Malaya Journal of Matematik

www.malayajournal.org

$\mathcal{MJM}$
an international journal of mathematical sciences with computer applications...

Comments on Jensen's Inequalities

Zlatko Pavić [a,*]

[a]Mechanical Engineering Faculty in Slavonski Brod, University of Osijek, Trg Ivane Brlić Mažuranić 2,35000 Slavonski Brod, Croatia.

Abstract

The paper gives generalizations of some Jensen type inequalities for convex functions of one variable. The work is based on the methods which use convex combinations in deriving inequalities. The main inequality is applied to the quasi-arithmetic means.

Keywords: Affine combination, convex combination, convex function, Jensen's inequality.

1 Introduction

1.1 Affine and Convex Combinations

The concept of affine and convex combinations refers to the sets of vectors. Through the paper we will only use the combinations

$$c = \sum_{i=1}^{n} p_i x_i \qquad (1.1)$$

of the points $x_i \in \mathbb{R}$ and the coefficients $p_i \in \mathbb{R}$. A combination in (1.1) is affine if $\sum_{i=1}^{n} p_i = 1$. A combination in (1.1) is convex if all $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. The point c itself is called the combination center. If $\mathcal{I} \subseteq \mathbb{R}$ is an interval, then any convex combination of the points $x_i \in \mathcal{I}$ belongs to the interval $\mathcal{I}$.

1.2 Affine and Convex Functions

A function $f : \mathbb{R} \to \mathbb{R}$ which is represented by the equation $f(x) = kx + l$ where k and l are real constants is affine, and it verifies the equality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = \sum_{i=1}^{n} p_i f(x_i) \qquad (1.2)$$

for all affine combinations $\sum_{i=1}^{n} p_i x_i$ from $\mathbb{R}$. A function $f : \mathcal{I} \to \mathbb{R}$ which satisfies the inequality $f(px + qy) \leq pf(x) + qf(y)$ for all binomial convex combinations $px + qy$ from $\mathcal{I}$ is convex, and it verifies the equality or inequality in (1.2) for all convex combinations $\sum_{i=1}^{n} p_i x_i$ from $\mathcal{I}$.

1.3 Recent Results

Theorem 1.1. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\sum_{i=1}^{n} p_i x_i$ be a convex combination from $[a, b]$.*

---

*Corresponding author.

E-mail address:* Zlatko.Pavic@sfsb.hr (Zlatko Pavić).

Then every convex function $f : [a,b] \to \mathbb{R}$ verifies the inequality

$$
\begin{aligned}
2f\left(\frac{a+b}{2}\right) - \sum_{i=1}^{n} p_i f(x_i) \quad &\leq \quad f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) \\
&\leq \quad f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i).
\end{aligned}
\tag{1.3}
$$

Theorem 1.2. *Let $[a,b] \subset \mathbb{R}$ and $[c,d] \subset \mathbb{R}$ be bounded closed intervals where $a < b$ and $c < d$. Let $p : [c,d] \to \mathbb{R}$ be a non-negative continuous function with $\int_c^d p(x)\,dx > 0$, and $g : [c,d] \to [a,b]$ be a continuous function.*

Then every convex function $f : [a,b] \to \mathbb{R}$ verifies the inequality

$$
\begin{aligned}
2f\left(\frac{a+b}{2}\right) - \frac{\int_c^d p(x)f(g(x))\,dx}{\int_c^d p(x)\,dx} \quad &\leq \quad f\left(a+b-\frac{\int_c^d p(x)g(x)\,dx}{\int_c^d p(x)\,dx}\right) \\
&\leq \quad f(a) + f(b) - \frac{\int_c^d p(x)f(g(x))\,dx}{\int_c^d p(x)\,dx}.
\end{aligned}
\tag{1.4}
$$

The right-hand side of the inequality in (1.3) was obtained in [3]. The left-hand side of the inequality in (1.3), and the inequality in (1.4) were obtained in [2]. Some new Jensen type inequalities have been recently derived in [4].

2 Three Methods of Deriving Convex Function Inequalities

2.1 Basic Method Using Affinity

If $a, b \in \mathbb{R}$ are different numbers, say $a < b$, then every number $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$
x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b.
\tag{2.1}
$$

The above binomial combination is convex if, and only if, the number x belongs to the interval $[a,b]$. Given the function $f : \mathbb{R} \to \mathbb{R}$, let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. Applying the affinity of $f_{\{a,b\}}^{\text{line}}$ to the combination in (2.1), we get

$$
f_{\{a,b\}}^{\text{line}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b).
\tag{2.2}
$$

Assume that the function f is convex. Applying its convexity to the combination in (2.1) and connecting it with the equation in (2.2), we get the basic inequalities of convex functions:

Lemma 2.1. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$.*

Then every convex function $f : \mathbb{R} \to \mathbb{R}$ verifies the inequality

$$
f(x) \leq f_{\{a,b\}}^{line}(x) \quad \text{if } x \in [a,b],
\tag{2.3}
$$

and the reverse inequality

$$
f(x) \geq f_{\{a,b\}}^{line}(x) \quad \text{if } x \notin (a,b).
\tag{2.4}
$$

If f is concave, then the reverse inequalities are valid in (2.3) and (2.4).

2.2 Discrete Method Using Common Center

The following lemma deals with two convex combinations of the same center (one convex combination with two "sub-combinations" has been studied in [5, Proposition 2]). Applying a convex function on such convex combinations, we obtain the Jensen type inequality:

Lemma 2.2. *Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, and $a, b \in \mathcal{I}$ be points such that $a \leq b$. Let $\sum_{i=1}^{n} p_i x_i$ be the convex combination with points $x_i \in [a, b]$. Let $\sum_{j=1}^{m} q_j y_j$ be the convex combination with points $y_j \in \mathcal{I} \setminus (a, b)$.*

If the convex combination center equality

$$\sum_{i=1}^{n} p_i x_i = \sum_{j=1}^{m} q_j y_j \tag{2.5}$$

is satisfied, then every convex function $f : \mathcal{I} \to \mathbb{R}$ verifies the inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leq \sum_{j=1}^{m} q_j f(y_j). \tag{2.6}$$

If f is concave, then the reverse inequality is valid in (2.6).

Proof. Prove the convexity case. If $a < b$, relying on the convexity of f and the affinity of $f_{\{a,b\}}^{\text{line}}$, we get the series of inequalities

$$
\begin{aligned}
\sum_{i=1}^{n} p_i f(x_i) &\leq \sum_{i=1}^{n} p_i f_{\{a,b\}}^{\text{line}}(x_i) = f_{\{a,b\}}^{\text{line}} \left(\sum_{i=1}^{n} p_i x_i \right) \\
&= f_{\{a,b\}}^{\text{line}} \left(\sum_{j=1}^{m} q_j y_j \right) = \sum_{j=1}^{m} q_j f_{\{a,b\}}^{\text{line}}(y_j) \\
&\leq \sum_{j=1}^{m} q_j f(y_j)
\end{aligned}
$$

derived applying the inequality in (2.3) to x_i, and the inequality in (2.4) to y_j. If $a = b$, we use any support line $f_{\{a\}}^{\text{line}}$ instead of the chord line $f_{\{a,b\}}^{\text{line}}$. $\qquad \square$

Remark 2.1. *Lemma 2.2 is the generalization of Jensen's inequality. Applying the lemma to the convex combination center equality*

$$1c = \sum_{i=1}^{n} p_i x_i, \tag{2.7}$$

with the assumption $a = b = c$, we come to the Jensen inequality

$$f \left(\sum_{i=1}^{n} p_i x_i \right) = 1 f(c) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{2.8}$$

Respecting the Jensen inequality and our purposes in the main section, we give the following consequence of Lemma 2.2:

Corollary 2.1. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\sum_{i=1}^{n} p_i x_i$ be a convex combination from $[a, b]$.*

If the convex combination center equality

$$\sum_{i=1}^{n} p_i x_i = \alpha a + \beta b \tag{2.9}$$

is satisfied, then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} p_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{2.10}$$

If f is concave, then the reverse inequality is valid in (2.10).

Let us show the immediate application of the above corollary. Rewrite the inequality in (1.3) of Theorem 1.1 in the form

$$f \left(\frac{a + b}{2} \right) \leq \frac{1}{2} f \left(a + b - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \frac{p_i}{2} f(x_i) \leq \frac{f(a) + f(b)}{2}, \tag{2.11}$$

and observe the the convex combination center equality

$$\frac{1}{2}\left(a + b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{p_i}{2} x_i = \frac{1}{2}a + \frac{1}{2}b. \tag{2.12}$$

The middle member in (2.12) is the $(n+1)$-membered convex combination of the points $\bar{x}_1 = a + b - \sum_{i=1}^{n} p_i x_i$ and $\bar{x}_{i+1} = x_i$ from $[a,b]$ with the coefficients $\bar{p}_1 = 1/2$ and $\bar{p}_{i+1} = p_i/2$, including all $i = 1,\ldots,n$. The right member in (2.12) is the two-membered convex combination, in fact the arithmetic center, of the endpoints a and b. So, we can apply the inequality in (2.10) of Corollary 2.1 to the equality in (2.12) to obtain the inequality in (2.11).

2.3 Integral Method Using Convex Combinations

Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $f : [a,b] \to \mathbb{R}$ be the Riemann integrable function. Given the positive integer n, let

$$[a,b] = \bigcup_{i=1}^{n} [a_{ni}, b_{ni}] \tag{2.13}$$

where $a = a_{n1}$, $a_{ni} < b_{ni} = a_{n\,i+1}$ for $i = 1,\ldots,n-1$ and $a_{nn} < b_{nn} = b$. It is assumed that every interval of the above union contracts to the point as n approaches infinity. Take one point $x_{ni} \in [a_{ni}, b_{ni}]$ for every index $i = 1,\ldots,n$. Then the limit of the sequence $(c_n)_n$ of the convex combination centers

$$c_n = \sum_{i=1}^{n} \frac{b_{ni} - a_{ni}}{b - a} f(x_{ni}), \tag{2.14}$$

as n approaches infinity, is the point

$$\frac{1}{b-a} \int_a^b f(x)\, dx.$$

As an application of the above procedure, insert the points $x_i = x_{ni}$ and the convex combination coefficients $p_i = (b_{ni} - a_{ni})/(b - a)$ in the inequality in (2.11). Letting n to infinity, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}f\left(a + b - \frac{1}{b-a}\int_a^b x\, dx\right) + \frac{1}{2(b-a)}\int_a^b f(x)\, dx \le \frac{f(a)+f(b)}{2}$$

which after arranging and using $(a+b)/2 = a + b - (a+b)/2$, gives the inequality

$$f\left(a + b - \frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\, dx \le f(a) + f(b) - f\left(\frac{a+b}{2}\right). \tag{2.15}$$

The integral method with convex combinations in deriving some variants of the known inequalities has been applied in [6].

3 Main Results

Lemma 3.3. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a \le b$, and $x_i \in [a,b]$ be points. Let $\alpha, \beta, p_i \in [0,1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

Then the affine combination

$$\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i \tag{3.1}$$

belongs to the interval $[a,b]$.

Proof. Take $\gamma = \sum_{i=1}^{n} p_i$, so $\alpha + \beta - \gamma = 1$ by the assumption. Note that $\gamma \leq \alpha$ and $\gamma \leq \beta$. In the case $\gamma = 0$, the combination in (3.1) is reduced to the convex combination $\alpha a + \beta b \in [a, b]$.

If $\gamma > 0$, then the convex combination $\sum_{i=1}^{n} (p_i/\gamma) x_i \in [a, b]$, so it is consequently equal to the binomial convex combination $\alpha_1 a + \beta_1 b$. In this case, we have

$$\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i = \alpha a + \beta b - \gamma(\alpha_1 a + \beta_1 b)$$
$$= (\alpha - \gamma\alpha_1)a + (\beta - \gamma\beta_1)b$$
$$= \alpha_2 a + \beta_2 b,$$

where the coefficients $\alpha_2 = \alpha - \gamma\alpha_1 \geq \alpha - \gamma \geq 0$ and $\beta_2 = \beta - \gamma\beta_1 \geq \beta - \gamma \geq 0$, and their sum $\alpha_2 + \beta_2 = \alpha + \beta - \gamma(\alpha_1 + \beta_1) = 1$. $\qquad\square$

Assigning the convex function to the affine combinations of the above lemma, our main result reads as follows:

Theorem 3.3. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $x_i \in [a, b]$ be points. Let $\alpha, \beta, p_i \in [0, 1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

Then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$f\left(\frac{\alpha a + \beta b}{\alpha + \beta}\right) \leq \frac{1}{\alpha + \beta}\left[f\left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(x_i)\right]$$
$$\leq \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta}. \tag{3.2}$$

Proof. Briefly, since the convex combination center equality

$$\frac{1}{\alpha + \beta}\left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{p_i}{\alpha + \beta} x_i = \frac{\alpha a + \beta b}{\alpha + \beta} \tag{3.3}$$

is satisfied, we can apply the inequality in (2.10) of Corollary 2.1 to obtain the inequality in (3.2). Namely, the middle member in (3.3) should be taken as the $(n+1)$-membered convex combination from $[a, b]$, and similarly the right member as the two-membered convex combination of the endpoints. $\qquad\square$

The inequality in (3.2) with $\alpha = \beta = 1$ reduces to the inequality in (1.3). By application the integral method with convex combinations the inequality in (3.2) can be transferred to integrals.

Corollary 3.2. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$. Let $\alpha, \beta \in [0, 1]$ be coefficients such that $\alpha + \beta > 1$.*

Then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$\frac{\alpha + \beta}{\gamma} f\left(\frac{\alpha a + \beta b}{\alpha + \beta}\right) - \frac{1}{\gamma} f(c) \leq \frac{1}{b - a} \int_a^b f(x)\, dx \leq \frac{\alpha}{\gamma} f(a) + \frac{\beta}{\gamma} f(b) - \frac{1}{\gamma} f(c) \tag{3.4}$$

where $\gamma = \alpha + \beta - 1$ and

$$c = \frac{\alpha - \beta + 1}{2} a + \frac{\beta - \alpha + 1}{2} b.$$

Proof. Using the inequality in (3.2) with $x_i = x_{ni}$ and $p_i = \gamma(b_{ni} - a_{ni})/(b - a)$ in which case $\sum_{i=1}^{n} p_i x_i$ approaches

$$\frac{\gamma}{b - a} \int_a^b x\, dx = \frac{\gamma}{2}(a + b),$$

and $\sum_{i=1}^{n} p_i f(x_i)$ approaches

$$\frac{\gamma}{b - a} \int_a^b f(x)\, dx$$

as n approaches infinity, we get the inequality in (3.4). $\qquad\square$

Corollary 3.3. *Let $[a, b] \subset \mathbb{R}$ and $[c, d] \subset \mathbb{R}$ be bounded closed intervals where $a < b$ and $c < d$. Let $p : [c, d] \to \mathbb{R}$ be a non-negative continuous function with $\int_c^d p(x)\, dx > 0$, and $g : [c, d] \to [a, b]$ be a continuous function. Let $\alpha, \beta \in [0, 1]$ be coefficients such that $\alpha + \beta > 1$.*

Then every convex function $f : [a, b] \to \mathbb{R}$ verifies the inequality

$$
\begin{aligned}
f\left(\frac{\alpha a + \beta b}{\alpha + \beta} \right) &\leq \frac{1}{\alpha + \beta} \left[f\left(\alpha a + \beta b - \gamma \frac{\int_c^d p(x)g(x)\, dx}{\int_c^d p(x)\, dx} \right) + \gamma \frac{\int_c^d p(x)f(g(x))\, dx}{\int_c^d p(x)\, dx} \right] \\
&\leq \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta}
\end{aligned}
\tag{3.5}
$$

where $\gamma = \alpha + \beta - 1$.

Proof. The inequality in (3.5) follows from the inequality in (3.2) with the points $x_i = g(x_{ni})$ and the coefficients $p_i = \gamma(d_{ni} - c_{ni})p(x_{ni}) / \sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})$. For that matter, the combination

$$
\sum_{i=1}^n p_i x_i = \gamma \sum_{i=1}^n \frac{(d_{ni} - c_{ni})p(x_{ni})}{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})} g(x_{ni}) = \gamma \frac{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})g(x_{ni})}{\sum_{i=1}^n (d_{ni} - c_{ni})p(x_{ni})}
$$

passes to the integral quotient

$$
\gamma \frac{\int_c^d p(x)g(x)\, dx}{\int_c^d p(x)\, dx}
$$

as n approaches infinity. The same goes for the combination $\sum_{i=1}^n p_i f(x_i)$. $\qquad \square$

The inequality in (3.5) with $\gamma = 1$, and consequently $\alpha = \beta = 1$, reduces to the inequality in (1.4).

4 Applications

We want to apply the combination in (3.1), and the right-hand side of the inequality in (3.2),

$$
f\left(\alpha a + \beta b - \sum_{i=1}^n p_i x_i \right) \leq \alpha f(a) + \beta f(b) - \sum_{i=1}^n p_i f(x_i),
\tag{4.1}
$$

to discrete quasi-arithmetic means. The excellent book on means and their inequalities in [1] can always be recommended.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. In the applications of convexity, we often use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), that is, $f = \psi \circ \varphi^{-1}$ is convex on $\varphi(\mathcal{I})$. A similar notation is used for the concavity.

Let $\sum_{i=1}^n p_i x_i$ be a convex combination from $\mathcal{I}$. The discrete φ-quasi-arithmetic mean of the points x_i with the coefficients p_i is the point

$$
M_\varphi(x_i; p_i) = \varphi^{-1}\left(\sum_{i=1}^n p_i \varphi(x_i) \right)
\tag{4.2}
$$

which belongs to $\mathcal{I}$. The point $M_\varphi(x_i; p_i)$ can also be called the φ-quasi-center of the convex combination center $c = \sum_{i=1}^n p_i x_i$. The idea of the formula in (4.2) may be applied for a quasi-arithmetic mean of the affine combination $\alpha a + \beta b - \sum_{i=1}^n p_i x_i$ that belongs to $[a, b]$, in this way:

$$
M_\varphi(a, b, x_i; \alpha, \beta, p_i) = \varphi^{-1}\left(\alpha \varphi(a) + \beta \varphi(b) - \sum_{i=1}^n p_i \varphi(x_i) \right).
\tag{4.3}
$$

The mean defined in (4.3) belongs to $[a, b]$ because $\alpha \varphi(a) + \beta \varphi(b) - \sum_{i=1}^n p_i \varphi(x_i)$ belongs to $\varphi([a, b])$.

We have the following application of the formula in (4.1) to the quasi-arithmetic means in (4.3):

Corollary 4.4. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$, and $\varphi, \psi : [a, b] \to \mathbb{R}$ be strictly monotone continuous functions. Let $x_i \in [a, b]$ be points, and $\alpha, \beta, p_i, \in [0, 1]$ be coefficients such that $\alpha + \beta - \sum_{i=1}^{n} p_i = 1$.*

If ψ is either φ-convex and increasing or φ-concave and decreasing, then the inequality

$$M_\varphi(a, b, x_i; \alpha, \beta, p_i) \leq M_\psi(a, b, x_i; \alpha, \beta, p_i) \tag{4.4}$$

holds.

If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in (4.4).

Proof. Prove the case that ψ is φ-convex and increasing. Since φ is monotone, the endpoints of the interval $[c, d] = \varphi([a, b])$ are $\varphi(a)$ and $\varphi(b)$. Using the inequality in (4.1) with the convex function $f = \psi \circ \varphi^{-1} : [c, d] \to \mathbb{R}$, we get

$$\psi \circ \varphi^{-1} \left(\alpha \varphi(a) + \beta \varphi(b) - \sum_{i=1}^{n} p_i \varphi(x_i) \right) \leq \alpha \psi(a) + \beta \psi(b) - \sum_{i=1}^{n} p_i \psi(x_i),$$

and assigning the increasing function ψ^{-1} to the above inequality, we attain the mean inequality in (4.4).   □

Using the pairs of functions $\varphi(x) = x^{-1}$, $\psi(x) = \ln x$ and $\varphi(x) = \ln x$, $\psi(x) = x$ in the inequality in (4.4) with $a, b > 0$, we get the harmonic-geometric-arithmetic inequality for the means defined in (4.3):

$$\left(\frac{\alpha}{a} + \frac{\beta}{b} - \sum_{i=1}^{n} \frac{p_i}{x_i} \right)^{-1} \leq a^\alpha b^\beta \prod_{i=1}^{n} x_i^{-p_i} \leq \alpha a + \beta b - \sum_{i=1}^{n} p_i x_i. \tag{4.5}$$

A further application of the inequality in (4.1) could be related to the definition of the variant of Jensen's functional by the formula

$$J_f(a, b, x_i; \alpha, \beta, p_i) = \alpha f(a) + \beta f(b) - \sum_{i=1}^{n} p_i f(x_i) - f \left(\alpha a + \beta b - \sum_{i=1}^{n} p_i x_i \right). \tag{4.6}$$

Some new results relating to the bounds of Jensen's functional have been latterly achieved in [7].

References

[1] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht, NL, 1988.

[2] A. E. Farissi, B. Belaidi, and Z. Latreuch, A variant of Jensen's inequalities, *Malaya Journal of Matematik*, 4(2013), 54-60.

[3] A. McD. Mercer, A variant of Jensen's inequality, *Journal of Inequalities in Pure and Applied Mathematics*, 4(2003), Article 73, 2 pages.

[4] C. P. Niculescu, and C. I. Spiridon, New Jensen-type inequalities, arXiv:1207.6877 [math.CA], 7 pages, 2012.

[5] Z. Pavić, J. Pečarić, and I. Perić, Integral, discrete and functional variants of Jensen's inequality, *Journal of Mathematical Inequalities*, 5(2011), 253-264.

[6] Z. Pavić, Convex combinations, barycenters and convex functions, *Journal of Inequalities and Applications*, 2013(2013), Article 61, 13 pages.

[7] J. Sándor, On global bounds for generalized Jensen's inequality", *Annals of Functional Analysis*, 4(2013), 18-24.