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Asymptotic behavior of solutions to a nonautonomous semilinear evolution equation

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Abstract

In this paper, we shall deal with μ-pseudo almost automorphic solutions to the nonautonomous semilinear evolution equations: $u'(t) = A(t)u(t) + f(t, u(t-h))$, $t \in \mathbb{R}$ in a Banach space $\mathbb{X}$, where $A(t)$, $t \in \mathbb{R}$ generates an exponentially stable evolution family $\{U(t,s)\}$ and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a μ-pseudo almost automorphic function satisfying some suitable conditions. We obtain our main results by properties of μ-pseudo almost automorphic functions combined with theories of exponentially stable evolution family.

Keywords: μ-pseudo almost automorphic function, nonautonomous semilinear evolution equations, fixed point.

2010 MSC: 34K14, 60H10, 35B15, 34F05.

1 Introduction

This paper is mainly concerned with the existence of μ-pseudo almost automorphic mild solutions to the following nonautonomous semilinear evolution equations such as

$$u'(t) = A(t)u(t) + f(t, u(t-h)),\ t \in \mathbb{R}, \tag{1.1}$$

where $h \geq 0$ is a fixed constant, and $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace-Terreni* condition in [1].

The concept of almost automorphy was first introduced in the literature by Bochner [2, 3], it is an important generalization of the classical almost periodicity. For more details about this topic we refer the reader to [4–6]. Since then, there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [7]. Liang, Xiao and Zhang in [8, 9] presented the concept of pseudo almost automorphy. In [10], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [11], which generalizes that of pseudo-almost automorphic functions. Zhang et al. investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [12, 13]. Recently, Blot, Cieutat and Ezzinbi in [14] applied the measure theory to define an ergodic function and they investigate many powerful properties of μ-pseudo almost automorphic functions, and thus the classical theory of pseudo almost automorphy becomes a particular case of this approach.

In [15], the authors studied the existence and uniqueness of Stepanov-like almost solutions to Eq. (1.1). However, few results are available for μ-pseudo asymptotic behavior of solutions to the nonautonomous semilinear evolution equation (1.1). Inspired by the methods in [14, 15], the main aim of the present paper is

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to investigate μ-pseudo behavior of solutions to the problem (1.1). Some sufficient conditions are established via composition theorems of μ-pseudo almost automorphic functions combined with theories of exponentially stable evolution family.

The rest of this paper is organized as follows. In section 2, we introduce some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In section 3, we prove the existence of μ-pseudo almost automorphic mild solutions to the nonautonomous semilinear evolution equation (1.1).

2 Preliminary

In this section, we fix some basic definitions, notations, lemmas and preliminary facts which will be used in the sequel. Throughout the paper, the notation $(\mathbb{X}, \|\cdot\|)$ is a complex Banach space and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$.

Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}(a < b)$.

Definition 2.1. *[3] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.

Definition 2.2. *[16] A continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is said to be bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t,s) := \lim_{n\to\infty} f(t + s_n, s + s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n, s - s_n) = f(t,s)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Define

$$PAA_0(\mathbb{R}, \mathbb{X}) := \left\{ \phi \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

In the same way, we define by $PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ as the collection of jointly continuous functions $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ which belong to $BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma, x)\| d\sigma = 0$$

uniformly in compact subset of $\mathbb{X}$.

Definition 2.3. *[16, 17] A continuous function $f : \mathbb{R} \to \mathbb{X}$ (respectively $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$(respectively $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) and $\phi \in PAA_0(\mathbb{R}, \mathbb{X})$(respectively $PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$). Denote by $PAA(\mathbb{R}, \mathbb{X})$ (respectively $PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) the set of all such functions.*

Definition 2.4. *[14] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.5. *[14] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Obviously, we have $AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X})$.

Lemma 2.1. *[14, Proposition 2.13] Let $\mu \in \mathcal{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.2. *[14, Theorem 4.1] Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. If $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of f).*

Lemma 2.3. *[14, Theorem 2.14] Let $\mu \in \mathcal{M}$ and I be the bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent.*
(i) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.
(ii) $\lim_{r\to+\infty} \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0$.
(iii)For any $\varepsilon > 0$, $\lim_{r\to+\infty} \frac{\mu(\{t\in[-r,r]\setminus I : \|f(t)\|>\varepsilon\})}{\mu([-r,r]\setminus I)} = 0$.

Lemma 2.4. *[14, Theorem 4.7] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is unique.*

Lemma 2.5. *[14, Theorem 4.9] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Theorem 2.1. *[18] Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*
(a1) $f(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
(a2) $g(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ if $\phi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now, we recall a useful compactness criterion.

Let $h' : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h'(t) \geq 1$ for all $t \in \mathbb{R}$ and $h'(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_{h'}(\mathbb{X}) = \left\{ u \in C(\mathbb{R}, \mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h'(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_{h'} = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h'(t)}$, it is a Banach space (see [19, 20]).

Lemma 2.6. *[19, 20] A subset $R \subseteq C_{h'}(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:*
(c-1) The set $R(t) = \{u(t) : u \in R\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.
(c-2) The set R is equicontinuous.
(c-3) For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leq \varepsilon h'(t)$ for all $u \in R$ and all $|t| > L$.

Lemma 2.7. *[21] (Leray-Schauder Alternative Theorem) Let D be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

Theorem 2.2. *[22] Assume that $A(t)$, $t \in \mathbb{R}$ is a bounded linear operator on a Banach space $\mathbb{X}$ and $t \to A(t)$ is continuous in the uniform operator topology, then for $-\infty < s \leq t < \infty$, $U(t, s)$ generated by $A(t)$, is a bounded linear operator satisfying the following:*
(i) $\|U(t, s)\| \leq \exp(\int_s^t \|A(\tau)\| d\tau)$.
(ii) $U(t, t) = I$, $U(t, s) = U(t, r)U(r, s)$, for $-\infty < s \leq r \leq t < \infty$.
(iii) $(t, s) \to U(t, s)$ is continuous in the uniform operator topology for $-\infty < s \leq t < \infty$.
(iv) $\partial U(t, s)/\partial t = A(t)U(t, s)$ for $-\infty < s \leq t < \infty$.
(v) $\partial U(t, s)/\partial s = -U(t, s)A(s)$ for $-\infty < s \leq t < \infty$.

3 Main results

In this paper we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace-Terreni* conditions introduced in [1, 23], that is,
(H1) There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $\mathcal{L}, \mathcal{K} \geq 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \ \ \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{\mathcal{K}}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \mathcal{L}|t-s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| \leq \theta\}$.

Remark 3.1. *[1, 24] If the condition (H1) holds, then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$ on $\mathbb{X}$, which satisfies the homogeneous equation $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$.*

We further suppose that
(H2) The evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there are constants K, $\omega > 0$ such that $\|U(t,s)\| \leq Ke^{-\omega(t-s)}$ for all $t \geq s$, $U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R}, \mathbb{X})$ uniformly for all x in any bounded subset of $\mathbb{X}$.

Consider the following abstract differential equation in the Banach space $(\mathbb{X}, \|\cdot\|)$:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R} \tag{3.1}$$

where $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the condition (H1) and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Lemma 3.8. *Let $\mu \in \mathcal{M}$. Assume that $(H1)$ and $(H2)$ hold. Then the Eq. (3.1) has a unique μ-pseudo almost automorphic mild solution given by*

$$u(t) = \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma \tag{3.2}$$

Proof. First, it is conducted similarly as in the proof of [15, Theorem 3.2], we can prove the uniqueness of the μ-pseudo almost automorphic solution.

Now let us investigate the existence of the μ-pseudo almost automorphic solution. Since $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, there exist $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ such that $f = g + h$. So

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma \\ &= \int_{-\infty}^{t} U(t,\sigma)g(\sigma)d\sigma + \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma \\ &= \Phi(t) + \Psi(t), \end{aligned}$$

where $\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)g(\sigma)d\sigma$, and $\Psi(t) = \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma$. We just need to verify $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$ and $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. In view of [15, Theorem 3.2], we see that $\Phi(t) \in AA(\mathbb{R}, \mathbb{X})$.

Next, we prove that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. It is obvious that $\Psi(t) \in BC(\mathbb{R}, \mathbb{X})$, the left task is to show that

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\Psi(t)\| d\mu(t) = 0.$$

For $r > 0$, we notice that

$$\begin{aligned} & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\Psi(t)\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{-\infty}^{t} U(t,\sigma)h(\sigma)d\sigma \right\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{0}^{\infty} U(t,t-\sigma)h(t-\sigma)d\sigma \right\| d\mu(t) \\ \leq\ & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{0}^{\infty} \|U(t,t-\sigma)\| \|h(t-\sigma)\| d\sigma d\mu(t) \\ \leq\ & K \int_{0}^{\infty} e^{-\omega\sigma} \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t-\sigma)\| d\mu(t) \right) d\sigma \\ =\ & K \int_{0}^{\infty} e^{-\omega\sigma} \Omega_r(\sigma) d\sigma, \end{aligned}$$

where $\Omega_r(\sigma) = \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|h(t-\sigma)\|d\mu(t)$.

By the fact that the space $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ is translation invariant, it follows that $t \to h(t-\sigma)$ belongs to $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ for each $\sigma \in \mathbb{R}$ and hence $\Omega_r(\sigma) \to 0$ as $r \to +\infty$. Since Ω_r is bounded ($\|\Omega_r\| \le \|h\|_\infty$) and $e^{-\omega\sigma}$ is integrable in $[0,\infty)$, using the Lebesgue dominated convergence theorem it follows that $\lim_{r\to+\infty}\int_0^\infty e^{-\omega\sigma}\Omega_r(\sigma)d\sigma = 0$. We deduce that $\Psi(t) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Therefore $u(t) = \Phi(t) + \Psi(t)$ is μ-pseudo almost automorphic.

Finally, let us prove that $u(t)$ is a mild solution of the Eq. (3.1). Indeed, if we let

$$u(s) = \int_{-\infty}^{s} U(s,\sigma)f(\sigma)d\sigma, \tag{3.3}$$

and multiply both sides of (3.3) by $U(t,s)$, then

$$U(t,s)u(s) = \int_{-\infty}^{s} U(t,\sigma)f(\sigma)d\sigma.$$

If $t \ge s$, then

$$\begin{aligned}\int_s^t U(t,\sigma)f(\sigma)d\sigma &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)d\sigma - \int_{-\infty}^{s} U(t,\sigma)f(\sigma)d\sigma \\ &= u(t) - U(t,s)u(s).\end{aligned}$$

It follows that

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma)f(\sigma)d\sigma.$$

This completes the proof of the theorem. □

Since the space $\varepsilon(\mathbb{R},\mathbb{X},\mu)$ is translation invariant, we can easily obtain the following lemma.

Lemma 3.9. *If $u \in PAA(\mathbb{R},\mathbb{X},\mu)$ and $h \ge 0$, then $u(\cdot - h) \in PAA(\mathbb{R},\mathbb{X},\mu)$.*

Let us list the following basic assumptions:

(H3) $f \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ and there exists a constant $L_f > 0$, such that

$$\|f(t,x) - f(t,y)\| \le L_f\|x-y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(H4) The function $f = g + \varphi \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$, where $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly on $t \in \mathbb{R}$ and $\varphi \in \varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$.

The following theorems are the main results of this section.

Theorem 3.3. *Let $\mu \in \mathcal{M}$ and suppose that the conditions (H1)-(H3) are satisfied. Then Eq. (1.1) has a unique μ-pseudo almost automorphic mild solution on $\mathbb{R}$ and provided that $\frac{KL_f}{\omega} < 1$.*

Proof. We define the nonlinear operator $\Gamma : PAA(\mathbb{R},\mathbb{X},\mu) \to PAA(\mathbb{R},\mathbb{X},\mu)$ by

$$(\Gamma u)(t) := \int_{-\infty}^{t} U(t,s)f(s,u(s-h))ds, \quad t \in \mathbb{R}.$$

First, let us prove that $\Gamma(PAA(\mathbb{R},\mathbb{X},\mu)) \subset PAA(\mathbb{R},\mathbb{X},\mu)$. For each $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, by using the fact that the range of an almost automorphic function is relatively compact combined with the above Theorem 2.1 and Lemma 3.9, one can easily see that $f(\cdot,u(\cdot-h)) \in PAA(\mathbb{R},\mathbb{X},\mu)$. Hence, from the proof of Lemma 3.8, we know that $(\Gamma u)(\cdot) \in PAA(\mathbb{R},\mathbb{X},\mu)$. That is, Γ maps $PAA(\mathbb{R},\mathbb{X},\mu)$ into $PAA(\mathbb{R},\mathbb{X},\mu)$.

Now it suffices to show that the operator Γ has a unique fixed point in $PAA(\mathbb{R},\mathbb{X},\mu)$. For this, let u and v be in $PAA(\mathbb{R},\mathbb{X},\mu)$, we have

$$\begin{aligned}\|(\Gamma u)(t) - (\Gamma v)(t)\|_\infty &= \sup_{t\in\mathbb{R}}\left\|\int_{-\infty}^{t} U(t,s)[f(s,u(s-h)) - f(s,v(s-h))]ds\right\| \\ &\le K\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,u(s-h)) - f(s,v(s-h))\|ds \\ &\le KL_f\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\omega(t-s)}\|u(s-h) - v(s-h)\|ds \\ &\le KL_f\int_{-\infty}^{t} e^{-\omega(t-s)}ds\|u-v\|_\infty \\ &\le \frac{KL_f}{\omega}\|u-v\|_\infty.\end{aligned}$$

So $\|\Gamma u - \Gamma v\|_\infty \le \frac{KL_f}{\omega}\|u-v\|_\infty$. By the Banach contraction principle with $\frac{KL_f}{\omega} < 1$, Γ has a unique fixed point u in $PAA(\mathbb{R},\mathbb{X},\mu)$, which is the μ-pseudo almost automorphic solution to Eq. (1.1). The proof is complete. □

We next study the existence of μ-pseudo almost automorphic mild solutions of Eq. (1.1) when the perturbation f is not necessarily Lipschitz continuous. For that, we require the following assumptions:
(H5) $f \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ and $f(t,x)$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $M \subset \mathbb{X}$, $\{f(\cdot,x) : x \in M\}$ is bounded in $PAA(\mathbb{R},\mathbb{X},\mu)$.
(H6) There exists a continuous nondecreasing function $W : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x)\| \le W(\|x\|) \quad \text{for all} \quad t \in \mathbb{R} \ \text{ and } \ x \in \mathbb{X}.$$

Remark 3.2. *For condition (H6), an interesting results (see Corollary 3.1) is given for the perturbation f satisfying the Hölder type condition.*

Theorem 3.4. *Let $\mu \in \mathcal{M}$ and conditions (H1) and (H2) hold. Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be a function satisfying conditions (H4)-(H6) and the following additional conditions:*
(i) For each $z \ge 0$, the function $t \to \int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))\,ds$ belongs to $BC(\mathbb{R})$. We set

$$\beta(z) = K\left\|\int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))ds\right\|_{h'},$$

where K is the constant given in (H2).
(ii) For each $\varepsilon > 0$ there is $\delta > 0$ such that for every u, $v \in C_{h'}(\mathbb{X})$, $\|u-v\|_{h'} \le \delta$ implies that

$$\int_{-\infty}^t e^{-\omega(t-s)}\|f(s,u(s-h)) - f(s,v(s-h))\|ds \le \varepsilon,$$

for all $t \in \mathbb{R}$.
(iii) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.
(iv) For all a, $b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s,h'(s-h)x) : a \le s-h \le b, x \in C_{h'}(\mathbb{X}), \|x\|_{h'} \le z\}$ is relatively compact in $\mathbb{X}$.
Then Eq. (1.1) has at least one μ-pseudo almost automorphic mild solution.

Proof. We define the nonlinear operator $\Gamma : C_{h'}(\mathbb{X}) \to C_{h'}(\mathbb{X})$ by

$$(\Gamma u)(t) := \int_{-\infty}^t U(t,s)f(s,u(s-h))ds,\ t \in \mathbb{R}.$$

We will show that Γ has a fixed point in $PAA(\mathbb{R},\mathbb{X},\mu)$. For the sake of convenience, we divide the proof into several steps.

(I) For $u \in C_{h'}(\mathbb{X})$, we have that

$$\begin{aligned}\|(\Gamma u)(t)\| &\le K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u(s-h)\|)ds\\ &\le K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u\|_{h'}h'(s-h))ds.\end{aligned}$$

It follows from condition (i) that Γ is well defined.

(II) The operator Γ is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (ii). If u, $v \in C_{h'}(\mathbb{X})$ and $\|u-v\|_{h'} \le \delta$, then

$$\|(\Gamma u)(t) - (\Gamma v)(t)\| \le K\int_{-\infty}^t e^{-\omega(t-s)}\|f(s,u(s-h)) - f(s,v(s-h))\|ds \le \varepsilon,$$

which shows the assertion.

(III) We will show that Γ is completely continuous. We set $B_z(\mathbb{X})$ for the closed ball with center at 0 and radius z in the space $\mathbb{X}$. Let $V = \Gamma(B_z(C_{h'}(\mathbb{X})))$ and $v = \Gamma(u)$ for $u \in B_z(C_{h'}(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from condition (i) that the function $s \to Ke^{-\omega s}W(zh'(t-s-h))$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $a \geq 0$ such that $K\int_a^\infty e^{-\omega s}W(zh'(t-s-h))ds \leq \varepsilon$. Since

$$v(t) = \int_0^a U(t,t-s)f(t-s,u(t-s-h))ds + \int_a^\infty U(t,t-s)f(t-s,u(t-s-h))ds$$

and

$$\left\|\int_a^\infty U(t,t-s)f(t-s,u(t-s-h))ds\right\| \leq K\int_a^\infty e^{-\omega s}W(zh'(t-s-h))ds \leq \varepsilon,$$

we get $v(t) \in \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{U(t,t-s)f(\xi,h'(\xi-h)x) : 0 \leq s \leq a, t-a \leq \xi - h \leq t, \|x\|_{h'} \leq z\}$. Using the strong continuous of $U(t,s)$ and property (iv) of f, we infer that N is a relatively compact set, and $V(t) \subseteq \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(s) &= \int_0^s U(t,t-\sigma)f(t+s-\sigma,u(t+s-h-\sigma))d\sigma \\ &\quad + \int_0^a [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma \\ &\quad + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} &\left\|\int_0^s U(t,t-\sigma)f(t+s-\sigma,u(t+s-h-\sigma))d\sigma \right. \\ &\left. + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))d\sigma\right\| \\ \leq\ & K\int_0^s e^{-\omega\sigma}W(zh'(t+s-h-\sigma))d\sigma + K\int_a^\infty (e^{-\omega(\sigma+s)} + e^{-\omega\sigma})W(zh'(t-h-\sigma))d\sigma \\ \leq\ & \frac{\varepsilon}{2} \end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma,u(t-h-\sigma)) : 0 \leq \sigma - h \leq a, x \in B_z(C_{h'}(\mathbb{X}))\}$ is a relatively compact set and $U(t,s)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[U(t,t-\sigma-s) - U(t,t-\sigma)]f(t-\sigma,u(t-h-\sigma))\| \leq \frac{\varepsilon}{2a}$ for $s \leq \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \leq \varepsilon$ for s small enough and independent of $u \in B_z(C_{h'}(\mathbb{X}))$.

Finally, applying condition (i), it is easy to see that

$$\frac{\|v(t)\|}{h'(t)} \leq \frac{K}{h'(t)}\int_{-\infty}^t e^{-\omega(t-s)}W(zh'(s-h))ds \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $x \in B_z(C_{h'}(\mathbb{X}))$. Hence, by Lemma 2.6, V is a relatively compact set in $(C_{h'}(\mathbb{X}))$.

(IV) Let us now assume that $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned} \left\|u^\lambda(t)\right\| &= \lambda\left\|\int_{-\infty}^t U(t,s)f(s,u^\lambda(s-h))\right\| \\ &\leq K\int_{-\infty}^t e^{-\omega(t-s)}W(\|u^\lambda\|_{h'}h'(s-h))ds \\ &\leq \beta(\|u^\lambda\|_{h'})h'(t). \end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_{h'}}{\beta(\|u^\lambda\|_{h'})} \leq 1$$

and combining with condition (iii), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(V) It follows from assumption (H5), Theorem 2.1 and Lemma 3.9 that the function $t \to f(t,u(t-h))$ belongs to $PAA(\mathbb{R},\mathbb{X},\mu)$, whenever $u \in PAA(\mathbb{R},\mathbb{X},\mu)$. Moreover, from Lemma 3.8 we infer that $\Gamma(PAA(\mathbb{R},\mathbb{X},\mu)) \subset PAA(\mathbb{R},\mathbb{X},\mu)$ and noting that $PAA(\mathbb{R},\mathbb{X},\mu)$ is a closed subspace of $C_{h'}(\mathbb{X})$, consequently, we can consider $\Gamma : PAA(\mathbb{R},\mathbb{X},\mu) \to PAA(\mathbb{R},\mathbb{X},\mu)$. Using properties (I)-(III), we deduce that this map is completely continuous. Applying Lemma 2.7 we infer that Γ has a fixed point $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, which completes the proof. □

Corollary 3.1. *Let $\mu \in \mathcal{M}$. Assume that conditions (H1)-(H2) hold. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfying assumption (H4)-(H5) and the Hölder type condition*

$$\|f(t,u) - f(t,v)\| \leq \gamma \|u - v\|^{\alpha},\ 0 < \alpha < 1,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $\gamma > 0$ is a constant. Moreover, assume the following conditions are satisfied:
(a) $f(t,0) = q$.
(b) $\sup_{t\in\mathbb{R}} K \int_{-\infty}^{t} e^{-\omega(t-s)} h'(s-h)^{\alpha} ds = \gamma_2 < \infty$.
(c) For all $a, b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s, h'(s-h)x) : a \leq s - h \leq b, x \in C_{h'}(\mathbb{X}), \|x\|_{h'} \leq z\}$ is relatively compact in $\mathbb{X}$.
Then Eq.(1.1) has a μ-pseudo almost automorphic mild solution.

Proof. Let $\gamma_0 = \|q\|$, $\gamma_1 = \gamma$. We take $W(\xi) = \gamma_0 + \gamma_1 \xi^{\alpha}$. Then condition (H6) is satisfied. It follows from (b), we can see that function f satisfies (i) in Theorem 3.4. To verify condition (ii), note that for each $\varepsilon > 0$ there is $0 < \delta^{\alpha} < \frac{\varepsilon}{\gamma_1 \gamma_2}$ such that for every $u, v \in C_{h'}(\mathbb{X})$, $\|u - v\|_{h'} \leq \delta$ implies that $K \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s, u(s-h)) - f(s, v(s-h))\| ds \leq \varepsilon$ for all $t \in \mathbb{R}$. On the other hand, the hypothesis (iii) in the statement of Theorem 3.4 can be easily verified using the definition of W. This completes the proof. □

Example 3.1. *Consider the following problem:*

$$\begin{cases} \frac{\partial u}{\partial t} u(t,x) = \frac{\partial^2 u}{\partial^2 x} u(t,x) + u(t,x) \sin \frac{1}{2+\cos t + \cos \sqrt{2} t} + f(t, u(t-h, x)), \\ u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{3.4}$$

where $h > 0$, $\mathbb{X} = L^2(0,1)$, and

$D(B) := \{x \in C^1[0,1]; x'$ *is absolutely continuous on* $[0,1], x'' \in \mathbb{X}, x(0) = x(1) = 0\}$, $Bx(r) = x''(r)$, $r \in (0,1)$, $x \in D(B)$.

Then B generates a C_0-semigroup $T(t)$ on $\mathbb{X}$ given by

$$(T(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} < x, e_n >_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2} \sin n\pi r$, $n = 1, 2, \cdots$. Moreover, $\|T(t)\| \leq e^{-\pi^2 t}$, $t \geq 0$.
Define a family of linear operators $A_1(t)$ by

$$\begin{cases} D(A_1(t)) = D(B), \ \ t \in \mathbb{R} \\ A_1(t)x = \left(B + \sin \frac{1}{2+\cos t + \cos \sqrt{2} t}\right) x, \quad x \in D(A_1(t)). \end{cases}$$

Then, $\{A_1(t), t \in \mathbb{R}\}$ generates an evolution family $\{U_1(t,s)\}_{t \geq s}$ such that

$$U_1(t,s)x = T(t-s) e^{\int_s^t \sin \frac{1}{2+\cos t + \cos \sqrt{2} t} d\tau} x.$$

Hence

$$\|U_1(t,s)\| \leq e^{-(\pi^2 - 1)(t-s)}, \ \ t \geq s.$$

It is easy to see that $U_1(t,s)$ satisfies conditions (H1)-(H2) with $K = 1$, $\omega = \pi^2 - 1$.
Set

$$f(t,u) = u \sin \frac{1}{\cos^2 t + \cos^2 \pi t} + \max_{k \in \mathbb{Z}} \{e^{-(t \pm k^2)^2}\} \sin u, \ \ t \in \mathbb{R}.$$

According to [16, 17], f clearly satisfies conditions (H3) and (H4). From Theorem 3.3, the problem (3.4) has a unique μ-pseudo almost automorphic mild solution.

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