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Distributive Lattice: A Rough Set Approach

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Abstract

This paper studies the distributive lattice under the rough set environment and thereby forms a concept of rough distributive lattice. We have discussed the properties of lattice theory in the approximation space and defined rough lattice, rough sublattice and complete rough lattice. We have defined approximation space by using an equivalence relation and then present rough set as a pair of two ordinary sets namely lower and upper approximation sets in the approximation space. The objective of this paper is to study the lattice theory based on rough set by using indiscernibility relation. Some important result are proved. Finally we cite some examples to illustrate the definitions and theories.

Keywords: Distributive Lattice, Rough Set, Approximation Space, Rough Lattice.

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1 Introduction

The concept of rough set was introduced by Pawlak [1] which is a mathematical tool for systematic study of incomplete knowledge. The basic notions of rough set include rough set approximations and information systems. In Pawlak's rough set the approximations are constructed by equivalence classes of an equivalence relation. For every rough set we associate two crisp sets, called lower and upper approximation sets and viewed as the set of elements which certainly and possibly belong to the set. J. Pomykala and J. A. Pomykala [7] showed that set of rough sets form a stone algebra. Davey and Priestly [4] introduced the concept of lattice theory and order. Iwinski [2] defined rough lattice and order without using any concept of indiscernibility of rough set. Rana and Roy [8] introduced rough set approach on lattice. Most of these are motivated to form lattice by inducing some order relation on rough sets. That is rough set is used as an example of lattice. But rough set is the generalization of ordinary set and therefore one of the main directions of the paper is to study the algebraic structure (lattice) based on rough set.

2 Basic Concept and Definitions

In this section, we give some basic properties and definitions related to lattice under the light of rough set environment which will be needed in the following sections. Let ρ be an equivalence relation defined over a set U.

Definition 2.1. *An equivalence class of $x(\in U)$ is denoted by $[x]_\rho$ and defined as follows: $[x]_\rho = \{y \in U : (x, y) \in \rho\}$.*

Definition 2.2. *The sets $A_\star(X) = \{x \in U : [x]_\rho \subseteq X\}$ and $A^\star(X) = \{x \in U : [x]_\rho \cap X \neq \phi\}$ are respectively called lower and upper approximations of $X \subseteq U$. The pair $S = (U, \rho)$ is called an approximation space and the pair $(A_\star(X), A^\star(X))$ is called the rough set of X in S and is denoted by $A(X)$. The difference $B(X) = A^\star(X) - A_\star(X)$ is called boundary region of X and treated as the area of uncertainty.*

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Definition 2.3. *The cartesian product of two rough sets* $A(X) = (A_\star(X), A^\star(X))$ *and* $A(Y) = (A_\star(Y), A^\star(Y)$ *is denoted by* $A(X) \times A(Y)$ *and defined as follows:*
$A(X) \times A(Y) = \{(x, y) : x \in A^\star(X)$ *and* $y \in A^\star(Y)\}$

Definition 2.4. *A rough set* $A(Y)$ *is said to be rough subset of a rough set* $A(X)$ *if* $A_\star(Y) \subseteq A_\star(X)$ *and* $A^\star(Y) \subseteq A^\star(X)$ *and it is denoted by* $A(Y) \subseteq A(X)$.

Definition 2.5. *[3] A poset* $(L, \leq)$ *is called a meet-semilattice if for all* $a, b \in L$, $Inf\{a, b\}$ *exists. The definition of join-semilattice is dual one.*

Definition 2.6. *[3] A non-empty set L together with two binary operations '*$\vee$*' and '*$\wedge$*' is said to form a lattice if* $\forall a, b, c \in L$, *the following conditions hold:*
1. $a \wedge a = a,\ \ a \vee a = a$ *(Idempotency)*
2. $a \wedge b = b \wedge a,\ \ a \vee b = b \vee a$ *(Commutativity)*
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c,\ \ a \vee (b \vee c) = (a \vee b) \vee c$ *(Associativity)*
4. $a \wedge (a \vee b) = a,\ \ a \vee (a \wedge b) = a$ *(Absorption).*

Definition 2.7. *[3] A lattice L is said to be complete lattice if for every non-empty subset X of L has a least upper bound and greatest lower bound in X.*

Definition 2.8. *[3] A lattice L is said to be modular lattice if* $\forall x, y, z \in L$ *with* $x \geq y$ *such that* $x \wedge (y \vee z) = y \vee (x \wedge z)$.

Definition 2.9. *[3] A lattice L is said to be distributive lattice if* $\forall x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Now we state two well known lemmas without proof.

Lemma 2.1. *A sublattice of a distributive lattice is distributive.*

Lemma 2.2. *A distributive lattice is always modular.*

2.1 Rough Lattice

In this section we present rough lattice and some properties of them based on Pawlak's notion of roughness. Let $< L, \vee, \wedge >$ be a lattice with two binary operations '$\vee$' and '$\wedge$' defined over L and also let $S = (L, \rho)$ be an approximation space. Let $X \subseteq U$ and $A(X) = (A_\star(X), A^\star(X))$ be the rough set of X in S.

Definition 2.10. $A(X)$ *is said to be rough join semi-lattice if* $x \vee y \in A^\star(X)$, $\forall\, x, y \in X$.
$A(X)$ *is said to be rough meet semi-lattice if* $x \wedge y \in A^\star(X)$, $\forall x, y \in X$.

Definition 2.11. $A(X)$ *is said to be rough lattice in* $S[= (L, \rho)]$ *with respect to the operations '*$\vee$*' and '*$\wedge$*' if* $\forall\, x, y \in X$
(i) $x \vee y \in A^\star(X)$
(ii) $x \wedge y \in A^\star(X)$

A rough lattice $< A(X), \vee, \wedge >$ satisfy the following properties $\forall x, y, z \in X$:
(i) $x \vee x = x$ and $x \wedge x = x$ (Idempotency)
(ii) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (Commutativity)
(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (Associativity)
(iv) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (Absorption)
(v) $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$ (Consistency)

Proposition 2.1. *If* $A(X) = (A_\star(X), A^\star(X))$ *is a rough lattice in an approximation space* $S[= (L, \rho)]$ *such that* $A^\star(X) = X$, *then* $A^\star(X)$ *is a sublattice of L.*

Proof. Since $A(X) = (A_\star(X), A^\star(X))$ is a rough lattice in the approximation space $S[= (L, \rho)]$, therefore clearly, $\forall x, y \in A^\star(X)$, $x \vee y \in A^\star(X)$ and $x \wedge y \in A^\star(X)$. Hence the proved. □

Proposition 2.2. *If L is a distributive lattice and* $A(X)$ *is a rough lattice in* $S[= (L, \rho)]$ *such that* $A^\star(X) = X$ *then* $A^\star(X)$ *is a distributive lattice.*

Proof. Since $A^\star(X) = X$ and $A(X)$ is a rough lattice, therefore by **Proposition** 2.1, $A^\star(X)$ is a sublattice of L and hence by **Lemma 2.1**, $A^\star(X)$ is a distributive lattice. □

Definition 2.12. *A rough subset $A(Y)$ of a rough lattice $A(X)$ in an approximation space $S[= (L, \rho)]$ is said to be rough sublattice if $A(Y)$ itself form a rough lattice with respect to the same operations.*

Definition 2.13. *A rough lattice $A(X)$ under an approximation space $S = (L, \rho)$ is said to be a complete rough lattice if every non-empty subset of X has least upper bound and greatest lower bound in $A^\star(X)$.*

Proposition 2.3. *A rough lattice $A(X)$ under an approximation space $S[= (L, \rho)]$ is complete rough lattice if $A^\star(X)$ is a complete sublattice of L.*

Proof. Let $A^\star(X)$ is a complete sublattice of L. Then every non-empty subset of $A^\star(X)$ has a least upper bound and greatest lower bound in $A^\star(X)$. Since X is a non-empty subset of $A^\star(X)$, therefore X has a least upper bound and greatest lower bound in $A^\star(X)$. Hence $A(X)$ is a complete rough lattice. □

Definition 2.14. *Let $< A(X), \vee, \wedge >$ is a rough lattice under an approximation space $S[= (L, \rho)]$, then $< A(X), \vee, \wedge >$ is said to be rough modular lattice (RML) if $\forall\ x, y, z \in A^\star(X)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$.*

2.2 Rough Distributive Lattice

Definition 2.15. *Let $< A(X), \vee, \wedge >$ is a rough lattice under an approximation space $S[= (L, \rho)]$, then $< A(X), \vee, \wedge >$ is said to be rough distributive lattice (RDL) if $\forall\ x, y, z \in A^\star(X)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

Example 2.1. *The set $L = \{1, 2, 4, 5, 10, 20\}$ of factors of 20 form a lattice where the operators '$\vee$' and '$\wedge$' are defined as $a \vee b$ =least common multiple of $\{a, b\}$ and $a \wedge b$ =greatest common divisor of $\{a, b\}$ and the order relation is divisibility. Let us consider an equivalence relation ρ on L by $x\rho y$ iff "x is congruent to y modulo 2" $\forall x, y \in L$. Let $X = \{2, 4\}$. Then $A_\star(X) = \phi$ and $A^\star(X) = \{2, 4, 10, 20\}$. Clearly $A(X)$ is rough lattice. Also $\forall x, y, z \in A^\star(x)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore $A(X)$ is a RDL.*

Proposition 2.4. *Rough sublattice of a RDL is RDL.*

Proof. Let $A(X)$ is a RDL and $A(Y)$ is a rough sublattice of $A(X)$. Therefore $A^\star(Y) \subseteq A^\star(X)$ and hence if $x, y, z \in A^\star(Y)$ then $x, y, z \in A^\star(X)$ and since $A(X)$ is RDL, therefore,$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore if $x, y, z \in A^\star(Y)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. □

Proposition 2.5. *If L is a distributive lattice and $A(X)$ is a rough lattice then $A(X)$ is a RDL.*

Proof. Since $A(X)$ is a rough lattice, $A^\star(X) \subseteq L$. Therefore $\forall x, y, z \in A^\star(X)$ imply, $x, y, z \in L$ and since L is distributive, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore $\forall x, y, z \in A^\star(X)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ i.e, $A(X)$ is a RDL. □

Proposition 2.6. *If $A(X)$ is a RDL in $S = (L, \rho)$ and if$A^\star(X) = X$ then X is distributive sublattice of L and vice-versa.*

Proof. Since $A(X)$ is RDL and $A^\star(X) = X$, therefore $\forall x, y, z \in A^\star(X) = X$,
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Also by **proposition 2.1** $A^\star(X)$ is sublattice of L and since sublattice of a distributive lattice is distributive, therefore $A^\star(X) = X$ is distributive sublattice of L.
Conversely, let $A^\star(X) = X$ is distributive sublattice of L. Therefore $A^\star(X)$ is rough lattice with $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z \in A^\star(X)$. So $A(X)$ is RDL. □

Proposition 2.7. *Two rough lattices $A(X)$ and $A(Y)$ are RDL iff $A(X) \times A(Y)$ is RDL.*

Proof. Let $A(X)$ and $A(Y)$ be RDL and also let $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$. Then $x_1, x_2, x_3 \in A^\star(X)$ and $y_1, y_2, y_3 \in A^\star(Y)$. Therefore

$$\begin{aligned}
(x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} &= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) \\
&= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3)) \\
&= (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\
&= \{(x_1, y_1) \wedge (x_2, y_2)\} \vee \{(x_1, y_1) \wedge (x_3, y_3).
\end{aligned}$$

Hence $A(X) \times A(Y)$ is RDL.
Conversely, let $A(X) \times A(Y)$ be RDL. Let $x_1, x_2, x_3 \in A^\star(X)$ and $y_1, y_2, y_3 \in A^\star(Y)$. Then $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^\star(X) \times A^\star(Y)$. As $A(X) \times A(Y)$ is RDL, therefore

$$\begin{aligned} & (x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} && = \{(x_1, y_1) \wedge (x_2, y_2)\} \vee \{(x_1, y_1) \wedge (x_3, y_3)\} \\ i.e \quad & (x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) && = (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3) \\ or, \quad & (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) && = ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2)g \vee (y_1 \wedge y_3)). \end{aligned}$$

Which gives $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = y_1 \wedge y_2) \vee (y_1 \wedge y_3)$. This imply $A(X)$ and $A(Y)$ are RDL. □

Proposition 2.8. *Every RDL is RML but converse is not true.*

Proof. Let $A(X)$ is a RDL. Therefore $x, y, z \in A^\star(Y)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. If $x \geq y$ and $x, y, z \in A^\star(Y)$, then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = y \vee (x \wedge z)$. Hence $A(X)$ is a RDL.
The converse is not true which is illustrated by the following example: □

Example 2.2. *Let $K_4 = \{e, a, b, c\}$ be the Klein's four group. Let L be the set of all subgroups of K_4. Then $L = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, K_4\}$. L forms a lattice under set inclusion and the operations '$\vee$' and '$\wedge$' are defined by $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, $\forall A, B \in L$. Let us consider an equivalence relation ρ on L defined by $A \rho B$ iff $O(A) = O(B)$ $\forall A, B \in L$. Let $X = \{\{e\}, \{e, a\}, \{K_4\}\}$. Then $A_\star(X) = \{\{e\}, \{K_4\}\}$ and $A^\star(X)) = \{\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, \{K_4\}\}$. Clearly, $A(X)$ is a rough modular lattice but $A(X)$ is not rough distributive lattice.*

3 Conclusion

In this paper the concept of rough distributive lattice is introduced based on Pawlak's indiscernibility relation. At first we construct rough lattice in an approximation space and then we study various properties of them compare to ordinary lattice. Our rough lattice(as we defined) is a rough set with two binary operations and it behaves in a lattice like manner within the rough boundary. As we defined RDL it is seen that the distributivity property of lattice is extended to the area of uncertainty. So we may use lattice structure when the elements of the set are not crisp.

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