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New representation of a fuzzy set

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Abstract

A new representation of a fuzzy set is introduced. Moreover, the decomposition theorem for the new representation is proved. Fuzzy number is defined using this definition and some properties are established.

Keywords: New representation of a fuzzy set, decomposition theorem, fuzzy number.

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1 Introduction

Fuzzy sets have been introduced by LoftiA.Zadeh(1965)[5]. Fuzzy set theory permits the gradual assessment of the membership of elements in a set which is described in the interval[0,1]. It can be used in a wide range of domains where information is incomplete and imprecise.

The representation of fuzzy set in terms of their α-cuts was introduced by Zadeh(1971)[6] in the form of the decomposition theorem. The extension principal is an important tool by which classical mathematical theories can be fuzzified.

The concept of fuzzy numbers have been introduced by Chang and Zadeh(1972)[1].The thory of fuzzy numbers has been studied by Fuller, Majlender[3] and Fodor, Bede[2].

In this paper, some new representation of a fuzzy set is introduced based on α-cut and then some related theorems are proved.

2 Preliminaries

Definition 2.1. *Let X be a universe of discourse, then a fuzzy set is defined as* $A = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, *which is characterized by a membership function* $\mu_A(x) : X \to [0,1]$, *where* $\mu_A(x)$ *denotes the degree of membership of the element x to the set A.*

Definition 2.2. *The* α*-cut* ${}^{\alpha}A$ *of a set A is the crisp subset of A with membership grades of at least* α. *So,* ${}^{\alpha}A = \{x|\mu_A(x) \geq \alpha\}$.

Definition 2.3. *Define for each* $x \in X$, *a fuzzy set* ${}_{\alpha}A(x) = \alpha.{}^{\alpha}A(x)$, *where* ${}^{\alpha}A$ *is the* α*- cut of the fuzzy set A.*

Definition 2.4. *A fuzzy set* $\tilde{A}$ *is convex if* $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$, *for* $x_1, x_2 \in X, \lambda \in [0,1]$ *Alternatively, a fuzzy set is convex if all* α*-level sets are convex.*

Definition 2.5. *A fuzzy subset A of a classical set X is called normal if there exists an* $x \in X$ *such that* $A(x) = 1$. *Otherwise A is subnormal.*

Definition 2.6. *The support of a set A is the crisp subset of A with nonzero membership grades. So,* $\sup(A) = \{x|\mu_A(x) > 0\}$.

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Definition 2.7. *A fuzzy number A must possess the following three properties:*

1 . *A must be a normal fuzzy set,*

2 . *The alpha levels $^{\alpha}A$ must be closed for every $\alpha \in (0,1]$.*

3 . *The support of A, ^{0+}A, must be bounded.*

3 New representation of a fuzzy set

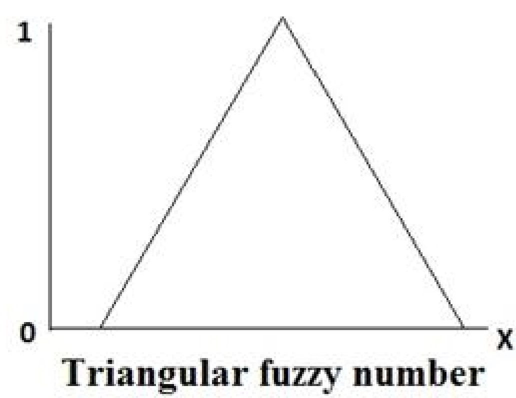
Definition 3.8. *(Power Level Fuzzy Set)*

For a fuzzy set A, $\alpha \in (0,1]$ define a fuzzy set $A^{(\alpha)}$, for each $x \in X$, as follows.
$A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_{\alpha} A(x) \quad and \quad x \in^{\alpha} A \quad for \quad \alpha \in [0,1]\}$
where $_{\alpha}A(x) = \alpha.^{\alpha}A$ and $^{\alpha}A$ is the α-cut of the fuzzy set A.

Theorem 3.1. *(Decomposition theorem for the new representation)*

For every $A \in \mathcal{F}(X)$, $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$ where $A^{(\alpha)} = \sum \alpha_i / x$, where $\alpha_i \geq_{\alpha} A(x)$ and $x \in^{\alpha} A$.

Proof. If $A(X) = a$, then choose $\alpha = a$, for $x \in^{\alpha} A$ and for all $\alpha_i \geq A(x)$,
Clearly $a \in A^{(\alpha)}$, hence, $A \subseteq \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$
Suppose $\gamma' \in \bigcup_{\alpha \in [0,1]} A^{(\alpha)}, \gamma' \in A(x_\gamma) = \gamma$
$\gamma' \in A^{(\alpha)}$ for some α where $\gamma' = (x_\gamma, \gamma)$ then $x_\gamma \in^{\alpha} A$ and $A(x_\gamma) \leq \alpha_i$
Therefore $A(x_\gamma) \geq \alpha$ and $\gamma \leq \alpha_i$ and so $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$ □



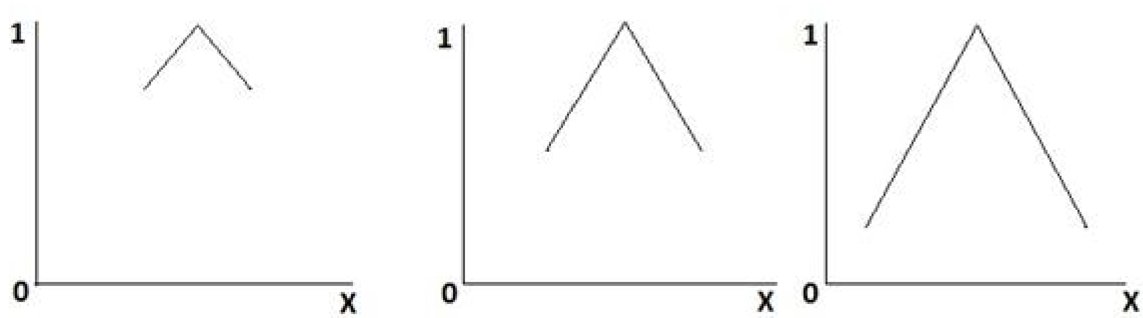


Figure 1: Various $A^{(\alpha)}$ for $\alpha \in (0,1]$

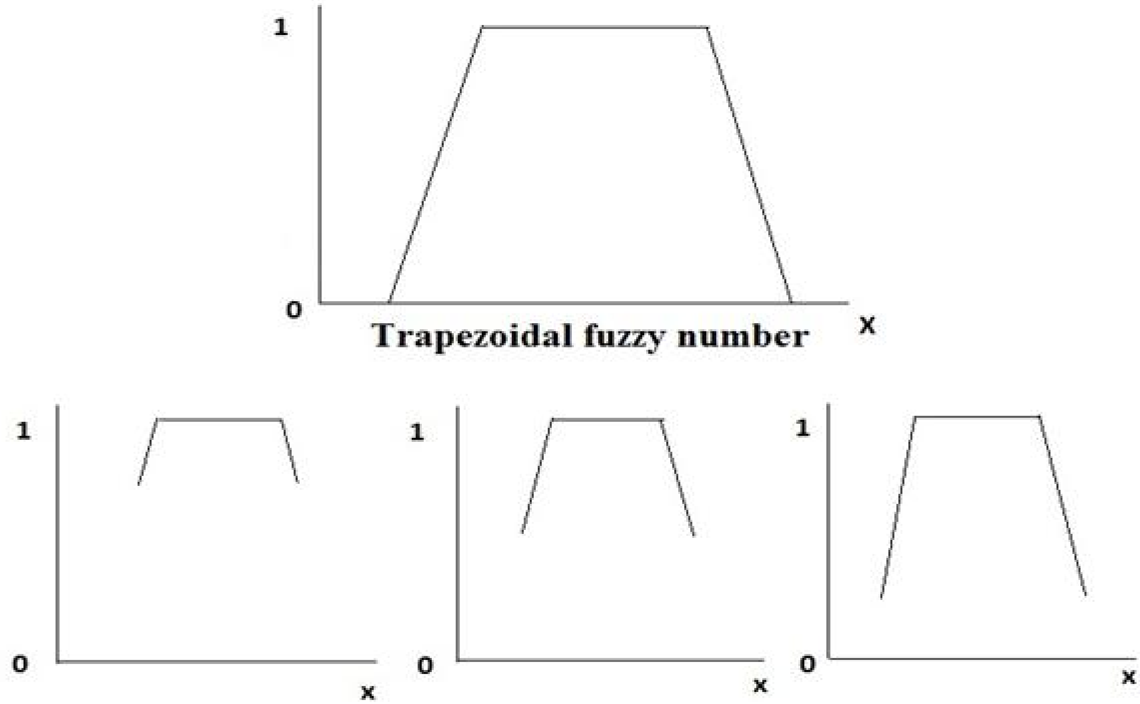


Figure 2: Various $A^{(\alpha)}$ for $\alpha \in (0,1]$

3.1 Extension Principle

Any given function $f : X \to Y$ induces two functions, $f : \mathcal{F}(X) \to \mathcal{F}(Y), f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$, which are defined by $[f(A)](Y)sup_{x|y=f(x)}A(x)$ for all $A \in \mathcal{F}(X)$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in \mathcal{F}(X)$.

Theorem 3.2. *Let $f : X \to Y$ be an arbitrary crisp function, for any $A \in \mathcal{F}(X)$ and all $\alpha \in [0,1]$ the following property of f fuzzified by extension principle satisfies the equation $[f(A)]^{(\alpha+)} = f(A^{(\alpha+)})$.*

Proof. For all $y \in Y$,

$$\begin{aligned} y \in [f(A)]^{(\alpha+)} &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(y)] \quad \text{and} \quad y \in^\alpha [f(A)] \quad \text{for} \quad \alpha \in [0,1] \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and} \quad f(A)(y) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and} \quad \sup_{x|y=f(x)} A(x) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and there exist} \quad x_0 \in X \quad \text{with} \quad y = f(x_0) \\ &\qquad \text{and} \quad A(x_0) \geq \alpha \\ &\Leftrightarrow \alpha_i \geq_\alpha [f(A)(Y)] \quad \text{and there exist} \quad x_0 \in X \quad \text{with} \quad y = f(x_0) \\ &\qquad \text{and} \quad x_0 \in^{\alpha+} A \end{aligned}$$

hence $[f(A)]^{(\alpha+)} = f(A^{(\alpha+)})$.

□

Theorem 3.3. *Let $f : X \to Y$ be an arbitrary crisp function. Then for any $A \in \mathcal{F}(X)$,f fuzzified by the extension principle satisfies the equation, $f(A) = \bigcup_{\alpha \in [0,1]} f(A^{(\alpha+)})$.*

Proof. Applying theorem(3.1) to f(A), which is a fuzzy set on Y,
we obtain $f(A) = \bigcup_{\alpha \in [0,1]} [f(A)]^{(\alpha+)}$

by theorem(3.2)

$$[f(A)]^{(\alpha+)} = f(A^{(\alpha+)}),$$

$$\text{Hence,} \quad f(A) = \bigcup_{\alpha\in[0,1]} f(A^{(\alpha+)}).$$

□

Definition 3.9. *Define $A^{c(\alpha)}$,for each $x \in X$ as $A^{c(\alpha)} = \{(x, \alpha_i) | \alpha_i <_\alpha A(x)$ and $x \in^\alpha A^c\}$. where ${}^\alpha A^c = \{x | \mu_A(x) \geq 1 - \alpha\}$.*

Theorem 3.4. *For every $A \in \mathcal{F}(X)$,then $A^c = \bigcup_{\alpha\in[0,1]} A^{c(\alpha)}$*

Proof. If $A^{c(\alpha)} = a$, then choose $\alpha = a$, for $x \in^\alpha A^c$, then for all $\alpha_i <_a A(x)$,where $a \in A^{c(\alpha)}$ hence,

$$A^c \subseteq \bigcup_{\alpha\in[0,1]} A^{c(\alpha)} \tag{1}$$

Conversely, suppose $x \in \bigcup_{\alpha\in[0,1]} A^{c(\alpha)}$
then $x \in A^{c(\alpha)}$ for some $\alpha \in [0,1]$ this is true when $x \notin A^{(\alpha)}$ which means that $A^c(x) \leq \alpha$ and $\alpha_i <_\alpha A(x)$, by the definition, we get $x \in A^c$ and hence

$$A^c \supseteq \bigcup_{\alpha\in[0,1]} A^{c(\alpha)} \tag{2}$$

From (1) and (2) we have $A^c = \bigcup_{\alpha\in[0,1]} A^{c(\alpha)}$. □

Definition 3.10.

$$A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_\alpha A(x) \quad and \quad x \in^\alpha A \quad for \quad \alpha \in (0,1]\}$$

$A^{(\alpha)}$ is convex if and only if ${}_\alpha A$ and ${}^\alpha A$ are convex for any $\alpha \in (0,1]$.

Theorem 3.5. *A fuzzy set $A^{(\alpha)}$ is convex if and only if $A^{(\alpha)}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A^{(\alpha)}(x_1), A^{(\alpha)}(x_2)\}$ for all $x_1, x_2 \in \Re$ and all $\lambda \in [0,1]$.*

Proof. Assume that $A^{(\alpha)}$ is convex if and only if ${}_\alpha A$ and ${}^\alpha A$ are convex for any $\alpha \in (0,1]$.
By definition, $A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_\alpha A(x)$ and $x \in^\alpha A$ for $\alpha \in (0,1]\}$.
Let $x_1^{(1)} = (x_1, \alpha_{1i})$ and $x_2^{(1)} = (x_2, \alpha_{2i}) \in A^{(\alpha)}$,
to show that $\lambda(x_1, \alpha_{1i}) + (1-\lambda)(x_2, \alpha_{2i})$ belongs to $A^{(\alpha)}$
We have to prove that $\lambda x_1 + (1-\lambda)x_2 \in A^{(\alpha)}$ and $\left(\frac{\alpha_1 i}{\lambda}\right) + \left(\frac{\alpha_2 i}{1-\lambda}\right) \geq_\alpha A$
as $(x_1, \alpha_{1i}), (x_2, \alpha_{2i})$, by definition $x_1, x_2 \in^\alpha A$ and $\alpha_{1i} \geq_\alpha A(x)$ and $\alpha_{2i} \geq_\alpha A(x)$
If $x_1, x_2 \in^\alpha A$, then $A(x_1) \geq \alpha$ and $A(x_2) \geq \alpha$ and for any $\lambda \in [0,1]$, by the given condition

$$A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}, A^{(\alpha)}(x_2^{(1)})\} \tag{1}$$

Consider $\lambda(x_1, \alpha_{1i}) + (1-\lambda)(x_2, \alpha_{2i}) = \left(\lambda x_1 + (1-\lambda)x_1, \frac{\alpha_{1i}}{(\lambda)} + \frac{\alpha_{2i}}{(1-\lambda)}\right)$
From(1), we get $A(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A(x_1^{(1)}), A(x_2^{(1)})\} \geq \min\{\alpha, \alpha\} = \alpha$
Therefore, $\lambda x_1 + (1-\lambda)x_2 \in^\alpha A$, since $\alpha_{1i} \geq_\alpha A(x)$ and $\alpha_{2i} \geq_\alpha A(x)$

From(1),

$$A^{(\alpha)}\big(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}\big) \geq \min\{\alpha_{1i}, \alpha_{2i}\} \tag{2}$$

Suppose $\alpha_{1i} \leq \alpha_{2i}$, then(2) becomes
$\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq \alpha_{1i}$
and so,$\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq \alpha_{1i} \geq_{\alpha} A(x)$
hence $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in A^{(\alpha)}$
Let $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in A^{(\alpha)}$
then $\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)} \in^{\alpha} A$ and $\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) \geq_{\alpha} A(x)$
to prove $A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}), A^{(\alpha)}(x_2^{(1)})\}$
We already prove that $A(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A(x_1^{(1)}), A(x_2^{(1)})\} \geq \min\{\alpha, \alpha\} = \alpha$
Now let $\alpha_{1i} \leq \alpha_{2i}$, we get

$$\begin{aligned}\left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) &\geq \left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{1i}}{1-\lambda}\right)\\ &\geq \left(\frac{(1-\lambda)\alpha_{1i} + \lambda\alpha_{1i}}{\lambda(1-\lambda)}\right)\\ &= \left(\frac{\alpha_{1i}}{\lambda(1-\lambda)}\right)\\ \left(\frac{\alpha_{1i}}{\lambda}\right) + \left(\frac{\alpha_{2i}}{1-\lambda}\right) &\geq_{\alpha} A(x)\end{aligned}$$

hence,
$A^{(\alpha)}(\lambda x_1^{(1)} + (1-\lambda)x_2^{(1)}) \geq \min\{A^{(\alpha)}(x_1^{(1)}), A^{(\alpha)}(x_2^{(1)})\}$. □

Definition 3.11. *A fuzzy set A on $\Re$ is a fuzzy number if,*

(i) *$A^{(\alpha)}$ is a normal fuzzy set,*

(ii) *α-cut of $\{A^{(\alpha)}|\alpha \in (0,1]\}$ must be nested sequence of closed intervals,*

(iii) *Support of $A^{(\alpha)}$ is bounded.*

Definition 3.12. *The fuzzy set $A^{(\alpha)}$ is normal if $supA^{(\alpha)}(x) = 1$ for $x \in X$.*

Definition 3.13. *By definition $A^{(\alpha)} = \{(x, \alpha_i)|\alpha_i \geq_{\alpha} A(x)$ and $x \in^{\alpha} A$ for $\alpha \in [0,1]\}$,then α-cut of $A^{(\alpha)}$ consists of all elements (x, α_i) such that $A^{(\alpha)}(x, \alpha_i) \geq \alpha$.*

Theorem 3.6. *$A \in \mathcal{F}(\Re)$ is a fuzzy number if and only if for each $\alpha \in (0,1]$*

$$A^{(\alpha)}(x) = \begin{cases} 1 \quad for \quad x \in [0,1] \\ l^{\alpha}(x) \quad for \quad x \in (-\infty, a) \\ r^{\alpha}(x) \quad for \quad x \in (b, +\infty) \end{cases}$$

l^{α} is a monotonically increasing function from $(-\infty, a)$ to $[0,1]$ such that $l^{\alpha}(x) = 0$ for $x \in (-\infty, a)$ and r^{α} is a monotonically decreasing function from $(b, +\infty)$ to $[0,1]$ such that $r^{\alpha}(x) = 0$ for $x \in (b, +\infty)$.

Proof. Necessity. Suppose A is a fuzzy number then $A = \bigcup_{\alpha \in [0,1]} A^{(\alpha)}$
and we obtain a sequence $A^{(\alpha)}$ of fuzzy set for each $\alpha \in [0,1]$.
So, $A^{(\alpha)}$ is a normal fuzzy set and sup $A^{(\alpha)}(x) = 1$ for $x \in X$
by definition, α-cut of $\{A^{(\alpha)}|\alpha \in (0,1]\}$ is a nested sequence of closed intervals
we know that, $A^{(\alpha)} = \{(x, \alpha_i)|\alpha_i \geq_{\alpha} A(x)$ and $x \in^{\alpha} A$ for $\alpha \in [0,1]\}$ and its α-cut, ${}^{\alpha}A(\alpha)$ is a closed interval, for $\alpha = 1$,

Consider ${}^{1}A^{(\alpha)} = \{x | A^{(\alpha)}(x) \geq 1\}$ and so, $\alpha_i \geq_1 A(x)$ with $x \in^1 A$ for $\alpha \in (0,1]$
for $x \in^1 A^{(\alpha)}, A(x) \geq 1$ with $\alpha_i \geq A(x)$, choose ${}^{1}A^{(\alpha)} = [a,b]$
define, $l^{\alpha}(x) = A^{\alpha}(x)$ for $(-\infty, a)$, then $0 \leq l^{\alpha}(x) < 1$
Corresponding to each $\alpha \in (0,1]$, there exists a sequence $\{l^{(\alpha)}\}$ of functions from $(-\infty, a)$ to $[0,1]$
Similarly we obtain a sequence $\{r^{(\alpha)}\}$ of functions from $(b, +\infty)$ to $[0,1]$ for each $\alpha \in (0,1]$.
Sufficiency. Every fuzzy set $A^{(\alpha)}$ defined by (1) is clearly normal and support of $A^{(\alpha)}$ is bounded. Finally it remains to prove that α-cut of $A^{(\alpha)}$, for $\alpha \in (0,1]$ is a closed interval.
By definition, $A^{(\alpha)} = \{(x, \alpha_i) | \alpha_i \geq_\alpha A(x)$ and $x \in^\alpha A$ for $\alpha \in [0,1]\}$

$$\begin{aligned} Let \quad l_{x_\alpha}^{(\alpha)} &= \inf\{(x,\alpha_i)|^{\alpha} l^{\alpha}(x) \geq \alpha, x < a \quad \text{and} \quad \alpha_i \geq_\alpha l^{\alpha}(x)\} \\ l_{y_\alpha}^{(\alpha)} &= \sup\{(x,\alpha_i)|^{\alpha} r^{\alpha}(x) \geq \alpha, y > b \quad \text{and} \quad \alpha_i \geq_\alpha r^{\alpha}(x)\} \end{aligned}$$

we have

$$\begin{aligned} l_{x\alpha}^{(\alpha)} &= \left(l_{x_\alpha}, \alpha_{l_{x\alpha}}\right) \\ l_{y\alpha}^{(\alpha)} &= \left(l_{y_\alpha}, \alpha_{l_{y\alpha}}\right) \end{aligned}$$

To prove that $[l_{x_\alpha}, l_{y_\alpha}]$ and $[\alpha_{l_{x\alpha}}, \alpha l_{y\alpha}]$ are closed intervals.
If$\left(x_0, \alpha_{x0,i}\right)$ belongs to α-cut of $A^{(\alpha)}$ and if $x_0 < a$
then $l^{(\alpha)}(x_0) = A^{(\alpha)}(x_0)$ and so $x_0 \in^\alpha l^{(\alpha)}$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x)$
$\Rightarrow x_0 \in^\alpha A^{(\alpha)}$ and $\alpha_i \geq_\alpha A^{(\alpha)}$
$\Rightarrow x_0 \in l^{(\alpha)}(x_\alpha)$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x)$
$\Rightarrow \left(x_0, \alpha_{x_{0},i}\right) \geq l_{x_\alpha}^{(\alpha)}$
if $\left(x_0, \alpha_{x_{0},i}\right)$ belongs to α-cut of $A^{(\alpha)}$ and if $x_0 > b$
then $r^{(\alpha)}(x_0) = A^{(\alpha)}(x_0)$ and so $x_0 \in^\alpha r^\alpha$ and $\alpha_i \geq_\alpha l^{(\alpha)}$
$\Rightarrow x_0 \in^\alpha A^{(\alpha)}$ and $\alpha_i \geq_\alpha A^{(\alpha)}$
$\Rightarrow x_0 \in l^{(\alpha)}(y_\alpha)$ and $\alpha_i \geq_\alpha l^{(\alpha)}(y)$
$\Rightarrow \left(x_0, \alpha_{x_{0},i}\right) \leq l_{ya}^{(\alpha)}$
Therefore, $x_0 \in [l_{x_\alpha}^{(\alpha)}, l_{y_\alpha}^{(\alpha)}]$ and hence α-cut of $A^{(\alpha)}$ is a subset of $[l_{x_\alpha}^{(\alpha)}, l_{y_\alpha}^{(\alpha)}]$
by the definition of $l_{x_\alpha}^{(\alpha)}$ there must exist a sequence $\left(x_n, \alpha_{i,n}\right)$ in
$\{(x,\alpha_i)|\alpha_i \geq_\alpha A(x)$ and $x \in^\alpha A$ for $\alpha \in [0,1]\}$ such that $\lim\limits_{n\to\infty} \left(x_n, \alpha_{i,n}\right) = l_{x_\alpha}^{(\alpha)}$ where $x_n \geq x_\alpha$ for any α, since $l^{(\alpha)}$ is right continuous,$l(l_{x_\alpha}^{(\alpha)}) = l\left(\lim\limits_{n\to\infty}(x_n, \alpha_{i,n})\right) = \lim\limits_{n\to\infty} l(x_n, \alpha_{i,n})$
Then ${}^{\alpha}l^{(\alpha)}(x_n) \geq \alpha$ and $\alpha_i \geq_\alpha l^{(\alpha)}(x_n)$ and so $l_{x_\alpha}^{(\alpha)}$ lies in the α-cut of $A^{(\alpha)}$
Hence both the necessary and sufficient part of the theorem is proved. □

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