Phi_0 - L_y^{-1} g(x, y),$$
$$u_1(x, y) = -L_y^{-1} L_x u_0(x, y) - L_y^{-1} R(u_0(x, y)) - L_y^{-1} A_0,$$
$$u_2(x, y) = -L_y^{-1} L_x u_1(x, y) - L_y^{-1} R(u_1(x, y)) - L_y^{-1} A_1,$$
$$u_3(x, y) = -L_y^{-1} L_x u_2(x, y) - L_y^{-1} R(u_2(x, y)) - L_y^{-1} A_2,$$
$$u_4(x, y) = -L_y^{-1} L_x u_3(x, y) - L_y^{-1} R(u_3(x, y)) - L_y^{-1} A_3,$$
$$.$$
$$.$$
$$.$$

where each component can be determined by using the preceding component. Having the calculated the components $u_n(x, y)$, $n \geq 0$, the solution in a series form is readily obtained.

3 Exact solutions for(2+1)-dimensional hyperbolic Schrödinger equation

In this section we obtain exact solution of (2+1)-dimensional hyperbolic nonlinear Schrödinger equation by using the decomposition method. The hyperbolic nonlinear Schrödinger equation given by[1] is

$$iu_t + \frac{1}{2}u_{xx} - \frac{1}{2}u_{yy} + |u|^2 u = 0 \tag{3.1}$$

where u is a complex valued function, while x, y and t are the independent variables. In order to seek exact solution, we assume that $u(x,y,0) = e^{i(mx+ny)}$ Multiplying Eq.(3.1) by i, we may express this equation in an operator form as follows

$$L_t u(x,y,t) = \frac{i}{2}L_{xx}u(x,y,t) - \frac{i}{2}L_{yy}u(x,y,t) + i|u(x,y,t)|^2 u(x,y,t) \tag{3.2}$$

where L_t is defined by $L_t = \frac{\partial}{\partial t}$ and the inverse operator L_t^{-1} is identified by

$$L_t^{-1}(\cdot) = \int\limits_0^t (\cdot)dt$$

Applying L_t^{-1} to both sides of (3.2) and using the initial condition we obtain

$$u(x,y,t) = e^{i(mx+ny)} + \frac{i}{2}L_t^{-1}(u(x,y,t))_{xx} - \frac{i}{2}L_t^{-1}(u(x,y,t))_{yy} + iL_t^{-1}|u(x,y,t)|^2 u(x,y,t), \tag{3.3}$$

where $|u(x,y,t)|^2 u(x,y,t)$ is nonlinear term.
Substituting

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) \tag{3.4}$$

and nonlinear term

$$|u(x,y,t)|^2 u(x,y,t) = \sum_{n=0}^{\infty} A_n \tag{3.5}$$

into (3.3) gives

$$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{i(mx+ny)} + \frac{i}{2}L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y,t)\right)_{xx} - \frac{i}{2}L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y,t)\right)_{yy} + iL_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{3.6}$$

Adomian's analysis introduces the recursive relation

$$\begin{aligned} u_0(x,y,t) &= e^{i(mx+ny)}, \\ u_{k+1}(x,y,t) &= \frac{i}{2}L_t^{-1}(u_k)_{xx} - \frac{i}{2}L_t^{-1}(u_k)_{yy} + iL_t^{-1}(A_k), k \geq 0. \end{aligned} \tag{3.7}$$

since u is a complex function so we can write

$$|u|^2 = u\bar{u} \tag{3.8}$$

where $\bar{u}$ is the conjugate of u. this means that (3.5) can be written as

$$u^2\bar{u} = \sum_{n=0}^{\infty} A_n \tag{3.9}$$

By using formal technique to find adomian polynomial used in [3] we find that (3.9)has the following polynomial representation

$$\begin{aligned} A_0 &= u_0^2\bar{u}_0, \\ A_1 &= 2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1, \\ A_2 &= 2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_1 + u_0^2\bar{u}_2, \\ A_3 &= 2u_0u_3\bar{u}_0 + 2u_1u_2\bar{u}_0 + 2u_0u_2\bar{u}_1 + u_1^2\bar{u}_1 + 2u_0u_1\bar{u}_2 + u_0^2\bar{u}_3 \end{aligned} \tag{3.10}$$

that in turn gives the first few components by

$$
\begin{aligned}
u_0(x,y,t) &= e^{i(mx+ny)}, \\
u_1(x,y,t) &= \tfrac{i}{2}L_t^{-1}(u_{0_{xx}}) - \tfrac{i}{2}L_t^{-1}\left(u_{0_{yy}}\right) + iL_t^{-1}(A_0), \\
u_2(x,y,t) &= \tfrac{i}{2}L_t^{-1}(u_{1_{xx}}) - \tfrac{i}{2}L_t^{-1}\left(u_{1_{yy}}\right) + iL_t^{-1}(A_1), \\
u_3(x,y,t) &= \tfrac{i}{2}L_t^{-1}(u_{2_{xx}}) - \tfrac{i}{2}L_t^{-1}\left(u_{2_{yy}}\right) + iL_t^{-1}(A_2),
\end{aligned}
\tag{3.11}
$$

we obtain

$$
\begin{aligned}
u_0(x,y,t) &= e^{i(mx+ny)}, \; A_0 = u_0^2\bar{u}_0 = e^{i(mx+ny)}, \\
u_1(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(-m^2 e^{i(mx+ny)}\right) - \tfrac{i}{2}L_t^{-1}\left(-n^2 e^{i(mx+ny)}\right) + iL_t^{-1}(e^{i(mx+ny)}) = it(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1)e^{i(mx+ny)}, \\
u_2(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(u_{1_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{1_{yy}}\right) + iL_t^{-1}(A_1) = \tfrac{(it)^2}{2!}\left(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1\right)^2 e^{i(mx+ny)} \\
u_3(x,y,t) &= \tfrac{i}{2}L_t^{-1}\left(u_{2_{xx}}\right) - \tfrac{i}{2}L_t^{-1}\left(u_{2_{yy}}\right) + iL_t^{-1}(A_2) = \tfrac{(it)^3}{3!}\left(\tfrac{n^2}{2} - \tfrac{m^2}{2} + 1\right)^3 e^{i(mx+ny)}
\end{aligned}
\tag{3.12}
$$

Accordingly, the series solution is given by

$$
u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_1 + u_2 + u_3 + \dots
$$

$$
u(x,y,t) = e^{i(mx+ny)}\left[1 + \frac{it}{1!}\left(\frac{n^2}{2} - \frac{m^2}{2} + 1\right) + \frac{(it)^2}{2!}\left(\frac{n^2}{2} - \frac{m^2}{2} + 1\right)^2 + \dots\right]
\tag{3.13}
$$

that gives exact solution of (3.1) in closed form

$$
u(x,y,t) = e^{i\left(mx+ny+\left(\frac{n^2}{2} - \frac{m^2}{2} + 1\right)t\right)}
\tag{3.14}
$$

.

4 Conclusion

The Adomian decomposition method is successfully used to establish new exact solution. The performance of this method is found to be reliable and effective and can give more solutions, which may be important for the explanation of some nonlinear complex physical phenomena.

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