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Existence of solution of a Coupled system of differential equation with nonlocal conditions

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Abstract

In this paper, we study the existence of at least one solution of the coupled system of differential equations with nonlocal conditions. Also, a coupled system of differential equations with the nonlocal integral conditions will be considered.

Keywords: Coupled systems, nonlocal conditions, at least one solution, integral conditions.

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1 Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last decades.The reader is referred to([2]-[20]) and references therein.
In [13] the authors studied nonlocal cauchy problem

$$\dot{x} = f(t, x(t)), \; t \in [0, T]$$

$$\sum_{j=1}^{m} b_j x(\eta_j) = x_1, \; \eta_j \in (0, a) \subset [0, T].$$

Also, in [7] the authors studied the local and global existence of solutions of the nonlocal problem

$$\frac{dx}{dt} = f_1(t, y(t)), \;\; t \in (0, T] \tag{1.1}$$

$$\frac{dy}{dt} = f_2(t, x(t)), \;\; t \in (0, T] \tag{1.2}$$

with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0, \;\; a_k > 0, \; \tau_k \in (0, \; T) \tag{1.3}$$

$$y(0) + \sum_{j=1}^{m} b_j y(\eta_j) = y_0, \;\; b_j > 0, \; \eta_j \in (0, \; T) \tag{1.4}$$

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Here we are studied the existence of at least one solution of the nonlocal problem (1.1)-(1.4), the problem with nonlocal integral conditions

$$x(0) + \int_0^T x(s)ds = x_0, \quad (1.5)$$

$$y(0) + \int_0^T y(s)ds = y_0. \quad (1.6)$$

are studied.

2 Preliminaries

we need the following definitions.

Definition 2.1. *[19] Let $F = \{f_i : X \to Y, i \in I\}$ be a family of functions with Y being a set of real (or complex) numbers, then we call F uniformly bounded if there exists a real number c such that $| f_i(x) | \leq c \ \forall i \in I, x \in X$.*

Definition 2.2. *[19] Let $F = \{f(x)\}$ is the class of functions defined on A where $A = [a, b] \subset R$, the class of functions $F = \{f(x)\}$ is equicontinuous if $\forall \epsilon > 0$, $\exists\delta(\epsilon)$ such that $| x - y |< \delta$, implies that $| f(x) - f(y) |< \epsilon \ \forall f \in F, \ x, y \in A$.*

Theorem 2.1. *[1] The function $f(x) = (f_1(x), f_2(x), \ldots\ldots\ldots, f_n(x))$ is uniformly continuous in $I = [a, b]$ if and only if each f_i is uniformly continuous in $[a, b]$.*

Theorem 2.2. *[19](Lebesgue Dominated Convergence Theorem)*
let f_n be a sequence of functions converging to a limit f of A, and suppose that
$| f_n(t) | \leq \phi(t), t \in A, n = 1, 2, 3, \ldots\ldots$ where ϕ is integrable on A. Then

1. *f is integrable on A*
2. *$\lim_{n\to\infty} \int_A f_n(t)d\mu = \int_A f(t)d\mu$.*

Theorem 2.3. *[18](Schauder)*
Let Q be a convex subset of a Banach space X, $T : Q \to Q$ be a compact and continuous map, then T has at least one fixed point in Q.

3 Integral Representation

Let X be the class of all columns vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, $x, y \in C(0, T]$ with the norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_X = ||x|| + ||y|| = \sup_{t\in[0,T]} | x(t) | + \sup_{t\in[0,T]} | y(t) | .$$

Throughout the paper we assume that the following assumptions hold:

i. $f_i : [0, T] \times R \to R$ satisfies Caratheodory conditions, that is f_i is

1. measurable in $t \in (0, T]$, for any $x \in R$.
2. continuous in $x \in R$, for almost all $t \in (0, T]$.

ii. There exist two integrable functions $m_i \in L_1[0, T]$, $i = 1, 2$ such that
$|f_i(t, x)| \leq m_i(t)$,
$\int_0^t m_i(s)\, ds < k_i, i = 1, 2 \ \ \forall t \in [0, T]$.

Lemma 3.1. *The solution of the nonlocal problem (1.1)-(1.4) can be expressed by the system of the integral equations*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\, x_0 + \int_0^t f_1(s, y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s)) ds \\ \\ b\, y_0 + \int_0^t f_2(s, x(s))\, ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s)) ds \end{pmatrix},$$

where $\left(1 + \sum_{k=1}^{n} a_k\right)^{-1} = a, \quad \left(1 + \sum_{j=1}^{m} b_j\right)^{-1} = b.$

3.1 Existence of solution

Here, we study the existence of at least one solution of the nonlocal problem (1.1)-(1.4). Define the superposition operator F by

$$F \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s, y(s)) ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s)) ds \\ \\ by_0 + \int_0^t f_2(s, x(s)) ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s)) ds \end{pmatrix} = \begin{pmatrix} F_1 y \\ F_2 x \end{pmatrix}.$$

Now we have the following theorem.

Theorem 3.4. *Consider the assumptions (i)-(ii) are satisfied, then there exists at least one solution of the nonlocal problem (1.1)-(1.4).*

Proof. Define the operator $F\,(x, y) = (F_1 x, F_2 y)$, where

$$F_1 y = a\, x_0 + \int_0^t f_1(s, y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))\, ds,$$

$$F_2 x = b\, y_0 + \int_0^t f_2(s, x(s))\, ds - a \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s))\, ds.$$

Now

$$\begin{aligned} |\, F_1 y \,| &= \left| a\, x_0 + \int_0^t f_1(s, y(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))\, ds \right| \\ &\leq |\, ax_0 \,| + \int_0^t |\, f_1(s, y(s)) \,|\, ds + |\, a \,| \sum_{k=1}^{n} |\, a_k \,| \int_0^{\tau_k} |\, f_1(s, y(s)) \,|\, ds \\ &\leq a \,|\, x_0 \,| + \int_0^t m_1(s)\, ds + a \sum_{k=1}^{n} |\, a_k \,| \int_0^{\tau_k} m_1(s)\, ds \\ &\leq a \,|\, x_0 \,| + K_1 + a \sum_{k=1}^{n} a_k\, K_1 \leq a \,|\, x_0 \,| + K_1 (1 + a \sum_{k=1}^{n} a_k\,) \\ &\leq a \,|\, x_0 \,| + K_1 \left(1 + \frac{\sum_{k=1}^{n} a_k}{1 + \sum_{k=1}^{n} a_k} \right) \leq a \,|\, x_0 \,| + 2K_1 = M_1, \end{aligned}$$

then F_1 is uniformly bounded.
Similarly

$$| F_2 x | \leq b | y_0 | + 2K_2 = M_2,$$

then F_2 is uniformly bounded.
Hence $\| F(x,y) \|_X = \| F_1 y \| + \| F_2 x \| \leq M_1 + M_2 = M,$
and then F is uniformly bounded.
For $t_1, t_2 \in (0,T]$, $t_1 < t_2$, let $| t_2 - t_1 | < \delta$, then

$$\begin{aligned} | F x(t_2) - F x(t_1) | &= | F_1 y(t_2) - F_1 y(t_1) | \\ &= \left| \int_0^{t_2} f_1(s,y(s))\, ds - \int_0^{t_1} f_1(s,y(s))\, ds \right| \\ &= \left| \int_{t_1}^{t_2} f_1(s,y(s))\, ds \right| \\ &\leq \int_{t_1}^{t_2} | f_1(s,y(s)) |\, ds \\ &\leq \int_{t_1}^{t_2} m_1(s)\, ds \leq \epsilon, \end{aligned}$$

then $\{F_1 y\}$ is a class of equicontinuous functions.
Similarly

$$| F y(t_2) - F y(t_1) | = | F_2 x(t_2) - F_2 x(t_1) | \leq \int_{t_1}^{t_2} m_2(s)\, ds \leq \epsilon,$$

then $\{F_2 x\}$ is a class of equicontinuous functions.
Therefore the operator F is equicontinuous and uniformly bounded.
Let
$\{y_N(t)\} \in C[0,T]$, $y_N(t) \to y(t)$, $\{x_N(t)\} \in C[0,T]$, $x_N(t) \to x(t)$,
So,

$$\lim_{N\to\infty} F_1(y_N) = \lim_{N\to\infty} \left(a\, x_0 + \int_0^t f_1(s,y_N(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))\, ds \right),$$

but $| f_i(s, y_N(s)) | \leq m_i$, and $f_i(s, y_N(s)) \to f_i(s, y(s))$
applying Lebesgue dominated convergence theorem [19], then we deduce that

$$\lim_{N\to\infty} \int_0^t f_1(s,y_N(s))ds = \int_0^t \lim_{N\to\infty} f_1(s,y_N(s))ds = \int_0^t f_1(s, \lim_{N\to\infty} y_N(s))ds = \int_0^t f_1(s,y(s))\, ds,$$

and

$$\begin{aligned}
\lim_{N\to\infty} a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))ds =& a \sum_{k=1}^{n} a_k \lim_{N\to\infty} \int_0^{\tau_k} f_1(s,y_N(s))ds, \\
&= a \sum_{k=1}^{n} a_k \int_0^{\tau_k} \lim_{N\to\infty} f_1(s,y_N(s))\, ds, \\
&= a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, \lim_{N\to\infty} y_N(s))\, ds, \\
&= a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))\, ds,
\end{aligned}$$

then

$$\lim_{N\to\infty} F_1(y_N) = a\, x_0 + \int_0^t f_1(s,y_N(s))\, ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_N(s))\, ds = F_1 y.$$

This proves that $F_1 y$ is continuous operator,
Similarly, we can prove that

$$\lim_{N\to\infty} F_2(x_N) = a\, y_0 + \int_0^t f_2(s,x_N(s))\, ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x_N(s))\, ds = F_2 x,$$

then $F_2 x$ is continuous operator.
Then $F : X \to X$ is continuous and compact.
Now we show that X is convex,
let $(x_1, y_1), (x_2, y_2) \in X$

$$\| (x_i, y_i) \|_X = \| x_i \| + \| y_i \| < M, \; i = 1,2.$$

For $\lambda \in [0, T]$

$$\begin{aligned}
&\| \lambda (x_1, y_1) + (1-\lambda)(x_2, y_2) \|_X \\
&\quad = \| (\lambda x_1, \lambda y_1) + ((1-\lambda) x_2, (1-\lambda) y_2) \| \\
&\quad = \| (\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \| \\
&\quad \leq \| \lambda x_1 + ((1-\lambda) x_2 \| + \| \lambda y_1 + (1-\lambda) y_2) \| \\
&\quad \leq \lambda \| x_1 \| + (1-\lambda) \| x_2 \| + \lambda \| y_1 \| + (1-\lambda) \| y_2 \| \\
&\quad = \lambda [\| x_1 + \| y_1 \|] + (1-\lambda) [\| x_2 \| + \| y_2 \|] \\
&\quad \leq \lambda M + (1-\lambda) M = M,
\end{aligned}$$

this means that X is convex.
Then F has a fixed point $(x, y) \in X$ which proves that there exists at least one solution of the nonlocal problem (1.1)-(1.4). □

4 Nonlocal Integral Condition

Let $a_k = (t_k - t_{k-1})$, $\tau_k \in (t_{k-1}, t_k)$, and $b_j = (t_j - t_{j-1})$, $\eta_j \in (t_{j-1}, t_j)$,
where $0 < t_1 < t_2 < t_3 < .... < 1$.
Then, the nonlocal conditions (1.3)-(1.4) will be in the form

$$x(0) + \sum_{k=1}^{n} (t_k - t_{k-1})\, x(\tau_k) = x_0, \qquad y(0) + \sum_{j=1}^{m} (t_j - t_{j-1})\, x(\eta_j) = y_0.$$

From the continuity of the solution of the nonlocal problem (1.1)-(1.4), we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}\,(t_k - t_{k-1})\,x(\tau_k) = \int_0^T\,x(s)ds, \qquad \lim_{m\to\infty}\sum_{j=1}^{m}\,(t_j - t_{j-1})\,y(\eta_j) = \int_0^T\,y(s)ds,$$

that is, the nonlocal conditions (1.3)-(1.4) is transformed to the integral condition

$$x(0)\;+\;\int_0^T\,x(s)ds\;=\;x_0, \qquad y(0)\;+\;\int_0^T\,y(s)ds\;=\;y_0.$$

Now, we have the following theorem.

Theorem 4.5. *Let the assumption (i)-(ii) be satisfied, then the coupled system of differential equations (1.1) and (1.4) with the nonlocal integral condition (1.5)and(1.6) has at least one solution represented in the form*

$$U\;=\;\begin{pmatrix} x(t)\\ y(t)\end{pmatrix}\;=\;\begin{pmatrix} a^\star\,x_0\;+\;\int_0^t\,f_1\,(\theta,\,y(\theta))\,d\theta\;-\;a^\star\int_0^T\int_0^s\,f_1\,(\theta,\,y(\theta))d\theta ds\\ \\ a^\star\,y_0\;+\;\int_0^t\,f_2\,(\theta,\,x(\theta))\,d\theta\;-\;a^\star\int_0^T\int_0^s\,f_2\,(\theta,\,x(\theta))d\theta ds\end{pmatrix},$$

where $a^\star\;=(1+T)^{-1}$.

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