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Square-mean asymptotically almost automorphic mild solutions to non-autonomous stochastic differential equations

Zhi-Hong Li[a,*] and Zhi-Han Zhao[b]

[a]Department of Mathematics, Lanzhou Jiaotong University, Lanzhou - 730070, P. R. China.

[b]Department of Information Engineering, Sanming University, Sanming - 365004, P. R. China.

Abstract

This paper is mainly concerned with square-mean asymptotically almost automorphic mild solutions to a class of non-autonomous stochastic differential equations in a real separable Hilbert space. Some existence results of square-mean asymptotically almost automorphic mild solutions have been established by properties and composition theroems of square-mean asymptotically almost automorphic functions and fixed point theorems.

Keywords: Non-autonomous differential equation, Square-mean asymptotically almost automorphic.

2010 MSC: 34K14, 60H10, 35B15, 34F05.

1 Introduction

In this paper, we study the existence of square-mean asymptotically almost automorphic solutions for the following non-autonomous stochastic differential equations in the form

$$\begin{cases} dx(t) = A(t)x(t)dt + f(t, B_1x(t))dt + g(t, B_2x(t))dW(t),\ t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A(t) : D(A(t)) \subset L^2(,) \to L^2(,)$ is a family of densely defined closed linear operators satisfying the so called "Acquistapace-Terreni" conditions, B_i, $i = 1, 2$ are bounded linear operators, and $W(t)$ is a two sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. x_0 is an $\mathcal{F}_0$-adapted, -valued random variable independent of the Wiener process W, and $f, g : [0, +\infty) \times L^2(,) \to L^2(,)$ are appropriate functions to be specified later.

The asymptotically almost automorphic functions were firstly introduced by G. M. N'Gu'er'ekata in [14]. Since then these functions have become of great interest to several mathematicians and generated lots of developments and applications, we refer the reader to [3, 11, 12] and the references therein.

Recently, the existence of almost periodic, almost automorphic and pseudo almost automorphic solutions to some stochastic differential equations have been considered in many publications such as [4, 5, 7, 8, 10, 18] and references therein. In a very recent paper [8], the authors introduced a new concept of S^2-almost automorphy for stochastic processes including a composition theorem. In paper [16], the authors introduced the notion of square-mean asymptotically almost automorphic stochastic process and established some basic results not only on the completeness of the space that consists of the square-mean asymptotically almost automorphic processes but also on the composition of such processes. They apply this new concept to investigate the existence of square-mean asymptotically almost automorphic mild solutions to the following abstract stochastic

*Corresponding author.

E-mail address: 1950290025@qq.com (Zhi-Hong Li), zhaozhihan841110@126.com (Zhi-Han Zhao).

integro-differential equations

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right]dt + f(t,x(t))dW(t),\ t \geq 0, \\ x(0) = x_0, \end{cases}$$

where A and $B(t), t \geq 0$ are densely defined and closed linear operators in a Hilbert space $L^2(,)$, and $W(t)$ is a two sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. x_0 is an $\mathcal{F}_0$-adapted, -valued random variable independent of the Wiener process W.

Motivated by the works [8, 11, 16, 17], the main purpose of this paper is to investigate the existence of square-mean asymptotically almost automorphic mild solutions to the problems (1.1). The obtained results can be seen as a contribution to this emerging field.

The present paper is organized as follows. In section 2, we introduce the notion of square-mean asymptotically almost automorphic processes and study some of their basic properties. In section 3, we prove the existence of existence of square-mean asymptotically almost automorphic mild solutions to the problem (1.1).

2 Preliminary

In this section, we introduce some basic definitions, notations, lemmas and technical results which will be used in the sequel. For more details on this section, we refer the reader to [7, 13].

Throughout the paper, we assume that $(, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(, \|\cdot\|, \langle\cdot,\cdot\rangle)$ are two real separable Hilbert spaces. Let $(\Omega, \mathcal{F},)$ be a complete probability space. The notation $L^2(,)$ stands for the space of all -valued random variable x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d < \infty.$$

For $x \in L^2(,)$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 d\right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(,)$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. We let $L(,)$ denote the space of all linear bounded operators from into , equipped with the usual operator norm $\|\cdot\|_{L(,)}$; in particular, this is simply denoted by $L()$ when $=$. The notation $C_0(\mathbb{R}^+; L^2(,))$ stands for the collection of all bounded continuous stochastic processes φ from $\mathbb{R}^+$ into $L^2(,)$ such that $\lim_{t\to+\infty} E\|\varphi(t)\|^2 = 0$. Similarly, $C_0(\mathbb{R}^+ \times L^2(,); L^2(,))$ stands for the space of the continuous stochastic processes $f : \mathbb{R}^+ \times L^2(,) \to L^2(,)$ such that

$$\lim_{t\to+\infty} E\|f(t,x)\|^2 = 0$$

uniformly for $x \in K$, where $K \subset L^2(,)$ is any bounded subset. In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, , \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$.

Definition 2.1. *[13] A stochastic process $x : \mathbb{R} \to L^2(,)$ is said to be stochastically continuous if*

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Definition 2.2. *[9] A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(,)$, $(t,x) \to f(t,x)$ is said to be square-mean almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(,)$ such that*

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \ \text{ and } \ \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(,)$ is denoted by $AA(\mathbb{R}; L^2(,))$.

Definition 2.3. *[9] A function $f : \mathbb{R} \times L^2(,) \to L^2(,)$, $(t,x) \to f(t,x)$, which is jointly continuous, is said to be square-mean almost automorphic if $f(t,x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$ is any bounded subset of $L^2(,)$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\widetilde{f} : \mathbb{R} \times L^2(,) \to L^2(,)$ such that*

$$\lim_{n\to\infty} E\|f(t+s_n,x) - \widetilde{f}(t,x)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f}(t-s_n,x) - f(t,x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in K$. Denote by $AA(\mathbb{R} \times L^2(,); L^2(,))$ the set of all such functions.

Lemma 2.1. *[13] $(AA(\mathbb{R}; L^2(,)), \|\cdot\|_\infty)$ is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA(\mathbb{R}; L^2(,))$.

Lemma 2.2. *[9] Let $f : \mathbb{R} \times L^2(,) \to L^2(,)$, $(t,x) \to f(t,x)$ be square-mean almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}$, that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $E\|x-y\|^2 < \delta$ imply that $E\|f(t,x) - f(t,y)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(,)$, the stochastic process $F : \mathbb{R} \to L^2(,)$ given by $F(\cdot) := f(\cdot, x(\cdot))$ is square-mean almost automorphic.*

Definition 2.4. *[16] A stochastically continuous process $f : \mathbb{R}^+ \to L^2(,)$ is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R}; L^2(,))$ and $h \in C_0(\mathbb{R}^+; L^2(,))$. Denote by $AAA(\mathbb{R}^+; L^2(,))$ the collection of all the square-mean asymptotically almost automorphic processes $f : \mathbb{R}^+ \to L^2(,)$.*

Definition 2.5. *[16] A function $f : \mathbb{R}^+ \times L^2(,) \to L^2(,)$, $(t,x) \to f(t,x)$, which is jointly continuous, is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R} \times L^2(,); L^2(,))$ and $h \in C_0(\mathbb{R}^+ \times L^2(,); L^2(,))$. Denote by $AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ the set of all such functions.*

Lemma 2.3. *[16] If f, f_1 and f_2 are all square-mean asymptotically almost automorphic stochastic processes, then the following hold true:*
(I) $f_1 + f_2$ is square-mean asymptotically almost automorphic ;
(II) λf is square-mean asymptotically almost automorphic for any scalar λ;
(III) There exists a constant $M > 0$ such that $\sup_{t\in\mathbb{R}^+} E\|f(t)\|^2 \leq M$.

Lemma 2.4. *[16] Suppose that $f \in AAA(\mathbb{R}^+; L^2(,))$ admits a decomposition $f = g + h$, where $g \in AA(\mathbb{R}; L^2(,))$ and $h \in C_0(\mathbb{R}^+; L^2(,))$. Then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$.*

Corollary 2.1. *[16] The decomposition of a square-mean asymptotically almost automorphic process is unique.*

Lemma 2.5. *[16] $AAA(\mathbb{R}^+; L^2(,))$ is a Banach space when it is equipped with the norm:*

$$\|f\|_{AAA(\mathbb{R}^+;L^2(,))} := \sup_{t\in\mathbb{R}} \|g(t)\|_2 + \sup_{t\in\mathbb{R}^+} \|h(t)\|_2,$$

where $f = g + h \in AAA(\mathbb{R}^+; L^2(,))$ with $g \in AA(\mathbb{R}; L^2(,))$, $h \in C_0(\mathbb{R}^+; L^2(,))$.

Lemma 2.6. *[16] $AAA(\mathbb{R}^+; L^2(,))$ is a Banach space with the norm:*

$$\|f\|_\infty := \sup_{t\in\mathbb{R}^+} \|f(t)\|_2 = \sup_{t\in\mathbb{R}^+} (E\|f(t)\|^2)^{\frac{1}{2}}.$$

Remark 2.1. *[16] In view of the previous Lemmas it is clear that the two norms are equivalent in $AAA(\mathbb{R}^+; L^2(,))$.*

Lemma 2.7. *[16] Let $f \in AA(\mathbb{R} \times L^2(,); L^2(,))$ and let $f(t,x)$ be uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}^+$. Then $f(t,x)$ is uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}$.*

Lemma 2.8. *[16] Let $f \in AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ and suppose that $f(t,x)$ be uniformly continuous in any bounded subset $K \subset L^2(,)$ uniformly for $t \in \mathbb{R}^+$. If $u(t) \in AAA(\mathbb{R}^+; L^2(,))$, then $f(\cdot, u(\cdot)) \in AAA(\mathbb{R}^+; L^2(,))$.*

Lemma 2.9. *Let $\mathcal{L} \in L(H)$ and assume that $f \in AAA(\mathbb{R}^+; L^2(,))$. Then $\mathcal{L}f \in AAA(\mathbb{R}^+; L^2(,))$.*

Proof. Since $f \in AAA\left(\mathbb{R}^+; L^2(,)\right)$, we have by definition that $f = g + h$, where $g \in AA\left(\mathbb{R}; L^2(,)\right)$ and $h \in C_0\left(\mathbb{R}^+; L^2(,)\right)$. Then, by [6, Lemma 2.4], we see that $\mathcal{L}g \in AA(\mathbb{R}; L^2(,))$. On the other hand, since $\mathcal{L} \in L(H)$, then we have

$$E\|\mathcal{L}h(t)\|^2 \leq \|\mathcal{L}\|^2_{L(H)} E\|h(t)\|^2$$

which shows that $\lim_{t\to+\infty} E\|\mathcal{L}h(t)\|^2 = 0$, since $h \in C_0\left(\mathbb{R}^+; L^2(,)\right)$. Thus, $\mathcal{L}f \in AAA(\mathbb{R}^+; L^2(,))$. This ends the proof. □

The following Lemma hold by [1, Theorem 2.3] and [2].

Lemma 2.10. *If the Acquistapace-Terreni conditions (ATCs) are satisfied, that is, there exists a constant $K_0 > 0$ and a set of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k, \beta_1, \cdots, \beta_k$ with $0 \leq \beta_i < \alpha_i \leq 2$, $i = 1, 2, \cdots, k$, such that*

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \leq K_0 \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1},$$

for $t, s \in \mathbb{R}$, $\lambda \in S_{\theta_0}\backslash\{0\}$, where

$$\rho(A(t)) \supset S_{\theta_0} = \{\lambda \in \mathbb{C} : |arg\lambda| \leq \theta_0\} \cup \{0\}, \theta_0 \in (\frac{\pi}{2}, \pi)$$

and there exists a constant $K_1 \geq 0$ such that

$$\|(\lambda - A(t))^{-1}\| \leq \frac{K_1}{1+|\lambda|}, \lambda \in S_{\theta_0}.$$

Then there exists a unique evolution family $\{U(t,s), t \geq s > -\infty\}$ on $L^2(,)$.

Throughout the rest of the paper we assume that (ATCs) are satisfied.

Definition 2.6. *An $\mathcal{F}_t$-adapted stochastic process $x : [0, \infty) \to L^2(,)$ is called a mild solution of problem (1.1) if $x(0) = x_0$ is $\mathcal{F}_0$-measurable and $x(t)$ satisfies the corresponding stochastic integral equation*

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s) f(s, B_1 x(s)) ds + \int_0^t U(t,s) g(s, B_2 x(s)) dW(s).$$

for all $t \geq 0$ and $0 \leq s \leq t$.

3 Extension Principle

In this section, we establish the existence of square-mean asymptotically automorphic mild solutions to (1.1). For that, we give the following assumptions:
(H1) The evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there exist $M \geq 1$ and $\delta > 0$ such that $\|U(t,s)\| \leq Me^{-\delta(t-s)}$ for all $t \geq s$.
(H2) $U(t,s)$, $t \geq s$, satisfies the condition that, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|U(t+s_n, s+s_n) - U(t,s)\| \leq \varepsilon e^{-\delta(t-s)},$$

for all $n > N$ and all $t \geq s$, moreover

$$\|U(t-s_n, s-s_n) - U(t,s)\| \leq \varepsilon e^{-\delta(t-s)},$$

for all $n > N$ and all $t \geq s$.
(H3) The operators $B_i : L^2(,) \to L^2(,)$ for $i = 1, 2$, are bounded linear operators and we let $\overline{\omega} := \max_{i=1,2}\left\{\|B_i\|_{\mathcal{L}(L^2(,))}\right\}$.
(H4) The functions $f, g \in AAA(\mathbb{R}^+ \times L^2(,); L^2(,))$ and there are positive numbers L_f, L_g such that

$$E\|f(t,\varphi) - f(t,\psi)\|^2 \leq L_f E\|\varphi - \psi\|^2,$$

and

$$E\|g(t,\varphi) - g(t,\psi)\|^2 \leq L_g E\|\varphi - \psi\|^2,$$

for all $t \in \mathbb{R}^+$ and $\varphi, \psi \in L^2(,)$.

Theorem 3.1. *Assume that the conditions (H1)-(H4) are satisfied, then the problem (1.1) has a unique square-mean asymptotically almost automorphic mild solution $x(\cdot) \in AAA(\mathbb{R}^+; L^2(,))$ provided that*

$$L_0 = M^2\overline{\omega}^2 \left[\frac{2}{\delta^2}L_f + \frac{1}{\delta}L_g\right] < 1.$$

Proof. Let $\Gamma : AAA(\mathbb{R}^+; L^2(,)) \to AAA(\mathbb{R}^+; L^2(,))$ be the operator defined by

$$\Gamma x(t) := U(t,0)x_0 + \int_0^t U(t,s)f(s, B_1x(s))ds + \int_0^t U(t,s)g(s, B_2x(s))dW(s), \quad t \geq 0.$$

Let us prove that Γx is well defined, for this, let $x \in AAA(\mathbb{R}^+; L^2(,))$. We need to prove that $\Gamma x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Let us consider the nonlinear operators Γ_0, Γ_1 and Γ_2 acting on the Banach space $AAA(\mathbb{R}^+; L^2(,))$ defined by $\Gamma_0 x(t) = U(t,0)x_0$,

$$\Gamma_1 x(t) = \int_0^t U(t,s)f(s, B_1x(s))ds \;\text{ and }\; \Gamma_2 x(t) = \int_0^t U(t,s)g(s, B_2x(s))dW(s)$$

respectively.

First, we will show that $\Gamma_1 x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Indeed, let $x \in AAA\left(\mathbb{R}^+; L^2(,)\right)$, then $s \to B_i x(s)$ is in $AAA\left(\mathbb{R}^+; L^2(,)\right)$ as $B_i \in L\left(L^2(,)\right)$, $i = 1, 2$. And hence, by Lemma 2.8, the functions $s \to f(s, B_1x(s))$ belongs to $AAA(\mathbb{R}^+; L^2(,))$. Then we let $F(t) = f(t, B_1x(t)) \in AAA(\mathbb{R}; L^2(,))$. Now we can write $F(t) = f_1(t) + f_2(t)$, where $f_1(t) \in AA(\mathbb{R}; L^2(,))$ and $f_2(t) \in C_0(\mathbb{R}^+; L^2(,))$. Observe

$$\begin{aligned}
\Gamma_1 x(t) &= \int_0^t U(t,s)f_1(s)ds + \int_0^t U(t,s)f_2(s)ds \\
&= \int_{-\infty}^t U(t,s)f_1(s)ds - \int_{-\infty}^0 U(t,s)f_1(s)ds + \int_0^t U(t,s)f_2(s)ds \\
&= \gamma_1(t) + \gamma_2(t),
\end{aligned}$$

where $\gamma_1(t) = \int_{-\infty}^t U(t,s)f_1(s)ds$ and $\gamma_2(t) = \int_0^t U(t,s)f_2(s)ds - \int_{-\infty}^0 U(t,s)f_1(s)ds$.

First we prove that $\gamma_1(t) \in AA(\mathbb{R}; L^2(,))$. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $f_1 \in AA(\mathbb{R}; L^2(,))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{f_1}$

$$\lim_{n\to\infty} E\|f_1(t+s_n) - \widetilde{f_1}(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f_1}(t-s_n) - f_1(t)\|^2 = 0 \tag{3.1}$$

hold for each $t \in \mathbb{R}$. By condition (H2), for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, it follows that $\|U(t+s_n, s+s_n)\| \leq \varepsilon e^{-\delta(t-s)}$ for all $t \geq s \in \mathbb{R}$. Moreover, if we let $\widetilde{\gamma_1}(t) = \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds$, then by using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& E\|\gamma_1(t+s_n) - \widetilde{\gamma_1}(t)\|^2 \\
= \;& E\left\|\int_{-\infty}^{t+s_n} U(t+s_n, s)f_1(s)ds - \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds\right\|^2 \\
= \;& E\left\|\int_{-\infty}^{t} U(t+s_n, s+s_n)f_1(s+s_n)ds - \int_{-\infty}^t U(t,s)\widetilde{f_1}(s)ds\right\|^2 \\
\leq \;& 2E\left\|\int_{-\infty}^{t} [U(t+s_n, s+s_n) - U(t,s)]f_1(s+s_n)ds\right\|^2 \\
& +2E\left\|\int_{-\infty}^{t} U(t,s)[f_1(s+s_n) - \widetilde{f_1}(s)]ds\right\|^2 \\
\leq \;& 2\varepsilon^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f_1(s+s_n)\|ds\right)^2 \\
& +2M^2 E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f_1(s+s_n) - \widetilde{f_1}(s)\|ds\right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq 2\varepsilon^2\left(\int_{-\infty}^{t}e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t}e^{-\delta(t-s)}E\|f_1(s+s_n)\|^2ds\right)\\
&\quad +2M^2\left(\int_{-\infty}^{t}e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t}e^{-\delta(t-s)}E\|f_1(s+s_n)-\widetilde{f}_1(s)\|^2ds\right)\\
&\leq 2\varepsilon^2\left(\int_{-\infty}^{t}e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)\|^2\\
&\quad +2M^2\left(\int_{-\infty}^{t}e^{-\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)-\widetilde{f}_1(t)\|^2\\
&\leq \frac{2}{\delta^2}\varepsilon^2\sup_{t\in\mathbb{R}}E\|f_1(t)\|^2+\frac{2}{\delta^2}M^2\sup_{t\in\mathbb{R}}E\|f_1(t+s_n)-\widetilde{f}_1(t)\|^2
\end{aligned}$$

for all $t \geq s \in \mathbb{R}$ and all $n > N$. Since $f_1(\cdot)$ is bounded and satisfies (3.1), then we immediately obtain that

$$\lim_{n\to\infty}E\|\gamma_1(t+s_n)-\widetilde{\gamma_1}(t)\|^2=0 \ \text{ for each } \ t\in\mathbb{R}.$$

A similar reasoning establishes that

$$\lim_{n\to\infty}E\|\widetilde{\gamma_1}(t-s_n)-\gamma_1(t)\|^2=0 \ \text{ for each } \ t\in\mathbb{R}.$$

Thus we conclude that $\gamma_1(\cdot) \in AA(\mathbb{R}; L^2(,))$.

Next, let us show that $\gamma_2 \in C_0(\mathbb{R}^+; L^2(,))$. Since $f_2 \in C_0(\mathbb{R}^+; L^2(,))$, for any sufficiently small $\varepsilon_0 > 0$, there exists a constant $T > 0$ such that $E\|f_2(s)\|^2 \leq \varepsilon_0$ for all $s \geq T$. Then, for all $t \geq 2T$, we obtain

$$\begin{aligned}
E\|\gamma_2(t)\|^2 &= E\left\|\int_0^{\frac{t}{2}}U(t,s)f_2(s)ds+\int_{\frac{t}{2}}^{t}U(t,s)f_2(s)ds-\int_{-\infty}^{0}U(t,s)f_1(s)ds\right\|^2\\
&\leq 3E\left\|\int_0^{\frac{t}{2}}U(t,s)f_2(s)ds\right\|^2+3E\left\|\int_{\frac{t}{2}}^{t}U(t,s)f_2(s)ds\right\|^2+3E\left\|\int_{-\infty}^{0}U(t,s)f_1(s)ds\right\|^2\\
&\leq 3EM^2\left(\int_0^{\frac{t}{2}}e^{-\delta(t-s)}\|f_2(s)\|ds\right)^2+3EM^2\left(\int_{\frac{t}{2}}^{t}e^{-\delta(t-s)}\|f_2(s)\|ds\right)^2\\
&\quad +3EM^2\left(\int_{-\infty}^{0}e^{-\delta(t-s)}\|f_1(s)\|ds\right)^2\\
&\leq 3M^2\left(\int_0^{\frac{t}{2}}e^{-\delta(t-s)}ds\right)\left(\int_0^{\frac{t}{2}}e^{-\delta(t-s)}E\|f_2(s)\|^2ds\right)\\
&\quad +3M^2\left(\int_{\frac{t}{2}}^{t}e^{-\delta(t-s)}ds\right)\left(\int_{\frac{t}{2}}^{t}e^{-\delta(t-s)}E\|f_2(s)\|^2ds\right)\\
&\quad +3M^2\left(\int_{-\infty}^{0}e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{0}e^{-\delta(t-s)}E\|f_1(s)\|^2ds\right)\\
&\leq 3M^2\left(\int_0^{\frac{t}{2}}e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}^+}E\|f_2(t)\|^2+3M^2\varepsilon_0\left(\int_{\frac{t}{2}}^{t}e^{-\delta(t-s)}ds\right)^2\\
&\quad +3M^2\left(\int_{-\infty}^{0}e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}E\|f_1(t)\|^2\\
&\leq \frac{3M^2}{\delta^2}\left[e^{-\frac{\delta t}{2}}-e^{-\delta t}\right]\sup_{t\in\mathbb{R}^+}E\|f_2(t)\|^2+\frac{3M^2\varepsilon_0}{\delta^2}\left[1-e^{-\frac{\delta t}{2}}\right]+\frac{3M^2e^{-\delta t}}{\delta^2}\sup_{t\in\mathbb{R}^+}E\|f_1(t)\|^2\\
&\leq \frac{3M^2}{\delta^2}e^{-\frac{\delta t}{2}}M_f+\frac{3M^2\varepsilon_0}{\delta^2}+\frac{3M^2}{\delta^2}M_g e^{-\delta t}.
\end{aligned}$$

where $M_f = \sup_{t\in\mathbb{R}^+} E\|f_2(t)\|^2$ and $M_g = \sup_{t\in\mathbb{R}} E\|f_1(t)\|^2$. Therefore, the last estimation converges to zero as $t \to +\infty$, since ε_0 is arbitrary. Thus, it leads to $\lim_{t\to+\infty} E\|\gamma_2(t)\|^2 = 0$. Recalling that $\Gamma_1 x(t) = \gamma_1(t) + \gamma_2(t)$ for all $t \geq 0$, we get $\Gamma_1 x(t) \in AAA(\mathbb{R}^+; L^2(,))$.

Now we prove that $\Gamma_2 x(t) \in AAA(\mathbb{R}^+; L^2(,))$. Similarly, by using Lemma 2.8 one can easily see that $s \to g(s, B_2 x(s))$ is in $AAA(\mathbb{R}^+; L^2(,))$ whenever $B_2 x \in AAA(\mathbb{R}^+; L^2(,))$. Then we let $G(t) = g(t, B_2 x(t)) = g_1(t) + g_2(t) \in AAA(\mathbb{R}^+; L^2(,))$ where $g_1 \in AA(\mathbb{R}; L^2(,))$ and $g_2 \in C_0(\mathbb{R}^+; L^2(,))$, then

$$\begin{aligned}
\Gamma_2 x(t) &= \int_0^t U(t,s) g_1(s) dW(s) + \int_0^t U(t,s) g_2(s) dW(s) \\
&= \int_{-\infty}^t U(t,s) g_1(s) dW(s) - \int_{-\infty}^0 U(t,s) g_1(s) dW(s) + \int_0^t U(t,s) g_2(s) dW(s) \\
&= M_1(t) + N_1(t),
\end{aligned}$$

where $M_1(t) = \int_{-\infty}^t U(t,s) g_1(s) dW(s)$ and $N_1(t) = \int_0^t U(t,s) g_2(s) dW(s) - \int_{-\infty}^0 U(t,s) g_1(s) dW(s)$.

The next step we prove that $M_1(t) \in AA(\mathbb{R}; L^2(,))$. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $g_1 \in AA(\mathbb{R}; L^2(,))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{g}_1$

$$\lim_{n\to\infty} E\|g_1(t+s_n) - \widetilde{g}_1(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{g}_1(t-s_n) - g_1(t)\|^2 = 0 \tag{3.2}$$

hold for each $t \in \mathbb{R}$ and by condition (H2), for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, it follows that $\|U(t+s_n, s+s_n) - U(t,s)\| \le \varepsilon e^{-\delta(t-s)}$. Now, let $\widetilde{W}(\sigma) := W(\sigma + s_n) - W(s_n)$ for each $\sigma \in \mathbb{R}$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $\widetilde{M}_1(t) = \int_{-\infty}^t U(t,s)\widetilde{g}_1(s) dW(s)$, then by making a change of variable $\sigma = s - s_n$ to get

$$\begin{aligned}
& E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 \\
= \; & E\left\| \int_{-\infty}^{t+s_n} U(t+s_n, s) g_1(s) dW(s) - \int_{-\infty}^t U(t,s)\widetilde{g}_1(s) dW(s) \right\|^2 \\
= \; & E\left\| \int_{-\infty}^{t} U(t+s_n, \sigma+s_n) g_1(\sigma+s_n) d\widetilde{W}(\sigma) - \int_{-\infty}^t U(t,\sigma)\widetilde{g}_1(\sigma) d\widetilde{W}(\sigma) \right\|^2 \\
\le \; & 2E\left\| \int_{-\infty}^{t} [U(t+s_n, \sigma+s_n) - U(t,\sigma)] g_1(\sigma+s_n) d\widetilde{W}(\sigma) \right\|^2 \\
& +2E\left\| \int_{-\infty}^{t} U(t,\sigma)[g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)] d\widetilde{W}(\sigma) \right\|^2 .
\end{aligned}$$

Thus using an estimate on the Ito integral established in [15], we obtain that

$$\begin{aligned}
& E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 \\
\le \; & 2\int_{-\infty}^t \|U(t+s_n, \sigma+s_n) - U(t,\sigma)\|^2 E\|g_1(\sigma+s_n)\|^2 d\sigma \\
& +2\int_{-\infty}^t \|U(t,\sigma)\|^2 E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2 d\sigma \\
\le \; & 2\varepsilon^2 \int_{-\infty}^t e^{-2\delta(t-\sigma)} E\|g_1(\sigma+s_n)\|^2 d\sigma \\
& +2M^2 \int_{-\infty}^t e^{-2\delta(t-\sigma)} E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2 d\sigma \\
\le \; & \frac{1}{\delta}\varepsilon^2 \sup_{t\in\mathbb{R}} E\|g_1(t)\|^2 + \frac{1}{\delta} M^2 \sup_{t\in\mathbb{R}} E\|g_1(\sigma+s_n) - \widetilde{g}_1(\sigma)\|^2,
\end{aligned}$$

for all $t \ge s$ and all $n > N$. Since $g_1(\cdot)$ is bounded and satisfies (3.2), then we immediately obtain that

$$\lim_{n\to\infty} E\|M_1(t+s_n) - \widetilde{M}_1(t)\|^2 = 0 \text{ for all } t \in \mathbb{R}.$$

Arguing in a similar way, we infer that $\lim_{n\to\infty} E\|\widetilde{M}_1(t-s_n) - M_1(t)\|^2 = 0$, for all $t \in \mathbb{R}$. This implies that $M_1(t) \in AA(\mathbb{R}; L^2(,))$.

The next step consists of showing that $N_1(t) \in C_0(\mathbb{R}^+; L^2(,))$, since $g_2 \in C_0(\mathbb{R}^+; L^2(,))$, for any sufficient small $\varepsilon_0 > 0$, there exists a constant $T > 0$ such that $E\|g_2(s)\|^2 \le \varepsilon_0$ for all $s \ge T$. Then, for all $t \ge 2T$, we obtain

$$
\begin{aligned}
& E\|N_1(t)\|^2 \\
= \; & E\left\|\int_0^{\frac{t}{2}} U(t,s)g_2(s)dW(s) + \int_{\frac{t}{2}}^t U(t,s)g_2(s)dW(s) - \int_{-\infty}^0 U(t,s)g_1(s)dW(s)\right\|^2 \\
\le \; & 3E\left(\int_0^{\frac{t}{2}} \|U(t,s)g_2(s)\|^2 ds\right) + 3E\left(\int_{\frac{t}{2}}^t \|U(t,s)g_2(s)\|^2 ds\right) \\
& +3E\left(\int_{-\infty}^0 \|U(t,s)g_1(s)\|^2 ds\right) \\
\le \; & 3M^2\int_0^{\frac{t}{2}} e^{-2\delta(t-s)}E\|g_2(s)\|^2 ds + 3M^2\int_{\frac{t}{2}}^t e^{-2\delta(t-s)}E\|g_2(s)\|^2 ds \\
& +3M^2\int_{-\infty}^0 e^{-2\delta(t-s)}E\|g_1(s)\|^2 ds \\
\le \; & \frac{3M^2}{2\delta}[e^{-\delta t} - e^{-2\delta t}]\sup_{t\in\mathbb{R}^+} E\|g_2(s)\|^2 ds + \frac{3M^2}{2\delta}[1 - e^{-\delta t}]\varepsilon_0 \\
& +\frac{3M^2}{2\delta}e^{-2\delta t}\sup_{t\in\mathbb{R}} E\|g_1(s)\|^2 ds \\
\le \; & \frac{3M^2}{2\delta}e^{-\delta t}M_u + \frac{3M^2}{2\delta}\varepsilon_0 + \frac{3M^2}{2\delta}M_v e^{-2\delta t}
\end{aligned}
$$

where $M_u = \sup_{t\in\mathbb{R}^+} E\|g_2(t)\|^2$ and $M_v = \sup_{t\in\mathbb{R}} E\|g_1(t)\|^2$. Therefore, the last estimation converges to zero as $t \to +\infty$ since ε_0 is arbitrary. Thus, it leads to $\lim_{t\to+\infty} E\|N_1(t)\|^2 = 0$. Recalling that $\Gamma_2 x(t) = M_1(t) + N_1(t)$ for all $t \ge 0$, we get $\Gamma_2 x(t) \in AAA(\mathbb{R}^+; L^2(,))$.

On the other hand, since the evolution family $U(t,s)$ is exponentially stable, it follows that $\lim_{t\to+\infty} E\|\Gamma_0 x(t)\|^2 = 0$. Thus, $\Gamma x(\cdot) \in AAA(\mathbb{R}^+; L^2(,))$. Hence, in view of the above, it is clear that Γ maps $AAA(\mathbb{R}^+; L^2(,))$ into itself.

Now to complete the proof, we have to prove that Γ is a contraction mapping on $AAA(\mathbb{R}^+; L^2(,))$. Indeed, for each $x(t), y(t) \in AAA(\mathbb{R}^+; L^2(,))$, we see that

$$
\begin{aligned}
& E\|(\Gamma x)(t) - (\Gamma y)(t)\|^2 \\
= \; & E\left\|\int_0^t U(t,s)[f(s,B_1x(s)) - f(s,B_1y(s))]ds + \int_0^t U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]dW(s)\right\|^2 \\
\le \; & 2E\left\|\int_0^t U(t,s)[f(s,B_1x(s)) - f(s,B_1y(s))]ds\right\|^2 \\
& +2E\left\|\int_0^t U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]dW(s)\right\|^2 \\
\le \; & 2M^2E\left(\int_0^t e^{-\delta(t-s)}\|f(s,B_1x(s)) - f(s,B_1y(s))\|\right)^2 \\
& +2E\left(\int_0^t \|U(t,s)[g(s,B_2x(s)) - g(s,B_2y(s))]\|^2 ds\right) \\
\le \; & 2M^2E\left[\left(\int_0^t e^{-\delta(t-s)}ds\right)\left(\int_0^t e^{-\delta(t-s)}\|f(s,B_1x(s)) - f(s,B_1y(s))\|^2 ds\right)\right] \\
& +2M^2\int_0^t e^{-2\delta(t-s)}E\|g(s,B_2x(s)) - g(s,B_2y(s))\|^2 ds \\
\le \; & 2M^2L_f\left(\int_0^t e^{-\delta(t-s)}ds\right)\left(\int_0^t e^{-\delta(t-s)}E\|B_1x(s) - B_1y(s)\|^2 ds\right) \\
& +2M^2L_g\int_0^t e^{-2\delta(t-s)}E\|B_2x(s) - B_2y(s)\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2M^2 L_f \overline{w}^2 \left(\int_0^t e^{-\delta(t-s)} ds \right)^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\quad +2M^2 L_g \overline{w}^2 \left(\int_0^t e^{-2\delta(t-s)} ds \right) \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\leq \frac{2M^2}{\delta^2} L_f \overline{w}^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 + \frac{M^2}{\delta} L_g \overline{w}^2 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2 \\
&\leq M^2 \overline{w}^2 \left[\frac{2}{\delta^2} L_f + \frac{1}{\delta} L_g \right] \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2,
\end{aligned}
$$

that is,

$$
\|(\Gamma x)(t) - (\Gamma y)(t)\|_2^2 \leq L_0 \sup_{t\in\mathbb{R}^+} E\|x(t)-y(t)\|^2. \tag{3.3}
$$

Note that

$$
\sup_{t\in\mathbb{R}^+} \|x(t)-y(t)\|_2^2 \leq \left(\sup_{t\in\mathbb{R}^+} \|x(t)-y(t)\|_2 \right)^2, \tag{3.4}
$$

and (3.3) together with (3.4) gives, for each $t \in \mathbb{R}$.

$$
\|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty.
$$

Hence, we obtain

$$
\|\Gamma x - \Gamma y\|_\infty = \sup_{t\in\mathbb{R}^+} \|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty.
$$

which implies that Γ is a contraction by (3.1). So by the Banach contraction principle, we conclude that there exists a unique fixed point $x(\cdot)$ for Γ in $AAA(\mathbb{R}^+; L^2(,))$ such that $\Gamma x = x$, that is

$$
x(t) = U(t,0)x_0 + \int_0^t U(t,s) f(s, B_1 x(s)) ds + \int_0^t U(t,s) g(s, B_2 x(s)) dW(s)
$$

for all $t \in \mathbb{R}^+$. It is clear that x is a square-mean asymptotically almost automorphic mild solution of Eq. (1.1). The proof is now completed.

□

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References

[1] P. Acquistapace, Evolution operators and strong solution of abstract linear parabolic equations, *Differential Integral Equations*, 1(1988), 433-457.

[2] P. Acquistapace, B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, *Rend. Sem. Mat. Univ. Padova*, 78(1987), 47-107.

[3] D. Bugajewski and G. M. N'Guérékata, On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces, *Nonlinear Anal.*, 59(2004), 1333-1345.

[4] P. Bezandry and T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, *Appl. Anal.*, 86(2007), 819-827.

[5] P. Bezandry and T. Diagana, Square-mean almost periodic solutions to nonautonomous stochastic differential equations, *Electron. J. Differertial Equations*, 2007, 1-10.

[6] Y. K. Chang, Z. H. Zhao, G. M. N'Guérékata, Square-mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces, *Comput. Math. Appl.,* 61(2011), 384-391.

[7] Y. K. Chang, Z. H. Zhao, G. M. N'Gu'er'ekata, Square-mean almost automorphic mild solutions to some stochastic differential equations in Hilbert space, *Advances in Difference Equations*, 2011, 2011:9.

[8] Y. K. Chang, Z. H. Zhao, G. M. N'Guérékata and R. Ma, Stepanov-like almost automorphic for stochastic processes and applications to stochastic differential equations, *Nonlinear Anal. RWA*, 12(2011), 1130-1139.

[9] Y. K. Chang, Z. H. Zhao and G. M. N'Guérékata, A new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations, *Nonlinear Anal.*, 74(2011), 2210-2219.

[10] Z. Chen, W. Lin, Square-mean pseudo almost automorphic process and its application to stochastic evolution equations, *J. Funct. Anal.,* 261(2011), 69-89.

[11] H. S. Ding, T. J. Xiao and J. Liang, Asymptotically almost automorphic solutions for some integral-differential equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, 338(2008), 141-151.

[12] T. Diagana, E. Hernández and J. P. C. dos Santos, Existence of asymptotically almost automorphic solutions to some abstract partial neutral integral-differential equations, *Nonlinear Anal.*, 71(2009), 248-257.

[13] M. M. Fu and Z. X. Liu, Square-mean almost automorphic solutions for some stchastic differential equations, *Proc. Amer. Math. Soc.*, 138(2010), 3689-3701.

[14] G. M. N'Guérékata, Sur les solutions presqu'automorphes d''equations diff'erentielles abstraites, *Ann. Sci. Math. Qu'ebec*, 5(1981), 69-79.

[15] A. Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.,* 90(1982), 12-14.

[16] Z. H. Zhao, Y. K. Chang and J. J. Nieto, square-mean asymptotically almost automorphic process and its application to stochastic integro-differential equations, *Dynam. Syst. Appl.,* 22(2013), 269-284.

[17] Z. H. Zhao, Y. K. Chang and J. J. Nieto, Asymptotic behavior of solutions to abstract stochastic fractional partial integrodifferential equations, *Abstr. Appl. Anal.,* Vol. 2013, Article ID 138068, 8 pages, 2013.

[18] Z. H. Zhao, Y. K. Chang and J. J. Nieto, Almost automorphic solutions to some stochastic functional differential equations with delay, *Afr. Diaspora J. Math.,* 15(2013), 7-25.

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