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On Hermite-Hadamard type integral inequalities for functions whose second derivative are nonconvex

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Abstract

In this paper, we extend some estimates of the right hand side of a Hermite- Hadamard type inequality for nonconvex functions whose second derivatives absolute values are φ-convex, log-φ-convex, and quasi-φ-convex.

Keywords: Hermite-Hadamard's inequalities, φ-convex functions, log-φ-convex, quasi-φ-convex, Hölder's inequality.

2010 MSC: 26D07, 26D10, 26D99.

1 Introduction

It is well known that if f is a convex function on the interval $I = [a, b]$ and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

which is known as the Hermite-Hadamard inequality for the convex functions. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[4], [10]-[18]).

The following lemma was proved for twice differentiable mappings in [3]:

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a, b \in I$ with $a < b$ and f'' of integrable on $[a, b]$, the following equality holds:*

$$\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\,f(ta+(1-t)b)\,dt.$$

A simple proof of this equality can be also done by twice integrating by parts in the right hand side.

In [4], by using Lemma 1.1, Hussain et al. proved some inequalities related to Hermite-Hadamard's inequality for s-convex functions:

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Theorem 1.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If $|f''|$ is $s-$convex on $[a,b]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2 \times 6^{\frac{1}{p}}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{(s+2)(s+3)} \right]^{\frac{1}{q}}, \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1.1. *If we take $s = 1$ in (1.2), then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}.$$

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [10]).

Alomari, Darus and Dragomir in [1] introduced the following theorems for twice differentiable quasiconvex functions:

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a,b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|$ is quasiconvex on $[a,b]$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)^2}{12} \max\left\{ \left|f''(a)\right|, \left|f''(b)\right| \right\}.$$

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a,b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|^{\frac{p}{p-1}}$ is a quasiconvex on $[a,b]$, for $p > 1$, then the following inequality holds*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\ \leq\ & \frac{(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{p}} \left(\max\left\{ \left|f''(a)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, $a,b \in I^o$ with $a < b$ and f'' is integrable on $[a,b]$. If $|f''|^q$ is a quasiconvex on $[a,b]$, for $q \geq 1$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)^2}{12} \left(\max\left\{ \left|f''(a)\right|^q, \left|f''(b)\right|^q \right\} \right)^{\frac{1}{q}}.$$

2 Preliminaries

Let $f, \varphi : K \to \mathbb{R}$, where K is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. First of all, we recall the following well known results and concepts, which are mainly due to Noor and Noor [5] and Noor [9] as follows:

Definition 2.1. *Let $u, v \in K$. Then the set K is said to be $\varphi-$convex at u with respect to φ, if*

$$u + te^{i\varphi}(v-u) \in K,\ \forall u,v \in K,\ t \in [0,1].$$

Remark 2.2. *We would like to mention that Definition 2.1 of a φ-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points, $u, v \in K$, then $e^{i\varphi}(v-u) = v-u$ if and only if, $\varphi = 0$, and consequently φ-convexity reduces to convexity. Thus, it is true that every convex set is also an φ-convex set, but the converse is not necessarily true, see [5]-[9] and the references therein.*

Definition 2.2. *The function f on the φ-convex set K is said to be φ-convex with respect to φ, if*

$$f\left(u + te^{i\varphi}(v-u)\right) \leq (1-t) f(u) + tf(v), \ \forall u, v \in K, \ t \in [0,1].$$

The function f is said to be φ-concave if and only if $-f$ is φ-convex. Note that every convex function is a φ-convex function, but the converse is not true.

Definition 2.3. *The function f on the φ-convex set K is said to be logarithmic φ-convex with respect to φ, such that*

$$f\left(u + te^{i\varphi}(v-u)\right) \leq (f(u))^{1-t}(f(v))^{t}, \ u, v \in K, \ t \in [0,1]$$

where $f(.) > 0$.

Now we define a new definition for quasi-φ-convex functions as follows:

Definition 2.4. *The function f on the quasi φ-convex set K is said to be quasi φ-convex with respect to φ, if*

$$f\left(u + te^{i\varphi}(v-u)\right) \leq \max\{f(u), f(v)\}.$$

From the above definitions, we have

$$\begin{aligned} f\left(u + te^{i\varphi}(v-u)\right) &\leq (f(u))^{1-t}(f(v))^{t} \\ &\leq (1-t) f(u) + tf(v) \\ &\leq \max\{f(u), f(v)\}. \end{aligned}$$

Clearly, any φ-convex function is a quasi φ-convex function. Furthermore, there exist quasi φ-convex functions which are neither φ-convex nor continuous. For example, for

$$\varphi(v,u) = \begin{cases} 2k\pi, & u.v \geq 0, \ k \in \mathbb{Z} \\ k\pi, & u.v < 0, \ k \in \mathbb{Z} \end{cases}$$

the floor function $f_{loor}(x) = \lfloor x \rfloor$, is the largest integer not greater than x, is an example of a monotonic increasing function which is quasi φ-convex but it is neither φ-convex nor continuous.

In [7], Noor proved the Hermite-Hadamard inequality for the $\varphi-$convex functions as follows:

Theorem 2.5. *Let $f : K = \left[a, a + e^{i\varphi}(b-a)\right] \to (0, \infty)$ be a φ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then the following inequality holds:*

$$\begin{aligned} f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) &\leq \frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)\,dx \\ &\leq \frac{f(a) + f\left(a + e^{i\varphi}(b-a)\right)}{2} \leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{2.3}$$

This inequality can easily show that using the φ-convex function's definition and $f\left(a + e^{i\varphi}(b-a)\right) < f(b)$.

In [19] and [20], the authors proved some generalization inequalities connected with Hermite-Hadamard's inequality for diferentiable φ-convex functions.

In this article, using functions whose second derivatives absolute values are φ-convex, log-φ-convex and quasi-φ-convex, we obtained new inequalities related to the right side of Hermite-Hadamard inequality given with (2.3).

3 Hermite-Hadamard Type Inequalities

We will start the following theorem:

Theorem 3.6. *Let $K \subset \mathbb{R}$ be an open interval, $a, a+e^{i\varphi}(b-a) \in K$ with $a < b$ and $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ a twice differentiable mapping such that f'' is integrable and $0 \le \varphi \le \frac{\pi}{2}$. If $|f''|$ is φ-convex function on $\left[a, a+e^{i\varphi}(b-a)\right]$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\ \le\ & \frac{e^{2i\varphi}(b-a)^2}{24}\left[|f''(a)|+|f''(b)|\right].\end{aligned}$$

Proof. If the partial integration method is applied twice, then it follows that

$$\begin{aligned}&\frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)f''(a+te^{i\varphi}(b-a))dt \qquad (3.4)\\ =\ & \frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}.\end{aligned}$$

Thus, by φ-convexity function of $|f''|$, we have

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\ \le\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left|\int_0^1 (t-t^2)f''(a+te^{i\varphi}(b-a))dt\right| \\ \le\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left[(1-t)|f''(a)|+t|f''(b)|\right]dt \\ \le\ & \frac{e^{2i\varphi}(b-a)^2}{24}\left[|f''(a)|+|f''(b)|\right]\end{aligned}$$

which the proof is completed. □

Theorem 3.7. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a twice differentiable mapping on K^0 and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f''|^{p/p-1}$ is φ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a+e^{i\varphi}(b-a)$ and $0 \le \varphi \le \frac{\pi}{2}$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\ \le\ & \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}+|f''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.\end{aligned}$$

Proof. By assumption, Hölder's inequality and (3.4), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 \left|t-t^2\right|\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)\left|f''(a)\right|^{\frac{p}{p-1}} + t\left|f''(b)\right|^{\frac{p}{p-1}}\right]dt\right)^{\frac{p-1}{p}} \\
=\ & \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}+|f''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}
\end{aligned}$$

where we use the fact that

$$\int_0^1 (t-t^2)^p dt = \frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}$$

which completes the proof. □

Let us denote by $A(a,b)$ the arithmetic mean of the nonnegative real numbers, and by $L(a,b)$ the logaritmic mean of the same numbers.

Theorem 3.8. *Let $K \subset \mathbb{R}$ be an open interval, $a, a+e^{i\varphi}(b-a) \in K$ with $a < b$ and $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ a twice differentiable mapping such that f'' is integrable and $0 \leq \varphi \leq \frac{\pi}{2}$. If $|f''|$ is log φ-convex function on $\left[a, a+e^{i\varphi}(b-a)\right]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \left(\frac{e^{i\varphi}(b-a)}{\log|f''(b)|-\log|f''(a)|}\right)^2\left[A\left(\left|f''(b)\right|,\left|f''(a)\right|\right) - L\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right].
\end{aligned}$$

Proof. By using (3.4) and log φ-convexity of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left(\left|f''(a)\right|^{1-t}\left|f''(b)\right|^{t}\right)dt \\
=\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left[\frac{|f''(b)|+|f''(a)|}{(\log|f''(b)|-\log|f''(a)|)^2} - \frac{2\left(|f''(b)|-|f''(a)|\right)}{(\log|f''(b)|-\log|f''(a)|)^3}\right] \\
=\ & \left(\frac{e^{i\varphi}(b-a)}{\log|f''(b)|-\log|f''(a)|}\right)^2\left[A\left(\left|f''(b)\right|,\left|f''(a)\right|\right) - L\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right].
\end{aligned}$$

The proof of Theorem 3.8 is completed. □

Theorem 3.9. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a twice differentiable mapping on K^o and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f''|^{p/p-1}$ is log φ-convex function on the interval of real numbers*

K^o (the interior of K) and $a, b \in K^o$ with $a < a + e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then, the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}-|f''(b)|^{\frac{p}{p-1}}}{\log|f''(b)|-\log|f''(a)|}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Proof. By using (3.4) and the well known Hölder's integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1 |f''(a)|^{\frac{p}{p-1}(1-t)}|f''(b)|^{\frac{p}{p-1}t}dt\right)^{\frac{p-1}{p}} \\
=\ & \frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{|f''(a)|^{\frac{p}{p-1}}-|f''(b)|^{\frac{p}{p-1}}}{\log|f''(b)|-\log|f''(a)|}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

□

Theorem 3.10. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a differentiable mapping on K^0 and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. If $|f''|$ is a quasi φ-convex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a+e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{4}\max\{|f'(a)|, |f'(b)|\}.
\end{aligned}$$

Proof. By using (3.4) and the quasi φ-convexity of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1 (t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{2}\max\{|f''(a)|, |f''(b)|\}\int_0^1 (t-t^2)dt \\
\leq\ & \frac{e^{2i\varphi}(b-a)^2}{24}\max\{|f''(a)|, |f''(b)|\}.
\end{aligned}$$

□

Theorem 3.11. *Let $f : K = \left[a, a+e^{i\varphi}(b-a)\right] \to (0,\infty)$ be a differentiable mapping on $K^o$and f'' be integrable on $\left[a, a+e^{i\varphi}(b-a)\right]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f''|^{p/p-1}$ is a quasi φ-convex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a+e^{i\varphi}(b-a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then, the following inequality* *holds:*

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\max\{|f''(a)|^{\frac{p}{p-1}},|f''(b)|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}.\end{aligned}$$

Proof. By using (3.4) and the well known Hölder's integral inequality, we get

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1(t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt\\ \leq\ &\frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1(t-t^2)^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p-1}{p}}dt\right)^{\frac{p}{p-1}}\\ \leq\ &\frac{e^{i\varphi}(b-a)}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1\max\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\}dt\right)^{\frac{p}{p-1}}\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\max\{|f''(a)|^{\frac{p}{p-1}},|f''(b)|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}.\end{aligned}$$

□

Theorem 3.12. *Let* $f:K=\left[a,a+e^{i\varphi}(b-a)\right]\to(0,\infty)$ *be a differentiable mapping on* K^o*and* f'' *be integrable on* $\left[a,a+e^{i\varphi}(b-a)\right]$*. Assume* $q\in\mathbb{R}$ *with* $q\geq 1$*. If* $|f''|^q$ *is a quasi* φ*-convex function on the interval of real numbers* K^o *(the interior of K) and* $a,b\in K^o$ *with* $a<a+e^{i\varphi}(b-a)$ *and* $0\leq\varphi\leq\frac{\pi}{2}$*. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{12}\left[\max\{|f''(a)|^q,|f''(b)|^q\}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. By using (1.1) and the well known power mean integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)}f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right|\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{2}\int_0^1(t-t^2)\left|f''(a+te^{i\varphi}(b-a))\right|dt\\ \leq\ &\frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1(t-t^2)dt\right)^{\frac{1}{p}}\left(\int_0^1(t-t^2)\left|f'(a+te^{i\varphi}(b-a))\right|^qdt\right)^{\frac{1}{q}}\\ \leq\ &\frac{e^{i\varphi}(b-a)}{2}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left(\max\{|f'(a)|^q,|f'(b)|^q\}\int_0^1(t-t^2)dt\right)^{\frac{1}{q}}\\ \leq\ &\frac{e^{2i\varphi}(b-a)^2}{12}\left[\max\{|f''(a)|^q,|f''(b)|^q\}\right]^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. □

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