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Existence results for an impulsive neutral integro-differential equation with infinite delay via fractional operators

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Abstract

In this present work, we consider an impulsive neutral integro-differential equation with infinite delay in an arbitrary Banach space X. The existence of mild solution is established by using resolvent operator and Hausdorff measure of noncompactness.

Keywords: Resolvent operator, Impulsive differential equation, Neutral integro-differential equation, Measure of noncompactness.

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1 Introduction

In recent years, impulsive differential equations have become an active area of research due to their demonstrated applications in widespread fields of science and engineering such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and many others. Neutral differential equations arise in many areas of applied mathematics. The system of rigid heat conduction with finite wave spaces can be modeled in the form of the integro-differential equation of neutral type with delay. In addition, the development of the theory of the functional differential equation with infinite delay depends on a suitable choice of phase space. There are various phases spaces which have been studied in a book by Hale and Kato [9] and they introduced a common phase space $\mathfrak{B}$. For more detail on phase space, we refer to book by Hale and Kato [9] and Y. Hino et al. [20].

On the other hand, many real world processes and phenomena which are subjected during their development to short-term external influences can be modeled as impulsive differential equation with fractional order. Their duration is negligible compared to the total duration of the entire process or phenomena. Such process is investigated in various fields such as biology, physics, control theory, population dynamics, medicine and so on. For the general theory of such differential equations, we refer to the monographs [12], [18], and papers [5], [6], [14], [17], [19], [21]-[22], and references given therein.

The purpose of this paper is to study the following integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\begin{aligned} \frac{d}{dt}[u(t) - F(t,u_t)] &= A[u(t) + \int_0^t f(t-s)u(s)ds] + G(t,u_t,\int_0^t E(t,s,u_s)ds), \\ & \quad t \in J = [0,T_0],\ t \neq t_k,\ k = 1,2,\cdots,m, && (1.1) \\ u_0 &= \phi \in \mathfrak{B}, && (1.2) \\ \Delta u(t_i) &= I_i(u_{t_i}),\ i = 1,2,\cdots,m, && (1.3) \end{aligned}$$

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where $0 < T_0 < \infty$, A is a closed linear operator defined on a Banach space $(X; \| \cdot \|)$ with dense domain $D(A) \subset X$; $f(t), t \in [0, T_0]$ is a bounded linear operator. The functions $F : [0, T_0] \times \mathfrak{B} \to X$, $G : [0, T_0] \times \mathfrak{B} \times X \to X$, $E : [0, T_0] \times [0, T_0] \times \mathfrak{B} \to X$, $I_i : X \to X,\ i = 1, \cdots, m$ are appropriate functions to be specified later, where $\mathfrak{B}$ is the phase space defined axiomatically later in section 2 and $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T_0$ are pre-fixed numbers. The symbol $\Delta u(t) = u(t^+) - u(t^-)$ denotes the jump of the function u at t i.e., $u(t^-)$ and $u(t^+)$ denotes the end limits of the $u(t)$ at t. The history $u_t : (-\infty, 0] \to X$ is a continuous function defined as $u_t(s) = u(t+s), s \leq 0$ belongs to the abstract phase space $\mathfrak{B}$.
Hernandez et al, [4] has discussed the existence of solution for the neutral integro-differential problem

$$\frac{d}{dt}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \ \ t \in [0, T_0], \tag{1.4}$$
$$u_0 = \phi, \ \ \phi \in \mathfrak{B}, \tag{1.5}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup and $f,\ g : [0, T_0] \times \mathfrak{B} \to X$ are appropriate functions. The existence of the mild solution for impulsive neutral integro-differential inclusions with nonlocal conditions

$$\begin{aligned} \frac{d}{dt}[u(t) - F(t, u(h_1(t)))] &= A[u(t) + \int_0^t f(t-s)u(s)ds] \\ &\quad + G(t, u(h_2(t))), \ \ t \in [0, T_0], \ t \neq t_k, \end{aligned} \tag{1.6}$$
$$\Delta u(t_k) = I_k(u(t_k^-)), \ k = 1, \cdots, m, \tag{1.7}$$
$$u(0) + g(u) = u_0, \tag{1.8}$$

has been established by Chang and Nieto in [22]. Where A is the infinitesimal generator of a compact, analytic resolvent operator $R(t), t > 0$ on a Banach space X and F, G, g, I_k are appropriated functions.

In this work, our work is spurred by the works [4]-[7], [14], [17], [21]-[22] to establish some existence results for the system (1.1)-(1.3) by using measure of noncompactness and resolvent operator. The tool of measure of noncompactness has been used in linear operator theory, theory of differential and integral equations, the fixed point theory and many others. For an initial study of the theory of the measure of noncompactness, we refer to book of Józef Banas [10], Akhmerov et. al.[16] and references given therein.

The organization of the article is as follows: In section 2, we provide some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. In section 3, we prove the existence of mild solution to (1.1)-(1.3). An example is also considered at the end of the article.

2 Preliminaries

In this segment, we provide some fundamental definition, Lemmas and Theorems which will be utilized all around this paper.
Let X be a Banach space. The symbol $C([a, b]; X)$, $(a, b \in \mathbb{R})$ stands for the Banach space of all the continuous functions from $[a, b]$ into X equipped with the norm $\| z(t)\|_C = \sup_{t\in[a,b]} \| z(t)\|_X$ and $L^p((a, b); X)$ stands for Banach space of all Bochner-measurable functions from (a, b) to X with the norm

$$\| z\|_{L^p} = (\int_{(a,b)} \| z(s)\|_X^p ds)^{1/p}.$$

Let $0 \in \rho(A)$ i.e. A is invertible. Then it can be conceivable to characterize the fractional power A^α for $0 < \alpha \leq 1$ as a closed linear operator with domain $D(A^\alpha) \subset X$. It is easy to see that $D(A^\alpha)$ which is dense in X is a Banach space endowed with the norm $\| z\| = \| A^\alpha z\|$, for $z \in D(A^\alpha)$. Henceforth, we use X_α as notation of $D(A^\alpha)$. Also, we have that $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and therefore, the embedding is continuous. Then, we define $X_{-\alpha} = (X_\alpha)^*$, for each $\alpha > 0$. The space $X_{-\alpha}$ stands for the dual space of X_α, is a Banach space with the norm $\| z\|_{-\alpha} = \| A^{-\alpha} z\|$. For additional parts on the fractional powers of closed linear operators, we allude to book by Pazy [1].

For the differential equation with infinite delay, Kato and Hale [9] was proposed the phase space $\mathfrak{B}$ satisfying certain fundamental axioms.

Definition 2.1. *The linear space of all functions from $(-\infty, 0]$ into Banach space X with a seminorm $\| \cdot \|_{\mathfrak{B}}$ is known as phase space $\mathfrak{B}$. The fundamental axioms assumed on $\mathfrak{B}$ are the followings:*

(A) *If $u : (-\infty, d+T_0] \to X$, $T_0 > 0$ is a continuous function on $[d, d+T_0]$ such that $u_d \in \mathfrak{B}$ and $u|_{[d,d+T_0]} \in \mathfrak{B} \in \mathcal{PC}([d, d+T_0]; X)$, then for every $t \in [d, d+T_0)$, the following conditions are hold:*

(i) $u_t \in \mathfrak{B}$,

(ii) $H\| u_t\|_{\mathfrak{B}} \geq \| u(t)\|$,

(iii)$\| u_t\|_{\mathfrak{B}} \leq N(t+d)\| u_d\|_{\mathfrak{B}} + K(t-d)\sup\{\| u(s)\| : d \leq s \leq t\}$,

where H is a positive constant; N, $K : [0,\infty) \to [1,\infty)$, N is a locally bounded, K is continuous and K, H, N are independent of $u(\cdot)$.

(A1) *For the function u in $(A1)$, u_t is a $\mathfrak{B}$-valued continuous function for $t \in [d, d+T_0]$.*

(B) *The space $\mathfrak{B}$ is complete.*

To set the structure for our primary existence results, we have to introduce the following definitions.

Definition 2.2. *A family $\{R(t)\}_{t\in J}$ of bounded linear operators is said to be a resolvent operator (Fractional operators) for following equation*

$$x'(t) = A[x(t) + \int_0^t f(t-s)x(s)ds], \tag{2.9}$$

if the following conditions are satisfied

(i) $R(0) = I$, *where I is the identity operator on X.*

(ii) $R(t)$ *is strongly continuous for $t \in [0, T_0]$.*

(iii) $R(t) \in B(Z)$, $t \in [0, T_0]$. *For $z \in Z$ and $R(\cdot)z \in C([0,T_0]; Z) \cap C^1([0,T_0]; Z)$, we have*

$$\frac{d}{dt}R(t)z = A[R(t)z + \int_0^t f(t-s)R(s)zds], \tag{2.10}$$

$$= R(t)Az + \int_0^t R(t-s)Af(s)zds, \quad t \in [0, T_0]. \tag{2.11}$$

Where $B(Z)$ denotes the space of bounded linear operators defined on Z and Z is a Banach space formed from $D(A)$ with the graph norm.

We assume that A generates a resolvent operator $\{R(t)\}_{t\geq 0}$ on a Banach space X and there exists a positive constant M_1 such that $\| R(t)\| \leq M_1$. For any $0 \leq \alpha \leq 1$, there exists a positive constant M_α such that

$$\| A^\alpha R(t)\| \leq \frac{M_\alpha}{t^\alpha}, \quad t \in [0, T_0]. \tag{2.12}$$

To consider the mild solution for the impulsive problem, we propose the set $\mathcal{PC}([0,T_0]; X) = \{u : [0,T_0] \to X : u$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and $u(t_i^+)$ exists, for all $i = 1, \cdots, m\}$. Clearly, $\mathcal{PC}([0,T_0]; X)$ is a Banach space endowed the norm $\| u\|_{\mathcal{PC}} = \sup_{t\in[0,T_0]} \| u(s)\|$. For a function $u \in \mathcal{PC}([0,T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we define the function $\widetilde{u_i} \in C([t_i, t_{i+1}], X)$ such that

$$\widetilde{u_i}(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases} \tag{2.13}$$

For $W \subset \mathcal{PC}([0,T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we have $\widetilde{W_i} = \{\widetilde{u_i} : u \in W\}$ and following Accoli-Arzelà type criteria. Now, we discuss some basic definition of measure of noncompactness (MNC).

Lemma 2.1. *[3]. A set $W \subset \mathcal{PC}([0,T_0]; X)$ is relatively compact if and only if each set $\widetilde{W_i} \subset C([t_i, t_{i+1}], X)$ $(i = 0, 1 \cdots, m)$ is relatively compact.*

Definition 2.3. *The Hausdorff's measure of noncompactness (H'MNC) χ_Y is defined as*

$$\chi_Y(U) = \inf\{\varepsilon > 0 : \text{ U can be covered by finite number of balls with radius } \varepsilon\}, \tag{2.14}$$

for the bounded set $U \subset Y$, where Y is a Banach space.

Lemma 2.2. *For any bounded set $U,\ V \subset Y$, where Y is a Banach space. Then, the following properties are fulfilled:*

(i) $\chi_Y(U) = 0$ *if and only if U is pre-compact;*

(ii) $\chi_Y(U) = \chi_Y(conv\ U) = \chi_Y(\overline{U})$, *where conv U and $\overline{U}$ denotes the convex hull and closure of U respectively;*

(iii) $\chi_Y(U) \subset \chi_Y(V)$, *when* $U \subset V$;

(iv) $\chi_Y(U+V) \le \chi_Y(U) + \chi_Y(V)$, *where* $U + V = \{u + v : u \in U,\ v \in V\}$;

(v) $\chi_Y(U \cup V) \le \max\{\chi_Y(U), \chi_Y(V)\}$;

(vi) $\chi_Y(\lambda U) = \lambda \cdot \chi_Y(U)$, *for any* $\lambda \in \mathbb{R}$;

(vii) If the map $P : D(P) \subset Y \to \mathcal{Z}$ is continuous and satisfy the Lipschitsz condition with constant κ. Then, we have that $\chi_{\mathcal{Z}}(PU) \le \kappa\chi_Y(U)$ for any bounded subset $U \subset D(P)$, where Y and $\mathcal{Z}$ are Banach space;

Definition 2.4. *[10] A bounded and continuous map $P : D \subset Z \to Z$ is a χ_Z-contraction if there exists a constant $0 < \kappa < 1$ such that $\chi_Z(P(U)) \le \kappa\chi_Z(U)$, for any bounded closed subset $U \subset D$, where Z is a Banach space.*

Lemma 2.3. *[15] Let $D \subset Z$ be a closed, convex with $0 \in D$ and the continuous map $P : D \to D$ be a χ_Z-contraction. If the set $\{u \in D : u = \lambda Pu,\ for\ \ 0 < \lambda < 1\}$ is bounded, then the map P has a fixed point in D.*

Lemma 2.4. *(Darbo-Sadovskii)[10]. Let $D \subset Z$ be bounded, closed and convex. If the continuous map $P : D \to D$ is a χ_Z-contraction, then the map P has a fixed point in D.*

In this paper, we consider that χ denotes the Hausdorff's measure of noncompactness (H'MNC)in X, χ_C denotes the Hausdorff's measure of noncompactness in $C([0, T_0]; X)$ and $\chi_{\mathcal{PC}}$ denotes the Hausdorff's measure of noncompactness in $\mathcal{PC}([0, T_0]; X)$.

Lemma 2.5. *([10]. If U is bounded subset of $C([0, T_0]; X)$. Then, we have that $\chi(U(t)) \le \chi_C(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t); u \in U\} \subseteq X$. Furthermore, if U is equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is continuous on the interval $[0, T_0]$ and*

$$\chi_C(U) = \sup_{t\in[0,T_0]} \{\chi(U(t))\}. \tag{2.15}$$

Lemma 2.6. *[10] If $U \subset C([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is continuous and*

$$\chi(\int_0^t U(s)ds) \le \int_0^t \chi(U(s))ds,\ \forall\ \ t \in [0, T_0], \tag{2.16}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds, u \in U\}$.

Lemma 2.7. *[14]*

(1) If $U \subset \mathcal{PC}([0, T_0]; X)$ is bounded, then $\chi(U(t)) \le \chi_{\mathcal{PC}}(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t) : u \in U\} \subset X$;

(2) If U is piecewise equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is piecewise continuous for $t \in [0, T_0]$ and

$$\chi_{\mathcal{PC}}(U) = \sup\{\chi(U(t)) : t \in [0, T_0]\}; \tag{2.17}$$

(3) If $U \subset \mathcal{PC}([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is piecewise continuous for $t \in [0, T_0]$ and

$$\chi(\int_0^t U(s)ds) \le \int_0^t \chi(U(s))ds, \forall\ \ t \in [0, T_0], \tag{2.18}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds : u \in U\}$.

3 Main Results

In this segment, the existence of the mild solution for the equation (1.1)-(1.3) is studied. Now we introduce following conditions:

(HR) Since $R(t)$ is a resolvent operator and f is bounded operator. Without loss of generality we assume that there exist positive constants $N_1,\ N_2$ such that $\| R(t)\| \leq N_1,\ \| f(t)\| \leq N_2,\ t \in [0, T_0]$. We assume that $R(t), t \geq 0$ satisfies the following property;

(R_1) The map $t \mapsto R(t)$ is continuous from $(0, T_0]$ to $\mathcal{L}(X)$ with the uniform operator norm $\| \cdot \|_{\mathcal{L}(X)}$.

(HF) The function $F : [0, T_0] \times \mathfrak{B} \to X$ is Lipschitz continuous and there exist constants $L_F > 0$ and $0 < \beta \leq 1$ such that

$$\| A^{\beta}F(t, x_1) - A^{\beta}F(s, x_2)\| \leq L_F[\| x_1 - x_2\|_{\mathfrak{B}}], \tag{3.19}$$

and

$$\| A^{\beta}F(t, x)\| \leq C_1\| x\|_{\mathfrak{B}} + C_2, \tag{3.20}$$

for all $x, x_1, x_2 \in \mathfrak{B}$ and $t \in [0, T_0]$, where C_1, C_2 are positive constants.

(HG) $G : [0, T_0] \times \mathfrak{B} \times X \to X$ is a nonlinear function such that
(1) For each $u : (-\infty, T_0] \to X$, $u_0 = \phi \in \mathfrak{B}$, $G(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, T_0]$ and function $t \mapsto G(t, u_t, \int_0^t E(t, s, u_s)ds)$ is strongly measurable for $u \in \mathcal{PC}([0, T_0]; X)$.
(2) There is an integrable function $\alpha : J \to [0, \infty)$ and a monotone increasing continuous function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| G(\tau, x, y)\| \leq \alpha(\tau)\Omega(\| x\|_{\mathfrak{B}} + \| y\|),\ \tau \in [0, T_0],\ (x, y) \in \mathfrak{B} \times X. \tag{3.21}$$

(3) There is an integrable function $\eta : J \to [0, \infty)$ such that for any bounded subset $E_1 \subset \mathcal{PC}((-\infty, 0]; X)$, $E_2 \subset X$, we have that

$$\chi(R(\tau)G(\tau, E_1, E_2)) \leq \xi(\tau)\{\sup_{-\infty \leq \theta \leq 0} \chi(E_1(\theta)) + \chi(E_2)\}, \tag{3.22}$$

for a.e. $t \in [0, T_0]$. Where $E_1(\theta) = \{u(\theta) : u \in E_1\}$.

(HE) (1) There is a constant $E_1 > 0$ such that

$$\| \int_0^{\tau} [E(\tau, s, u) - E(\tau, s, v)]ds\| \leq E_1\| u - v\|_{\mathfrak{B}},\ \tau, s \in [0, T_0], u, v \in \mathfrak{B}. \tag{3.23}$$

(2)The map $E(t, s, \cdot) : \mathfrak{B} \to X$ is continuous for each $(t, s) \in [0, T_0] \times [0, T_0]$ and $E(\cdot, \cdot, u) : [0, T_0] \times [0, T_0] \to X$ is a strongly measurable function for each $u \in \mathfrak{B}$. There exist a constant $\zeta > 0$ and integrable function $m_E : J \to [0, \infty)$ such that

$$\| E(\tau, s, x)\| \leq \zeta m_E(s)\varphi(\| x\|),\ \tau, s \in [0, T_0], \tag{3.24}$$

where $\varphi \in C([0, \infty); [0, \infty))$ is a increasing function and $\int_0^{\infty} \zeta m_E(s)ds \leq L_0$.

(HI) (1) The functions $I_i : \mathfrak{B} \to X,\ i = 1, 2, \cdots, m$ are continuous and there are constant $L_i > 0$ $(i = 1, 2, \cdots, m)$ such that

$$\| I_i(x) - I_i(y)\| \leq L_i\| x - y\|_{\mathfrak{B}},\ \forall\ x, y \in \mathfrak{B}. \tag{3.25}$$

(2) There exist positive constant K_i^1 and K_i^2,$(i = 1, \cdots, m)$ such that

$$\| I_i(x)\| = K_i^1\| x\|_{\mathfrak{B}} + K_i^2,\ x \in \mathfrak{B}. \tag{3.26}$$

(H')

$$\begin{aligned}
\mu_1 &= [(K_{T_0}N_1H + M_{T_0}) + K_{T_0}N_1\| A^{-\beta}\|C_1]\| \phi\|_{\mathfrak{B}} + K_{T_0}[\| A^{-\beta}\|C_2 \\
&\quad + \frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_2 + N_2\frac{M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_2 + N_1\sum_{0<t_i<t}K_i^1], \qquad (3.27)\\
\mu_2 &= [\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1 + N_2\frac{M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1 \\
&\quad + N_1\sum_{0<t_i<t}K_i^1] < 1 \qquad (3.28)
\end{aligned}$$

and

$$\int_0^{T_0}\widehat{m_E}(s)ds \le \int_b^{\infty}\frac{ds}{\Omega(s)+\varphi(s)} \quad , \qquad (3.29)$$

where $b = \frac{\mu_1}{1-\mu_2}$.

Definition 3.5. *A piecewise continuous function $u : (-\infty, T_0] \to X$ is said to be a solution for the system (1.1)-(1.3) if $u_0 = \phi$, $u(\cdot)|_{[0,T_0]} \in \mathcal{PC}$ and following impulsive integral equation*

$$\begin{aligned}
u(t) &= R(t)[\phi(0) - F(0,\phi)] + F(t,u_t) + \int_0^t AR(t-s)F(s,u_s)ds \\
&\quad + \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau,u_\tau)d\tau ds \\
&\quad + \int_0^t R(t-s)G(s,u_s,\int_0^s E(s,\tau,u_\tau)d\tau)ds \\
&\quad + \sum_{0<t_i<t}R(t-t_i)I_i(u_{t_i}), \qquad t\in[0,T_0], \qquad (3.30)
\end{aligned}$$

is verified.

Let $z : (-\infty, T_0] \to X$ be a function defined by $z_0 = \phi$ and $z(t) = R(t)\phi(0)$ on $[0, T_0]$. It is clear that $\| z_t\| \le (K_{T_0}N_1H + M_{T_0})\| \phi\|_{\mathfrak{B}}$, where $K_{T_0} = \sup_{t\in[0,T_0]} K(t)$, $M_{T_0} = \sup_{t\in[0,T_0]} M(t)$.

Theorem 3.1. *Suppose $(HR), (HF), (HG), (HE), (HI), (H')$ holds and*

$$K_{T_0}[L_F + \frac{M_{1-\beta}T_0{}^{\beta}}{\beta}L_F + \frac{N_2L_FM_{1-\beta}T_0{}^{\beta+1}}{\beta} + N_1\sum_{i=1}^{m}L_i] + (1+L_0\Omega_1)\int_0^t\xi(s)ds \le 1. \qquad (3.31)$$

Then, the impulsive system (1.1)-(1.3) has a mild solution.

Proof. Let $S(T_0) = \{u : (-\infty, T_0] \to X,\ u_0 = 0,\ u|_{[0,T_0]} \in \mathcal{PC}\}$ endowed with the supremum norm $\| \cdot \|$ be the space. Define operator $P : S(T_0) \to S(T_0)$ as

$$Pu(t) = \begin{cases} 0, & t\in(-\infty,0], \\ -R(t)F(0,\phi) + F(t,u_t+z_t) + \int_0^t AR(t-s)F(s,u_s+z_s)ds \\ + \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau,u_\tau+z_\tau)d\tau ds \\ + \int_0^t R(t-s)G(s,u_s+z_s,\int_0^s E(s,\tau,u_\tau+z_\tau)d\tau)ds \\ + \sum_{0<t_i<t}R(t-t_i)I_i(u_{t_i}+z_{t_i}), & t\in[0,T_0]. \end{cases} \qquad (3.32)$$

Also we have $\| u_t + z_t\|_{\mathfrak{B}} \le (K_{T_0}N_1H + M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u\|_t$, where $\| u\|_t = \sup_{s\in[0,t]}\| u(s)\|$. From the axioms A, our assumptions and the strongly continuity of $R(t)$, we can see that $Pu \in \mathcal{PC}$. For $u \in S(T_0)$, we get

$$\begin{aligned}
\| AR(t-s)F(s,u_s+z_s)\| &= \| A^{1-\beta}R(t-s)A^{\beta}F(s,u_s+z_s)\|, \\
&\le \frac{M_{1-\beta}}{(t-s)^{1-\beta}}[C_1\| u_s+z_s\|_{\mathfrak{B}} + C_2], \qquad (3.33)
\end{aligned}$$

thus, from the Bocher theorem it takes after that $AR(t-s)F(s,u_s+z_s)$ is integrable. So, we obtain that P is well defined on $S(T_0)$. We give the demonstration of Theorem 3.1 in the numerous steps.

Step 1. The set $\{x \in \mathcal{PC}([0,T_0];X) : u(t) = \lambda Pu(t),\ \text{for } 0<\lambda<1\}$ is bounded.
For $1>\lambda>0$, let u_λ be a solution for $u=\lambda Pu$. We have that

$$\| u_{\lambda t}+z_t\| \leq (K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u_\lambda\|_t. \tag{3.34}$$

Let $\nu_\lambda(t) = (K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}} + K_{T_0}\| u_\lambda\|_t$, for each $t\in[0,T_0]$. Then, we have

$$\begin{aligned}
\| u_\lambda(t)\| &= \| \lambda Pu_\lambda(t)\| \leq \| Pu_\lambda(t)\|,\\
&\leq \| R(t)F(0,\phi)\| + \| F(t,u_{\lambda t}+z_t)\|\\
&\quad+\int_0^t \| A^{1-\beta}R(t-s)\|\| A^\beta F(t,u_{\lambda s}+z_s)\|ds\\
&\quad+\int_0^t \| A^{1-\beta}R(t-s)\| \int_0^s f(s-\tau)\|A^\beta F(\tau,u_\tau+z_\tau)\|d\tau ds\\
&\quad+\int_0^t \| R(t-s)G(s,u_s+z_s,\int_0^s E(s,\tau,u_\tau+z_\tau)d\tau)\|ds\\
&\quad+\sum_{0<t_i<t}\| R(t-t_i)I_i(u_{t_i}+z_{t_i})\|,\\
&\leq N_1\| A^{-\beta}\|[C_1\| \phi\|_{\mathfrak{B}}+C_2] + \| A^{-\beta}\|[C_1\nu_\lambda(t)+C_2]\\
&\quad+\frac{M_{1-\beta}{T_0}^\beta}{\beta}(C_1\nu_\lambda(s)+C_2) + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}(C_1\nu_\lambda(s)+C_2)\\
&\quad+N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds\\
&\quad+N_1\sum_{0<t_i<t}(K_i^1\nu_\lambda(t)+K_i^2),\\
&\leq N_1\| A^{-\beta}\|[C_1\| \phi\|_{\mathfrak{B}}+C_2] + \| A^{-\beta}\|C_2 + \frac{M_{1-\beta}b^\beta}{\beta}C_2 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_2\\
&\quad+N_1\sum_{0<t_i<t}K_i^2 + [\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}{T_0}^\beta}{\beta}C_1 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_1\\
&\quad+N_1\sum_{0<t_i<t}K_i^1]\nu_\lambda(t) + N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,
\end{aligned}$$

which gives that

$$\begin{aligned}
\nu_\lambda(t) &\leq [(K_{T_0}N_1H+M_{T_0}) + K_{T_0}N_1\| A^{-\beta}\|C_1]\| \phi\|_{\mathfrak{B}} + K_{T_0}[\| A^{-\beta}\|C_2\\
&\quad+\frac{M_{1-\beta}{T_0}^\beta}{\beta}C_2 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_2 + N_1\sum_{0<t_i<t}K_i^1] + [\| A^{-\beta}\|C_1\\
&\quad+\frac{M_{1-\beta}{T_0}^\beta}{\beta}C_1 + N_2\frac{M_{1-\beta}{T_0}^{\beta+1}}{\beta}C_1 + N_1\sum_{0<t_i<t}K_i^1]\nu_\lambda(t)\\
&\quad+N_1\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,\\
\nu_\lambda(t) &\leq \frac{\mu_1}{1-\mu_2} + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(v_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds,
\end{aligned}$$

Take $b=\frac{\mu_1}{1-\mu_2}$, therefore we get

$$\nu_\lambda(t) \leq b + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\ m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds, \tag{3.35}$$

Let $\beta_\lambda(t) = b + \frac{N_1K_{T_0}}{1-\mu_2}\int_0^t \alpha(s)\Omega(\nu_\lambda(s)+\int_0^s \zeta\cdot m_E(\tau)\varphi(\nu_\lambda(\tau))d\tau)ds$, then we have $\beta_\lambda(0)=b$ and

$$\nu_\lambda(t) \leq \beta_\lambda(t), \quad 0\leq t\leq T_0. \tag{3.36}$$

Also, we get

$$\beta_\lambda'(t) \leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(\nu_\lambda(t) + \int_0^t \zeta\ m_E(s)\varphi(\nu_\lambda(s))ds). \tag{3.37}$$

Since we have that Ω is nondecreasing. Therefore we get

$$\beta_\lambda'(t) \leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(\beta_\lambda(t) + \int_0^t \zeta\ m_E(s)\varphi(\beta_\lambda(s))ds). \tag{3.38}$$

Now we take $B_\lambda(t) = \beta_\lambda(t) + \int_0^t \zeta\ m_E(s)\varphi(\beta_\lambda(s))ds$ and we have $B_\lambda(0) = \beta_\lambda(0)$ and $B_\lambda(t) \leq \beta_\lambda(t)$.

$$\begin{aligned} B_\lambda'(t) &= \beta_\lambda'(t) + \zeta\ m_E(t)\varphi(\beta_\lambda(t)), \\ &\leq \frac{N_1 K_{T_0}}{1-\mu_2}\alpha(t)\Omega(B_\lambda(t)) + \zeta\ m_E(t)\varphi(B_\lambda(t)), \\ &\leq \widehat{m_E}(t)(\Omega(B_\lambda(t)) + \varphi(B_\lambda(t))), \end{aligned} \tag{3.39}$$

which gives that

$$\int_{B_\lambda(0)}^{B_\lambda(t)} \frac{1}{\Omega(s)+\varphi(s)}ds \leq \int_0^{T_0} \widehat{m_E}(s)ds \leq \int_b^\infty \frac{1}{\Omega(s)+\varphi(s)}ds. \tag{3.40}$$

It implies that functions $\beta_\lambda(t)$ are bounded on $[0, T_0]$. Therefore, the function $\nu_\lambda(t)$ are bounded on $[0, T_0]$ and $u_\lambda(\cdot)$ are bounded on $[0, T_0]$.

Step 2. P is χ-contraction.

We introduce the decomposition of $P = P_1 + P_2$ such that

$$\begin{aligned} P_1 u(t) &= R(t)[-F(0,\phi)] + F(t, u_t + z_t) + \int_0^t AR(t-s)F(s, u_s + z_s)ds \\ &\quad + \int_0^t AR(t-s)\int_0^s f(s-\tau)F(\tau, u_\tau + z_\tau)d\tau ds \\ &\quad + \sum_{0<t_i<t} R(t-t_i)I_i(u_{t_i} + z_{t_i}), \end{aligned} \tag{3.41}$$

$$P_2 u(t) = \int_0^t R(t-s)G(s, u_s + z_s, \int_0^s E(s,\tau,u_\tau + z_\tau)d\tau)ds. \tag{3.42}$$

To prove the result, firstly we show that P_1 satisfies the Lipschitz condition. For $u_1,\ u_2 \in S(T_0)$, we have $\| P_1u_1(t) - P_1u_2(t)\|$

$$\begin{aligned} &\leq \| A^\beta F(t, u_{1t} + z_t) - A^\beta F(t, u_{2t} + z_t)\| \\ &\quad + \int_0^t \|A^{1-\beta}R(t-s)\|\|A^\beta F(s, u_{1s} + z_s) - F(s, u_{2s} + z_s)\|ds \\ &\quad + \int_0^t \|A^{1-\beta}R(t-s)\| \int_0^s \|f(s-\tau)\|\|A^\beta F(\tau, u_{1\tau} + z_\tau) - F(\tau, u_{2\tau} + z_\tau)\|d\tau ds \\ &\quad + \sum_{0<t_i<t} \|R(t-t_i)\|\|\ I_i(u_{1t_i} + z_{t_i}) - I_i(u_{2t_i} + z_{t_i})\|, \\ &\leq L_F\|\ u_{1t} - u_{2t}\|_{\mathfrak{B}} + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F\|\ u_{1t} - u_{2t}\|_{\mathfrak{B}} \\ &\quad + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta}\|\ u_{1t} - u_{2t}\|_{\mathfrak{B}} + N_1\sum_{i=1}^m L_i\|\ u_{1t} - u_{2t}\|_{\mathfrak{B}}, \\ &\leq K_{T_0}[L_F + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta} + N_1\sum_{i=1}^m L_i]\|\ u_1 - u_2\|_{T_0}, \end{aligned} \tag{3.43}$$

it gives that P_1 is Lipschitz continuous with Lipschitz constant $L = K_{T_0}[L_F + \frac{M_{1-\beta}{T_0}^\beta}{\beta}L_F + \frac{N_2 L_F M_{1-\beta}{T_0}^{\beta+1}}{\beta} + N_1\sum_{i=1}^m L_i]$.

Let B be an arbitrary subset of $S(T_0)$. Since $R(t)$ is equicontinuous resolvent. Therefore, from the assumption (HG) and the strongly continuity of $R(t)$, we have that $R(t-s)G(s, x_s + y_s, \int_0^s E(s, \tau, x_\tau + y_\tau)d\tau)$ is piecewise equicontinuous. Then, by the Lemma 2.6 we have
$\chi(P_2(B(t)))$

$$\begin{aligned}
&\leq \chi(\int_0^t R(t-s)G(s, B_s + z_s, \int_0^s E(s, \tau, B_\tau + z_\tau)d\tau)ds),\\
&\leq \int_0^t \xi(s)\cdot(\sup_{-\infty<\theta\leq 0}\chi(B(s+\theta)+z(s+\theta)) + \chi(\int_0^s E(s, \tau, B_\tau + z_\tau)d\tau))ds,\\
&\leq \int_0^t \xi(s)\sup_{-\infty<\theta\leq 0}[\chi(B(s+\theta)+z(s+\theta)) + L_0\chi(\Omega(B(s+\theta)+z(s+\theta)))]ds,\\
&\leq \int_0^t \xi(s)\sup_{0\leq\tau\leq s}(\chi(B(\tau)) + L_0\chi(\Omega(B(\tau))))ds,\\
&\leq \chi_{\mathcal{PC}}(B)[1+\Omega_1 L_0]\int_0^t \xi(s)ds,\ [\chi(\Omega(B(\tau)))\leq \Omega_1\chi(B(\tau))], \qquad (3.44)
\end{aligned}$$

for every bounded set $B \subset \mathcal{PC}$. Where Ω_1 is a constant.
Now we can see that for any bounded subset $B \in \mathcal{PC}$

$$\begin{aligned}
\chi_{\mathcal{PC}}(P(B)) &= \chi_{\mathcal{PC}}(P_1 B + P_2 B),\\
&\leq \chi_{\mathcal{PC}}(P_1 B) + \chi_{\mathcal{PC}}(P_2 B),\\
&\leq (L + (1 + L_0\Omega_1)\int_0^t \xi(s)ds)\chi_{\mathcal{PC}}(B), \qquad (3.45)
\end{aligned}$$

from the above inequality we obtain that P is χ-contraction. Hence P has at least one fixed point in B by Darbo fixed point theorem. Let u be the fixed point of the map Q on $S(T_0)$. Thus $y = u + z$ is a mild solution for the problem (1.1)-(1.3). Therefore this completes the proof of the theorem. □

Theorem 3.2. *Suppose that $(HR), (HF), (HG), (HE), (HI)$ and (H') are satisfied and*

$$\begin{aligned}
&K_{T_0}[\| A^{-\beta}\|C_1 + \frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1 + \frac{N_2 M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1\\
&+N_1\sum_{i=1}^m K_i^1] + N_1 K_{T_0}\int_0^{T_0}\alpha(s)ds \lim_{\tau\to\infty}\sup\frac{\tau + L_0\varphi(\tau)}{\tau} < 1. \qquad (3.46)
\end{aligned}$$

Then, the impulsive system (1.1)-(1.3) has a mild solution.

Proof. Thus proof of the above theorem is like that of Theorem 3.1, We characterize the operator P as (3.32). Now, we show that there exist a $r > 0$ such that $Q(B_r) \subset B_r$, where B_r is a closed and convex ball with center at the origin and radius r i.e., $B_r = \{u \in S(T_0) : \| u\|_{T_0} \leq r\}$. To prove it, we assume that for any $r > 0$, there exists $u_r \in B_r$ and $t_r \in [0, T_0]$ such that $r < \| Qu_r(t_r)\|$. For $u_r \in B_r$ and $t_r \in [0, T_0]$, we have

$$\begin{aligned}
r &< \| Qu_r(t_r)\| \leq N_1\| F(0,\phi)\| + \| A^{-\beta}\|[C_1\| u_{rt_r} + z_{t_r}\|_{\mathfrak{B}} + C_2]\\
&+\int_0^{t_r}\|A^{1-\beta}R(t_r - s)\|\|A^{\beta}F(s, u_{rs} + z_s)\|ds\\
&+\int_0^{t_r}\|A^{1-\beta}R(t_r - s)\|\int_0^s\|f(s-\tau)\|\ \|A^{\beta}F(\tau, u_{r\tau} + z_s)\|d\tau ds\\
&+N_1\int_0^{t_r}\| G(s, u_{rs} + z_s, \int_0^s E(s, \tau, u_{r\tau} + z_s)\tau)\|ds\\
&+N_1\sum_{i=1}^m(K_i^1\| u_{rt} + z_t\|_{\mathfrak{B}} + K_i^2),
\end{aligned}$$

$$
\begin{aligned}
&\leq N_1\| A^{-\beta}\|(C_1\|\phi\|_{\mathfrak{B}}+C_2)+\| A^{-\beta}\|[C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2]\\
&\quad+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}(C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2)+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}(C_1\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+C_2)\\
&\quad+\int_0^{t_r}\alpha(s)\Omega(\| u_{rt_r}+z_{t_r}\|_{\mathfrak{B}}+\| \int_0^{s}E(s,\tau,u_{r\tau}+z_{\tau})d\tau\|)ds\\
&\quad+N_1\sum_{i=1}^{m}(K_i^1\| u_{rt}+z_t\|_{\mathfrak{B}}+K_i^2),\\
&\leq N_1\| A^{-\beta}\|(C_1\|\phi\|_{\mathfrak{B}}+C_2)+\| A^{-\beta}\|C_2+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_2+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_2\\
&\quad+N_1\sum_{i=1}^{m}K_i^2+[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^{m}K_i^1]\\
&\quad\times[(K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r]+\int_0^{t_r}\alpha(s)\Omega((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}\\
&\quad+K_{T_0}r+L_0\varphi((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r))ds,
\end{aligned}
\tag{3.47}
$$

it gives that

$$
\begin{aligned}
1 &< K_{T_0}[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^{m}K_i^1]\\
&\quad+N_1\int_0^{T_0}\alpha(s)ds\\
&\quad\times\limsup_{r\to\infty}\frac{\Omega((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r+L_0\varphi((K_{T_0}N_1H+M_{T_0})\| \phi\|_{\mathfrak{B}}+K_{T_0}r))}{r},\\
&\leq K_{T_0}[\| A^{-\beta}\|C_1+\frac{M_{1-\beta}T_0{}^{\beta}}{\beta}C_1+\frac{N_2M_{1-\beta}T_0{}^{\beta+1}}{\beta}C_1+N_1\sum_{i=1}^{m}K_i^1]\\
&\quad+N_1K_{T_0}\int_0^{T_0}\alpha(s)ds\limsup_{\tau\to\infty}\frac{\tau+L_0\varphi(\tau)}{\tau},
\end{aligned}
\tag{3.48}
$$

which is the contradiction of the inequality (3.46). Hence we conclude that $QB_r\subset B_r$.
As the proof of the Theorem 3.1, we obtain that there exists at least a mild solution for the problem (1.1)-(1.3). □

4 Example

In this section, we consider an example to illustrate the application of the theory. Here we take the space $C_0\times L^2(h,X)$ as phase space $\mathfrak{B}$(see, [5]).
We consider the following first order neutral integro-differential equation with unbounded delay

$$
\begin{aligned}
&\frac{d}{dt}[x(t,u)-\int_{-\infty}^{t}\int_0^{\pi}B(t-s,\xi,u)x(s,\xi)d\xi ds]=\frac{\partial^2}{\partial u^2}[x(t,u)+\int_0^{t}f(t-s,u)x(s,u)ds]\\
&\qquad+\int_0^{t}a(t,u,s-t)G(x(s,u),\int_0^{s}E(s,\tau,x_\tau)d\tau)ds,\quad t\in[0,T_0],\quad u\in[0,\pi],
\end{aligned}
\tag{4.49}
$$

$$x(t,0)=x(t,\pi)=0,\quad t\in[0,T_0], \tag{4.50}$$

$$x(\tau,u)=\phi(\tau,u),\quad \tau\leq 0,\ 0\leq u\leq\pi, \tag{4.51}$$

$$\Delta x(t_i)(u)=\int_{-\infty}^{t}c_i(t_i-s)x(s,u)ds, \tag{4.52}$$

where $\phi\in C_0\times L^2(h,X)$ and $0<t_1<t_2<\cdots<t_m<b$ are fixed numbers.
The function B,f,a,G,E,c_i are satisfied the following conditions:

(A1) The function $B(s,\xi,u)$, $\frac{\partial}{\partial u}B$ are measurable and $B(s,\xi,0) = B(s,\xi,\pi) = 0$. Also

$$L_B = \max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{h(s)}(\frac{\partial^i B(s,\xi,u)}{\partial u^i})d\xi ds du)^{1/2} : i = 0,1\} < \infty; \tag{4.53}$$

(A2) The operator $f(t), t \geq 0$ is bounded and $\| f(t,u)\| \leq N_2$;

(A3) $a(t,u,\tau)$ is continuous function on $[0,T_0]\times[0,\pi]\times(-\infty,0]$ with $\int_{-\infty}^{0} a(t,u,\tau)d\tau = n(t,u) < \infty$;

(A4) G is a continuous function, satisfying $G(x_1,x_2) \leq \Omega^{'}(\| x_1\| + \| x_2\|)$, where $\Omega^{'}(\cdot)$ is continuous, increasing and positive on $[0,\infty)$;

(A5) The function $E(\cdot)$ is a continuous function, satisfying $0 \leq E(t,s,u) \leq \omega(\| u\|)$, where ω is a positive increasing continuous function on $[0,\infty)$;

(A6) The functions $c_i \in C([0,\infty);\mathbb{R})$ and $K_i^3 = (\int_{-\infty}^{0}\frac{(c_i(s))^2}{h(s)}ds)^{1/2} < 0, \forall\, i = 1,\cdots,m$;

Let $Ax = x''$, $A : D(A) \subset X \to X$ is a linear operator with domain

$$D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}. \tag{4.54}$$

It is known that A is the infinitesimal generator of an analytic resolvent operator $R(t), t \geq 0$. We assume that the $(A1)-(A6)$ are established.
Now, the system (4.49)-(4.52) can be reformulated as the abstract impulsive Cauchy problem (1.1)-(1.3) giving by

$$F(t,y)(u) = \int_{-\infty}^{0}\int_0^{\pi} B(s,z,u)y(s,z)dzds, \tag{4.55}$$

$$G_1(t,w,y)(u) = \int_{-\infty}^{0} a(t,u,\tau)G(w(\tau,u),\int_0^{\tau} y(\tau,\theta,x_\theta)d\theta)d\tau, \tag{4.56}$$

$$I_i(y)(u) = \int_{-\infty}^{0} c_i(s)y(s,u)ds. \tag{4.57}$$

It is easy to see that $F(t,\cdot), G_1(t,\cdot,\cdot), I_i(i = 1,\cdots,m)$ are bounded linear operators. Applying the Theorem 3.1, we conclude that the problem (4.49)-(4.52) has at least one mild solution.

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