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Ostrowski inequality for generalized fractional integral and related inequalities

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Abstract

In this article we obtain new generalizations for ostrowski inequality by using generalized Riemann-Liouville fractional integral.

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1 Introduction

Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable in (a,b) and assume $|f'(x)| \leq M$ for all $x \in (a,b)$. Then the following holds [1]:

$$|f(x) - M(f;a,b)| \leq \frac{M}{b-a} \frac{(b-x)^2 + (x-a)^2}{2} \tag{1.1}$$

for all $x \in [a,b]$. Where $M(f;a,b) = \frac{1}{b-a} \int\limits_a^b f(x)dx$.

(1.1) inequality is well known in the literature as Ostrowski Inequality. Many researchers try to generalize this inequality. There are numerous generalizations, variants and extensions in the literature, see [4-17] and the references cited therein. Hu Yue makes the following generalizations by using Riemann-Liouville fractional integrals [4].

Definition 1.1. *([24])Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integral $J_{a^+}^{\alpha} f(x)$ and $J_{b^-}^{\alpha} f(x)$ of order $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad x > a \tag{1.2}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad x < b. \tag{1.3}$$

respectively. Where $\Gamma(\alpha)$ is Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

Many researches have studied various integral inequality types for Riemann-Liouville integral which are given in Definition 1 ($[18-21]$, $[23-31]$).

Grüss proved the following inequality [2]:

$$|M(fg;a,b) - M(f;a,b)M(g;a,b)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1.4}$$

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provided that f and g are two integral function on $[a,b]$ satisfying the condition $m_1 \leq f \leq M_1$ and $m_2 \leq g \leq M_2$ for all $x \in [a,b]$, where $m_1, m_2, M_1, M_2 \in R$. The constant $\frac{1}{4}$ is the best possible. So we call (1.4) the Grüss inequality.

Korkine's identity [3] states that if f and g are two integral function on $[a,b]$, then

$$M(fg;a,b) - M(f;a,b)M(g;a,b) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dsdt. \tag{1.5}$$

Hu Yue obtains new generalizations (the following theorems) for (1.1) by using (1.4) and (1.5).

Theorem 1.1. *([4])Let f be differentiable function on $[a,b]$ and $|f'(x)| \leq M$ for any $x \in [a.b]$. Then the following fractional inequality holds:*

$$\left|\frac{(x-a)^\alpha + (b-x)^\alpha}{\Gamma(\alpha+1)} f(x) - J_{x^+}^\alpha f(a) - J_{x^+}^\alpha f(b)\right| \leq M\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\Gamma(\alpha+2)} \tag{1.6}$$

for any $x \in [a,b]$ and $\alpha \geq 0$.

Theorem 1.2. *([4])Let $f : [a,b] \to R$ be a differentiable mapping and $f' \in L^2[a,b]$. If f' bounded on $[a,b]$ with $m \leq f'(x) \leq M$, then we have*

$$\begin{aligned}\left|\frac{\alpha f(x)+f(a)}{\Gamma(\alpha)(\alpha+1)}(x-a)^{\alpha-1} - \frac{\alpha}{x-a}J_{x^-}^\alpha f(a) + \frac{\alpha f(x)+f(b)}{\Gamma(\alpha)(\alpha+1)}(b-x)^{\alpha-1} - \frac{\alpha}{b-x}J_{x^+}^\alpha f(b)\right| \\ \leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}}\,\tfrac{(x-a)^\alpha K_1 + (b-x)^\alpha K_2}{\Gamma(\alpha)} \\ \leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}}\,\tfrac{(x-a)^\alpha + (b-x)^\alpha}{2\Gamma(\alpha)}(M-m)\end{aligned} \tag{1.7}$$

for all $x \in [a,b]$ and $\alpha \geq 0$. Where

$$\begin{aligned} K_1^2 &= M(f'^2;a,x) - M^2(f';a,x) \\ K_2^2 &= M(f'^2;x,b) - M^2(f';x,b). \end{aligned}$$

Now we will give some definitions for fractional integrals which are called generalized fractional integrals.

Definition 1.2. *([22])A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a complex number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$.*

Definition 1.3. *([22])A function $f(t) \in C_\mu$, $t > 0$ is said to be in the $L_{p,k}(a,b)$ space if*

$$L_{p,k}(a,b) = \left\{ f : \|f\|_{L_{p,k(a,b)}} = \left(\int_a^b |f(t)|^p\, t^k dt\right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty,\ k \geq 0 \right\}.$$

Definition 1.4. *([22],[27])Consider the space $X_c^p(a,b)$ $(c \in R, 1 \leq p < \infty)$ of those real-valued lebesque measurable functions f on $[a,b]$ for which*

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty,\ (1 \leq p < \infty,\ c \in R)$$

and for the case $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{a \leq t \leq b} [t^c f(t)],\ c \in R.$$

In particular, when $c = \frac{k+1}{p}$ $(1 \leq p < \infty, k \geq 0)$ the space $X_c^p(a,b)$ coincides with the $L_{p,k}(a,b)$-space and also if we take $c = \frac{1}{p}$ $(1 \leq p < \infty)$ the space $X_c^p(a,b)$ coincides with the classical $L^p(a,b)$-space.

Definition 1.5. *([22],[27])Let $f \in L_{1,k}[a,b]$. The Generalized Riemann-Liouville fractional integral $J_{a^+}^{\alpha,k} f(x)$ and $J_{b^-}^{\alpha,k} f(x)$ of order $\alpha \geq 0$ and $k \geq 0$ are defined by*

$$J_{a^+}^{\alpha,k} f(x) = \tfrac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{k+1} - t^{k+1}\right)^{\alpha-1} t^k f(t)\, dt \quad x > a \tag{1.8}$$

and

$$J_{b^-}^{\alpha,k} f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \left(t^{k+1} - x^{k+1}\right)^{\alpha-1} t^k f(t)\, dt \quad b > x. \tag{1.9}$$

Where $\Gamma(\alpha)$ *is Gamma function and* $J_{a^+}^{0,k} f(x) = J_{b^-}^{0,k} f(x) = f(x)$.

(1.8) and (1.9) integral formulas are called right Generalized Riemann Liouville Integral and left Generalized Riemann Liouville Integral respectively.

Remark 1.1. *Letting* $k = 0$ *for (1.8) and (1.9) formulas we obtain the equalities in Definition 1.*

In this paper we will generalize (1.1), (1.5), (1.6) and (1.7) expressions by using Generalized Riemann-Liouville Fractional Integrals.

2 MAIN RESULTS

Theorem 2.3. *If* $f, g \in L_{1,k}[a,b]$, $k \geq 0$ *then*

$$\begin{aligned} J_{a^+}^{\alpha,k}[f(b)\, g(b)] - \frac{\Gamma(\alpha+1)(k+1)^{\alpha}}{\left(b^{k+1}-a^{k+1}\right)^{\alpha}} J_{a^+}^{\alpha,k}[f(b)] J_{a^+}^{\alpha,k}[g(b)] = \frac{\alpha(k+1)^{2-\alpha}}{2\left(b^{k+1}-a^{k+1}\right)^{\alpha}\Gamma(\alpha)} \int_a^b \int_a^b (f(t)-f(s)) \\ \times (g(t)-g(s)) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt. \end{aligned} \tag{2.10}$$

Proof. We have the following equality by $(f(t)-f(s))(g(t)-g(s))$;

$$\begin{aligned} &\int_a^b \int_a^b (f(t)-f(s))(g(t)-g(s)) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &= \int_a^b \int_a^b f(t) g(t) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &- \int_a^b \int_a^b f(t) g(s) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &- \int_a^b \int_a^b f(s) g(t) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt \\ &+ \int_a^b \int_a^b f(s) g(s) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k s^k ds dt. \\ &= 2\left[\int_a^b \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} s^k dt \int_a^b f(t) g(t) \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k dt\right] \\ &-2\left[\int_a^b g(s) \left(b^{k+1}-s^{k+1}\right)^{\alpha-1} s^k dt \int_a^b f(t) \left(b^{k+1}-t^{k+1}\right)^{\alpha-1} t^k dt\right] \\ &= \frac{2\left(b^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)} \frac{\Gamma(\alpha)}{(k+1)^{1-\alpha}} J_{a^+}^{\alpha,k}[f(b)\, g(b)] - \frac{2\Gamma^2(\alpha)}{(k+1)^{2-2\alpha}} J_{a^+}^{\alpha,k}[f(b)] J_{a^+}^{\alpha,k}[g(b)]. \end{aligned} \tag{2.11}$$

So this proves theorem. □

Remark 2.2. *If we take* $\alpha = 1$ *in (2.1) we obtain the following identity:*

$$M_k(fg; a,b) - M_k(f; a,b) M_k(g; a,b) = \frac{(k+1)^2}{2\left(b^{k+1}-a^{k+1}\right)^2} \int_a^b \int_a^b (f(t)-f(s))(g(t)-g(s)) t^k s^k ds dt \tag{2.12}$$

where $M_k(f; a,b) = \frac{k+1}{b^{k+1}-a^{k+1}} \int_a^b f(t) t^k dt, k \geq 0.$

Remark 2.3. *For* $\alpha = 1$ *and* $k = 0$ *in (2.1), we obtain the Korkine's identity (1.5).*

Theorem 2.4. *Let f be differentiable function on* $[a,b]$ *and* $|f'(x)| \leq M$ *for any* $x \in [a,b]$. *Then the following generalized fractional inequality holds for* $\alpha \geq 0$ *and* $k \geq 0$

$$\begin{aligned} &\left| \frac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} [\left(x^{k+1}-a^{k+1}\right)^{\alpha} + \left(b^{k+1}-x^{k+1}\right)^{\alpha}] x^k f(x) \right. \\ &\left. - J_{x^-}^{\alpha,k}[a^k f(a)] - J_{x^+}^{\alpha,k}[b^k f(b)] - k\left[J_{x^-}^{\alpha+1,k}[\frac{f(a)}{a}] + J_{x^+}^{\alpha+1,k}[\frac{f(b)}{b}]\right] \right| \\ &\leq \frac{(k+1)^{-\alpha-1}}{\Gamma(\alpha+2)} M[\left(x^{k+1}-a^{k+1}\right)^{\alpha+1} + \left(b^{k+1}-x^{k+1}\right)^{\alpha+1}]. \end{aligned} \tag{2.13}$$

Proof. If we use integration by parts for fractional integrals in Definition 5, we have

$$J_{x^-}^{\alpha+1,k} f'(a) = \frac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} x^k \left(x^{k+1} - a^{k+1}\right)^{\alpha} f(x) - J_{x^-}^{\alpha,k}[a^k f(a)] - kJ_{x^-}^{\alpha+1,k}[\frac{f(a)}{a}] \tag{2.14}$$

and

$$J_{x^+}^{\alpha+1,k} f'(b) = \frac{-(k+1)^{-\alpha}}{\Gamma(\alpha+1)} x^k \left(b^{k+1} - x^{k+1}\right)^{\alpha} f(x) + J_{x^+}^{\alpha,k}[b^k f(b)] + kJ_{x^+}^{\alpha+1,k}[\frac{f(b)}{b}]. \tag{2.15}$$

By (2.5) and (2.6) we obtain

$$\begin{aligned} &J_{x^-}^{\alpha+1,k} f'(a) - J_{x^+}^{\alpha+1,k} f'(b) \\ &= \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} x^k f(x)[\left(x^{k+1} - a^{k+1}\right)^{\alpha} + \left(b^{k+1} - x^{k+1}\right)^{\alpha}] \\ &-J_{x^-}^{\alpha,k}[a^k f(a)] - J_{x^+}^{\alpha,k}[b^k f(b)] - k[J_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}] + J_{x-}^{\alpha+1,k}[\tfrac{f(b)}{b}]]. \end{aligned} \tag{2.16}$$

Using $|f'(x)| \leq M$, $x \in [a,b]$ for the left part of the (2.7) formula we have

$$\begin{aligned} &\left| \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} \int_a^x \left(t^{k+1} - a^{k+1}\right)^{\alpha} t^k f'(t)\,dt - \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} \int_x^b \left(b^{k+1} - t^{k+1}\right)^{\alpha} t^k f'(t)\,dt \right| \\ &\leq \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} M[\int_a^x \left(t^{k+1} - a^{k+1}\right)^{\alpha} t^k dt + \int_x^b \left(b^{k+1} - t^{k+1}\right)^{\alpha} t^k dt] \\ &\leq \tfrac{(k+1)^{-\alpha-1}}{\Gamma(\alpha+2)} M[\left(x^{k+1} - a^{k+1}\right)^{\alpha+1} + \left(b^{k+1} - x^{k+1}\right)^{\alpha+1}]. \end{aligned} \tag{2.17}$$

So the proof is completed. □

Remark 2.4. *If we take $k = 0$ in inequality (2.4) we obtain the inequality (1.6) in Theorem 1.*

Remark 2.5. *Also letting $k = 0$ and $\alpha = 1$, formula (2.4) reduces Ostrowski Inequality:*

$$|f(x) - M(f;a,b)| \leq \frac{M}{b-a} \frac{(b-x)^2 + (x-a)^2}{2}.$$

Theorem 2.5. *Let $f : [a,b] \to R$ be a differentiable mapping and $f' \in L_{2,k}[a,b]$. If f' bounded on $[a,b]$ with $m \leq f'(x) \leq M$, then the following inequality holds :*

$$\begin{aligned} &\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)} \left(x^{k+1} - a^{k+1}\right)^{\alpha-1} [x^k f(x) - \tfrac{1}{(\alpha+1)} \int_a^x f'(t)\, t^k dt] \\ &- \frac{\alpha(k+1)}{x^{k+1} - a^{k+1}} \left(J_{x^-}^{\alpha,k}[a^k f(a)] + kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]\right) \\ &+ \frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)} \left(b^{k+1} - x^{k+1}\right)^{\alpha-1} [x^k f(x) + \tfrac{1}{(\alpha+1)} \int_x^b f'(t)\, t^k dt] \\ &- \frac{\alpha(k+1)}{b^{k+1} - x^{k+1}} \left(J_{x^+}^{\alpha,k}[b^k f(b)] + kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]\right) \\ &\leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}} \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha)} \left(x^{k+1} - a^{k+1}\right)^{\alpha} K_1 + \left(b^{k+1} - x^{k+1}\right)^{\alpha} K_2 \\ &\leq \sqrt{\tfrac{1}{2\alpha+1} - \tfrac{1}{(\alpha+1)^2}} \tfrac{(k+1)^{-\alpha}}{\Gamma(\alpha)} \tfrac{\left(x^{k+1}-a^{k+1}\right)^{\alpha} + \left(b^{k+1}-x^{k+1}\right)^{\alpha}}{2} (M-m) \end{aligned} \tag{2.18}$$

for all $x \in [a,b]$ and $\alpha \geq 0$. Where

$$\begin{aligned} K_1^2 &= M_k(f'^2; a, x) - M_k^2(f'; a, x) \\ K_2^2 &= M_k(f'^2; x, b) - M_k^2(f'; x, b). \end{aligned}$$

Proof. From (1.8) and (1.9) we have

$$\begin{aligned} \frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(x^{k+1} - a^{k+1}\right)} \int_a^x \left(t^{k+1} - a^{k+1}\right)^{\alpha} t^k f'(t)\,dt &= \frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)} x^k \left(x^{k+1} - a^{k+1}\right)^{\alpha-1} f(x) \\ &- \frac{\alpha(k+1)}{x^{k+1} - a^{k+1}} [J_{x^-}^{\alpha,k}[a^k f(a)] + kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]] \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{-(k+1)^{-\alpha+1}}{\Gamma(\alpha)\left(b^{k+1} - x^{k+1}\right)} \int_x^b \left(b^{k+1} - t^{k+1}\right)^{\alpha} t^k f'(t)\,dt &= \frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)} x^k \left(b^{k+1} - x^{k+1}\right)^{\alpha-1} f(x) \\ &- \frac{\alpha(k+1)}{b^{k+1} - x^{k+1}} [J_{x^+}^{\alpha,k}[b^k f(b)] + kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]]. \end{aligned} \tag{2.20}$$

Then

$$\begin{aligned}
&\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}[x^k f(x)-\tfrac{1}{(\alpha+1)}\int_a^x f'(t)\,t^k dt] \\
&\quad -\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}[J_{x^-}^{\alpha,k}[a^k f(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]] \\
&\quad +\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}[x^k f(x)+\tfrac{1}{(\alpha+1)}\int_x^b f'(t)\,t^k dt] \\
&\quad +\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}[J_{x^+}^{\alpha,k}[b^k f(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]] \\
&=\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(x^{k+1}-a^{k+1})}\int_a^x\left(t^{k+1}-a^{k+1}\right)^{\alpha}t^k f'(t)\,dt \\
&\quad -\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(\alpha+1)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}\int_a^x f'(t)\,t^k dt \\
&\quad -\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(b^{k+1}-x^{k+1})}\int_x^b\left(b^{k+1}-t^{k+1}\right)^{\alpha}t^k f'(t)\,dt \\
&\quad +\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)(\alpha+1)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}\int_x^b f'(t)\,t^k dt.
\end{aligned} \tag{2.21}$$

If we use the Korkine's identity (2.3) for Generalized Riemann Liouville integral for (2.12), we obtain

$$\begin{aligned}
&\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(x^{k+1}-a^{k+1}\right)^{\alpha-1}[x^k f(x)-\tfrac{1}{(\alpha+1)}\int_a^x f'(t)\,t^k dt] \\
&\quad -\frac{\alpha(k+1)}{x^{k+1}-a^{k+1}}[J_{x^-}^{\alpha,k}[a^k f(a)]+kJ_{x^+}^{\alpha+1,k}[\tfrac{f(a)}{a}]] \\
&+\frac{(k+1)^{-\alpha+1}}{\Gamma(\alpha)}\left(b^{k+1}-x^{k+1}\right)^{\alpha-1}[x^k f(x)+\tfrac{1}{(\alpha+1)}\int_x^b f'(t)\,t^k dt] \\
&\quad +\frac{\alpha(k+1)}{b^{k+1}-x^{k+1}}[J_{x^+}^{\alpha,k}[b^k f(b)]+kJ_{x^-}^{\alpha+1,k}[\tfrac{f(b)}{b}]] \\
&=\frac{(k+1)^2(k+1)^{-\alpha}}{2\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)^2}\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt \\
&+\frac{(k+1)^2(k+1)^{-\alpha}}{2\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)^2}\int_x^b\int_x^b[\left(b^{k+1}-s^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt.
\end{aligned} \tag{2.22}$$

Using the Cauchy-Schwarz inequality for double integrals in (2.13), we obtain

$$\begin{aligned}
&\left|\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^{\frac{k}{2}}t^{\frac{k}{2}}s^{\frac{k}{2}}t^{\frac{k}{2}}ds dt\right| \\
&\le\left(\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}]^2 s^k t^k ds dt\right)^{\frac{1}{2}}\left(\int_a^x\int_a^x[f'(t)-f'(s)]^2 s^k t^k ds dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.23}$$

However

$$\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}]^2 s^k t^k ds dt=\frac{2\left(x^{k+1}-a^{k+1}\right)^{2\alpha+2}}{(k+1)^2}\left(\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}\right) \tag{2.24}$$

and

$$\int_a^x\int_a^x[f'(t)-f'(s)]^2 s^k t^k ds dt=\frac{2\left(x^{k+1}-a^{k+1}\right)^2}{(k+1)^2}\left[M_k(f'^2;a,x)-M_k^2(f';a,x)\right]. \tag{2.25}$$

By (2.14)-(2.16), we have

$$\begin{aligned}
&\left|\frac{(k+1)^{-\alpha+2}}{2\Gamma(\alpha)\left(x^{k+1}-a^{k+1}\right)}\int_a^x\int_a^x[\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt\right| \\
&\qquad\le\frac{(k+1)^{-\alpha}\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\left[M_k(f'^2;a,x)-M_k^2(f';a,x)\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.26}$$

Similarly we have

$$\begin{aligned}
&\left|\frac{(k+1)^{-\alpha+2}}{2\Gamma(\alpha)\left(b^{k+1}-x^{k+1}\right)}\int_x^b\int_x^b[\left(b^{k+1}-s^{k+1}\right)^{\alpha}-\left(b^{k+1}-t^{k+1}\right)^{\alpha}][f'(t)-f'(s)]s^k t^k ds dt\right| \\
&\qquad\le\frac{(k+1)^{-\alpha}\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\left[M_k(f'^2;x,b)-M_k^2(f';x,b)\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.27}$$

Using (2.13),(2.17) and (2.18) we obtain (2.9) inequality. Moreover if $m \leq f'(x) \leq M$ on $[a,b]$, then by Grüss inequality, we have

$$0 \leq \frac{k+1}{x^{k+1}-a^{k+1}} \left\|f'\right\|^2_{L_{2,k}(a,x)} - (M_k(f';a,x))^2 \leq \frac{1}{2}(M-m)^2 \tag{2.28}$$

$$0 \leq \frac{k+1}{b^{k+1}-x^{k+1}} \left\|f'\right\|^2_{L_{2,k(x,b)}} - (M_k(f';x,b))^2 \leq \frac{1}{2}(M-m)^2 \tag{2.29}$$

which proves the last inequality of (2.9) □

Remark 2.6. *Letting $k=0$ in (2.9) we obtain the inequality (1.7) in Theorem 2.*

Corollary 2.1. *Under the assumptions of Theorem 5 with $\alpha = 1$, then the following inequality holds*

$$\begin{aligned} 2x^k f(x) + \tfrac{1}{2}\left[\int_x^b f'(t)t^k dt - \int_a^x f'(t)t^k dt\right] - \tfrac{k+1}{x^{k+1}-a^{k+1}}\left[J^{1,k}_{x^-}[a^k f(a)] + kJ^{2,k}_{x^+}[\tfrac{f(a)}{a}]\right] \\ - \tfrac{k+1}{b^{k+1}-x^{k+1}}\left[J^{1,k}_{x^+}[b^k f(b)] + kJ^{2,k}_{x^-}[\tfrac{f(b)}{b}]\right] \\ \leq \tfrac{1}{4\sqrt{3}}\tfrac{1}{k+1}(b^{k+1}-a^{k+1})(M-m). \end{aligned} \tag{2.30}$$

Remark 2.7. *Letting $k=0$ in (2.21) we obtain the inequality*

$$\begin{aligned} \left| f(x) + \tfrac{f(a)+f(b)}{2} - \tfrac{1}{x-a}\int_a^x f(t)dt - \tfrac{1}{b-x}\int_x^b f(t)dt \right| \\ \leq \tfrac{1}{4\sqrt{3}}(b-a)(M-m). \end{aligned} \tag{2.31}$$

Remark 2.8. *Letting $x = \frac{a+b}{2}$ in (2.22) we obtain*

$$\begin{aligned} \left| \tfrac{1}{2}\left(f(\tfrac{a+b}{2}) + \tfrac{f(a)+f(b)}{2}\right) - \tfrac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq \tfrac{1}{8\sqrt{3}}(b-a)(M-m). \end{aligned} \tag{2.32}$$

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