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Nonparametric estimation of some characteristics of the conditional distribution in single functional index model

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Abstract

The aim of this paper is to establish a nonparametric estimation of some characteristics of the conditional distribution. Kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density are introduced of a scalar response variable Y given a Hilbertian random variable X when the observations are linked with a single-index structure. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. Asymptotic properties are stated for each of these estimates, and they are applied to the estimations of the conditional mode and conditional quantiles.

Keywords: Conditional single-index, Conditional cumulative distribution, Derivatives of conditional density, Nonparametric estimation, Conditional mode, Conditional quantile, Kernel estimator, semi-metric choice.

2010 MSC: 62G07, 62G5, 62G20.

1 Introduction

The single-index models are becoming increasingly popular because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial econometric and so on. The single-index model, a special case of projection pursuit regression, has proven to be a very efficient way of coping with the high dimensional problem in nonparametric regression. Härdle *et al.* [16], Hristache *et al.* [18]. Delecroix et al. [6] have studied the estimation of the single-index approach of regression function and established some asymptotic properties. The recent literature in this domain shows a great potential of these functional statistical methods. The most popular case of functional random variable corresponds to the situation when we observe random curve on different statistical units. The first work in the fixed functional single-model was given by Ferraty *et al.* [10], where authors have obtained almost complete convergence (with the rate) of the regression function in the i.i.d. case. Their results have been extended to dependent case by Aït Saidi *et al.* [1]. Aït Saidi *et al.* [2] studied the case where the functional single-index is unknown. The authors have proposed for this parameter an estimator, based on the the cross-validation procedure.

In the present work we study a single- index modeling in the case of the functional explanatory variable. More precisely, we consider the problem of estimating some characteristics of the conditional distribution of a real variable Y given a functional variable X when the explanation of Y given X is done through its projection on one functional direction. The conditional distribution plays an important role in prediction problems, such as the conditional mode the conditional median or the conditional quantiles. Nonparametric estimation of the conditional density has been widely studied, when the data are real. The first related result in nonparametric functional statistic was obtained by Ferraty *et al.* [12], the authors have established the almost complete convergence (with rate) in the independent and identically distributed (i.i.d.) random variables. The

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asymptotic normality of this kernel estimator has been studied in the dependent data by Ezzahrioui and Ould Saïd [9].

The goal of this paper is to establish a nonparametric estimation of some characteristics of the conditional distribution where Kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density in the single functional index model are introduced. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of this model. Asymptotic properties are stated for each of these estimates, and they are applied to the estimations of the conditional mode and conditional quantiles.
Now, let us outline the paper. At first, in section 2, we present general notations and some conditions necessary for our study, Then, in sections 3 we propose the estimator of the conditional cumulative distribution function and that of the conditional density derivatives, and we give their pointwise almost complete convergence (with rate). Then, in section 4, we study the uniform almost complete convergence of the conditional cumulative distribution function (resp. the conditional density derivatives) estimator given in section 3. Section 5 is devoted to some applications, in this part, we first consider the problem of the estimation of the conditional mode in functional single-index model, then we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are in single functional index model and data are independent and identically distributed (i.i.d.), after that the cross-validation method is given, which is so important in guarding against testing hypotheses suggested by the data, especially where further samples are hazardous, costly or impossible to collect.
In the end, we finish our paper by giving technical proofs of lemmas and corollary (Appendix).

2 General notations and conditions

All along the paper, when no confusion will be possible, we will denote by C, C' or/and $C_{\theta,x}$ some generic constant in $\mathbb{R}_+^*$, and in the following, any real function with an integer in brackets as exponent denotes its derivative with the corresponding order.

Let X be a functional random variable, *frv* its abbreviation. Let (X_i, Y_i) be a sample of independent pairs, each having the same distribution as (X, Y), our aim is to build nonparametric estimates of several functions related with the conditional probability distribution (*cond-cdf*) of Y given $< X,\theta >=< x,\theta >$.
Let

$$\forall y \in \mathbb{R},\ F(\theta, y, x) = (Y \leq y| < X,\theta >=< x,\theta >).$$

be the *cond-cdf* of Y given $< X,\theta >=< x,\theta >$, for $x \in \mathcal{H}$, which also shows the relationship between X and Y but is often unknown.
If this distribution is absolutely continuous with respect to the Lebesgues measure on $\mathbb{R}$, then we will denote by $f(\theta,\cdot,x)$. (*resp.* $f^{(j)}(\theta,\cdot,x)$) the conditional density (*resp.* its j^{th} order derivative) of Y given $< X,\theta >=< x,\theta >$. In Sections 3 and 4, we will give almost complete convergence[1] results (with rates of convergence[2]) for nonparametric estimates of both functions $F(\theta,\cdot,x)$ and $f^{(j)}(\theta,\cdot,x)$.

In the following, for any $x \in \mathcal{H}$ and $y \in \mathbb{R}$, let $\mathcal{N}_x$ be a fixed neighborhood of x in $\mathcal{H}$, $\mathcal{S}_{\mathbb{R}}$ will be a fixed compact subset of $\mathbb{R}$, and we will use the notation $B_\theta(x,h) = \{X \in \mathcal{H}/0 < | < x - X,\theta > | < h\}$. Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of $< \theta, X >$:

(H1) $(X \in B_\theta(x,h)) = \phi_{\theta,x}(h) > 0,$

together with some usual smoothness conditions on the function to be estimated. According to the type of estimation problem to be considered, we will assume either

(H2) $\forall(y_1,y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}},\ \forall(x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x,\ |F(\theta,y_1,x_1) - F(\theta,y_2,x_2)| \leq C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right),$

[1] Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n (|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, 1987).

[2] Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n (|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = O(u_n)$, *a.co.* (or equivalently by $T_n = O_{a.co.}(u_n)$).

$b_1 > 0, b_2 > 0,$

(H3) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x, |f^{(j)}(\theta, y_1, x_1) - f^{(j)}(\theta, y_2, x_2)| = C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right),$
$b_1 > 0, b_2 > 0.$

3 Pointwise almost complete estimation

In this section we give the pointwise almost complete estimation (with rate) of the conditional cumulative distribution as of the successive derivatives of the conditional density.

3.1 Conditional cumulative distribution estimation

The purpose of this section is to estimate the *cond-cdf* $F^x(\theta, \cdot, x)$. We introduce a kernel type estimator $\widehat{F}^x(\theta, \cdot, x)$ of $F^x(\theta, \cdot, x)$ as follows:

$$\widehat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{3.1}$$

where K is a kernel, H is a cumulative distribution function (*cdf*) and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers which goes to zero as n tends to infinity, and with the convention $0/0 = 0$. Note that a similar estimate was already introduced in the case where X is a valued in some semi-metric space which can be of infinite dimension by Ferraty *et al.* [11]. In our single functional index context, we need the following conditions for our estimate:

(H4) H is such that, for all $(y_1, y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| \leq C|y_1 - y_2|$
$\int |t|^{b_2} H^{(1)}(t)dt < \infty,$

(H5) K is a positive bounded function with support $[-1, 1]$,

(H6) $\lim_{n\to\infty} h_K = 0$ with $\lim_{n\to\infty} \frac{\log n}{n\phi_{\theta,x}(h_K)} = 0,$

(H7) $\lim_{n\to\infty} h_H = 0$ with $\lim_{n\to\infty} n^{\alpha} h_H = \infty$ for some $\alpha > 0$.

• **Comments on the assumptions**

Our assumptions are very standard for this kind of model. Assumptions (H1) and (H5) are the same as those given in Ferraty et al. [10]. Assumptions (H2) and (H3) is a regularity conditions which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results. Assumptions (H4) and (H6)-(H7) are technical conditions and are also similar to those done in Ferraty *et al.* [12].

Theorem 3.1. *Under the hypotheses (H1), (H2) and (H4)-(H7), and for any fixed y, we have*

$$|\widehat{F}(\theta, y, x) - F(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \qquad a.co. \tag{3.2}$$

Proof. For $i = 1, \ldots, n$, we consider the quantities $K_i(\theta, x) := K(h_K^{-1}(< x - X_i, \theta >))$ and, for all $y \in \mathbb{R}$ $H_i(y) = H\left(h_H^{-1}(y - Y_i)\right)$ and let $\widehat{F}_N(\theta, y, x)$ (resp. $\widehat{F}_D(\theta, x)$) be defined as

$$\widehat{F}_N(\theta, y, x) = \frac{1}{n\,(K_1(\theta, x))} \sum_{i=1}^{n} K_i(\theta, x) H_i(y) \qquad (\text{resp. } \widehat{F}_D(\theta, x) = \frac{1}{n\,(K_1(\theta, x))} \sum_{i=1}^{n} K_i(\theta, x)).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{F}(\theta,y,x)-F(\theta,y,x) &= \frac{1}{\widehat{F}_D(\theta,x)}\left\{\left(\widehat{F}_N(\theta,y,x)-\widehat{\bar{F}}_N(\theta,y,x)\right)-\left(F(\theta,y,x)-\widehat{\bar{F}}_N(\theta,y,x)\right)\right\}\\ &\quad+\frac{F(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left\{1-\widehat{F}_D(\theta,x)\right\} \end{aligned} \tag{3.3}$$

and on the following intermediate results.

Lemma 3.1. *([1]) Under the hypotheses (H1) and (H5)-H6), we have*

$$|\widehat{F}_D(\theta,x)-1| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \tag{3.4}$$

Corollary 3.1. *Under the hypotheses of Lemma 3.1, we have*

$$\sum_{n=1}^{\infty}\left(|\widehat{F}_D(\theta,x)| \le 1/2\right) \;<\; \infty. \tag{3.5}$$

Lemma 3.2. *Under the hypotheses (H1), (H2) and (H4)-(H.6), we have*

$$|F(\theta,y,x)-\widehat{\bar{F}}_N(\theta,y,x)| = O\left(h_K^{b_1}\right)+O\left(h_H^{b_2}\right), \tag{3.6}$$

Lemma 3.3. *Under the hypotheses (H1), (H2) and (H4)-(H7), we have*

$$|\widehat{F}_N(\theta,y,x)-\widehat{\bar{F}}_N(\theta,y,x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \tag{3.7}$$

□

3.2 Estimating successive derivatives of the conditional density

The main objective of this part is the estimation of successive derivatives of the conditional density of Y given $< X,\theta >=< x,\theta >$, denoted by $f(\theta,\cdot,x)$. It is well known that, in nonparametric statistics, this latter provides an alternative approach to study the links between Y and X and it can be also used, in single index modelling, to estimate the functional index θ if it is unknown.

So, at first, we propose to define the estimator $\widehat{f^{(j)}}(\theta,y,x)$ of $f^{(j)}(\theta,y,x)$ as follows:

$$\widehat{f^{(j)}}(\theta,y,x) = \frac{h_H^{-1-j}\sum_{i=1}^{n}K(h_K^{-1}(< x-X_i,\theta >))H^{(j+1)}(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n}K(h_K^{-1}(< x-X_i,\theta >))},\;\; y\in\mathbb{R} \tag{3.8}$$

Similar estimate was already introduced in the case where X is a valued in some semi-metric space which can be of infinite dimension; Ferraty *et al.* [11], then widely studied (see for instance by Attaoui *et al.* [3], for several asymptotic results and references). In addition to the conditions introduced along the previous section, we need the following ones, which are technical conditions and are also similar to those given in Ferraty *et al.* [12]:

(H8) $\begin{cases}\forall(y_1,y_2)\in\mathbb{R}^2,\; |H^{(j+1)}(y_1)-H^{(j+1)}(y_2)| \le C_{\theta,x}|y_1-y_2|\\ \exists\nu>0,\;\forall j'\le j+1,\; \lim_{y\to\infty}|y|^{1+\nu}\;\left|H^{(j'+1)}(y)\right| = 0.\end{cases}$

(H9) $\lim_{n\to\infty} h_K = 0$ with $\lim_{n\to\infty}\frac{\log n}{n h_H^{2j+1}\phi_{\theta,x}(h_K)} = 0.$

The next result concerns the asymptotic behavior of the kernel functional estimator $\widehat{f^{(j)}}(\theta,\cdot,x)$ of the j^{th} order derivative of the conditional density function.

Theorem 3.2. *Under Assumptions (H1), (H3)-(H5), and (H7)-(H9), and for any fixed y, we have, as n goes to infinity*

$$|\widehat{f}^{(j)}(\theta,y,x) - f^{(j)}(\theta,y,x)| = O\left(h_K^{b_1}\right) + O\left(h_H^{b_2}\right) + O\left(\sqrt{\frac{\log n}{nh_H^{2j+1}\phi_{\theta,x}(h_K)}}\right) \ a.c.o \tag{3.9}$$

Proof. This result is based on the same kind of decomposition as (3.3). Indeed, we can write:

$$\begin{aligned} \widehat{f}^{(j)}(\theta,y,x) - f^{(j)}(\theta,y,x) &= \frac{1}{\widehat{F}_D(\theta,x)}\left(\widehat{f}_N^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)\right) - \frac{1}{\widehat{F}_D(\theta,x)} \\ &\left(f^{(j)}(\theta,y,x) - \widehat{f}_N^{(j)}(\theta,y,x)\right) \\ &+ \frac{f^{(j)}(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left(1-\widehat{F}_D(\theta,x)\right) \end{aligned} \tag{3.10}$$

where

$$\widehat{f}_N^{(j)}(\theta,y,x) = \frac{1}{n\,h_H^{j+1}\,(K_1(\theta,x))}\sum_{i=1}^{n} K_i(\theta,x)H_i^{(j+1)}(y).$$

Then, Theorem 3.2 can be deduced from both following lemmas, together with Lemma 3.1 and Corollary 3.1.

Lemma 3.4. *Under the hypotheses (H1), (H2), (H3), (H5) and (H6) we have*

$$|f^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)| = O\left(h_K^{b_1}\right) + O\left(h_H^{b_2}\right),$$

Lemma 3.5. *Under the hypotheses (H1)-(H7), we have*

$$|\widehat{f}_N^{(j)}(\theta,y,x) - (\widehat{f}_N^{(j)}(\theta,y,x)| = O_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H^{2j+1}\,\phi_{\theta,x}(h_K)}}\right),$$

The proofs of the the above lemmas and corollary are given in the same manner as it was done in [12], since they are a special case of the Lemmas 2.3.2, 2.3.3, 2.3.4 and 2.3.5. It suffices to replace $\widehat{f}^{(j)}(y,x)$ (resp. $f^{(j)}(y,x)$) by $\widehat{f}^{(j)}(\theta,y,x)$ (resp. $f^{(j)}(\theta,y,x)$), and $\widehat{F}_D(x)$, (resp. $F_D(x)$) by $\widehat{F}_D(\theta,x)$ (resp. $F_D(\theta,x)$) with $d(x_1,x_2) =< x_1 - x_2, \theta >$ □

4 Uniform almost complete convergence

In this section we derive the uniform version of Theorem 3.1 and Theorem 3.2. The study of the uniform consistency is an indispensable tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, however, in the functional case, it requires some additional tools and topological conditions (see Ferraty *et al.*, 2009). Thus, in addition to the conditions introduced previously, we need the following ones. Firstly, Consider

$$\mathcal{S}_{\mathcal{H}} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}} B(x_k, r_n) \text{ and } \Theta_{\mathcal{H}} \subset \bigcup_{j=1}^{d_n^{\Theta_{\mathcal{H}}}} B(t_j, r_n) \tag{4.11}$$

with x_k (resp. t_j) $\in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity.

4.1 Conditional cumulative distribution estimation

In this section we propose to study the uniform almost complete convergence of our estimator defined above (3.1) for this, we need the following assumptions:

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in \mathcal{S}_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \le \phi_{\theta,x}(h) \le C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0,\ \forall \eta < \eta_0,\ \phi'(\eta) < C,$$

(A2) $\forall(y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall(x_1, x_2) \in \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}}$ and $\forall\theta \in \Theta_{\mathcal{H}}$,

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C_{(x,\theta)} \left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2} \right),$$

(A3) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x) - K(y)| \leq C\|x - y\|,$$

(A4) For $r_n = O\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy:

$$\frac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \frac{n\phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty$ for some $\beta > 1$

Remark 4.1. *Note that Assumptions (A1) and (A2) are, respectively, the uniform version of (H1) and (H2). Assumptions (A1) and (A4) are linked with the the topological structure of the functional variable, see Ferraty* et al. *[13].*

Theorem 4.3. *Under Assumptions (A1)-(A4) and (H4), as n goes to infinity, we have*

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{4.12}$$

In the particular case, where the functional single-index is fixed we get the following result.

Corollary 4.2. *Under Assumptions (A1)-(A4) and (H4), as n goes to infinity, we have*

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{4.13}$$

Clearly The proofs of these two results namely the Theorem 4.3 and Corollary 4.2 can be deduced from the following intermediate results which are only uniform version of Lemmas 3.1-3.3 and Corollary 3.1.

Lemma 4.6. *Under Assumptions (A1), (A3) and (A4), we have as $n \to \infty$*

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta, x) - 1| = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

Corollary 4.3. *Under the assumptions of Lemma 4.6, we have,*

$$\sum_{n=1}^{\infty} \left(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in\mathcal{S}_{\mathcal{H}}} \widehat{F}_D(\theta, x) < \frac{1}{2} \right) < \infty$$

Lemma 4.7. *Under Assumptions (A1), (A2) and (H4), we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |F(\theta, y, x) - (\widehat{F}_N(\theta, y, x))| = O(h_K^{b_1}) + O(h_H^{b_2}) \tag{4.14}$$

Lemma 4.8. *Under the assumptions of Theorem 4.3, we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}_N(\theta, y, x) - \left[\widehat{F}_N(\theta, y, x)\right]| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + log\ d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

4.2 Estimating successive derivatives of the conditional density

In this part we focus on the study of uniform almost complete convergence of our estimator defined above (3.8). Thus, in addition to the conditions introduced in the section 4, we need the following ones.

(A5) $\forall (y_1, y_2) \in \mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}, \forall (x_1, x_2) \in \mathcal{S}_{\mathcal{F}} \times \mathcal{S}_{\mathcal{F}}$ and $\forall \theta \in \Theta_{\mathcal{F}}$,

$$|f^{(j)}(\theta, y_1, x_1) - f^{(j)}(\theta, y_2, x_2)| \leq C\left(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}\right),$$

(A6) For some $\gamma \in (0,1)$, $\lim_{n\to\infty} n^{\gamma} h_H = \infty$, and for $r_n = O\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{F}}}$ and $d_n^{\Theta_{\mathcal{F}}}$ satisfy:

$$\frac{(log\ n)^2}{n h_H^{2j+1} \phi(h_K)} < log\ d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}} < \frac{n h_H^{2j+1} \phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{(3\gamma+1)/2} (d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}})^{1-\beta} < \infty$, for some $\beta > 1$

Theorem 4.4. *Under Hypotheses (A1), (A3) ,(A5)-(A6) and (H8), as n goes to infinity, we have*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |\widehat{f}^{(j)}(\theta, y, x) - f^{(j)}(\theta, y, x)| = O(h_K^{b_1}) + O(h_H^{b_2}) + O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}}}{n h_H^{2j+1} \phi(h_K)}}\right) \tag{4.15}$$

Proof. This result is based on the same kind of decomposition (3.10), therefore, Theorem 4.4 can be deduced from both following lemmas, together with Lemma 4.6 and Corollary 4.3.

Lemma 4.9. *Under Assumptions (A1), (A5) and (H8), we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} |f^{(j)}(\theta, y, x) - (\widehat{f}_N^{(j)}(\theta, y, x))| = O(h_K^{b_1}) + O(h_H^{b_2})$$

Lemma 4.10. *Under the assumptions of Theorem 4.4, we have, as n goes to infinity*

$$\sup_{\theta\in\Theta_{\mathcal{F}}} \sup_{x\in\mathcal{S}_{\mathcal{F}}} \sup_{y\in\mathcal{S}_{\mathbb{R}}} \left|\widehat{f}_N^{(j)}(\theta, y, x)] - \left[\widehat{f}_N^{(j)}(\theta, y, x)\right]\right| = O_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}}}{n h_H^{2j+1} \phi_{\theta,x}(h_K)}}\right)$$

□

5 Applications

5.1 The conditional mode in functional single-index model

In this section we will consider the problem of the estimation of the conditional mode in the functional single-index model. The main objective, here, is to establish the almost complete convergence of the kernel estimator of the conditional mode of Y given $< X, \theta > = < x, \theta >$ denoted by $M_\theta(x)$, uniformly on fixed subset $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{H}$. To this end, we suppose that $M_\theta(x)$ satisfies on $\mathcal{S}_{\mathcal{H}}$ the following uniform uniqueness property (see, Ould-saïd and Cai [23], for the multivariate case).

(A6) $\forall \varepsilon_0 > 0, \exists \eta > 0, \forall \varphi : \mathcal{S}_{\mathcal{H}} \longrightarrow \mathcal{S}_{\mathbb{R}}$,

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} |M_\theta(x) - \varphi(x)| \geq \varepsilon_0 \Longrightarrow \sup_{x\in\mathcal{S}_{\mathcal{H}}} |f(\theta, \varphi(x), x) - f(\theta, M_\theta(x), x)| \geq \eta.$$

We estimate the conditional mode $\widehat{M_\theta}(x)$ with a random variable M_θ such as

$$\widehat{M_\theta}(x) = \arg\sup_{y\in\mathcal{S}_{\mathbb{R}}} \widehat{f}(\theta, y, x). \tag{5.16}$$

Note that the estimate $\widehat{M_\theta}$ is not necessarily unique, and if this is the case all the remaining of our paper will concern any value $\widehat{M_\theta}$ satisfying (5.16). The difficulty of the problem is naturally linked with the flatness of the function $f(\theta, y, x)$ around the mode M_θ. This flatness can be controlled by the number of vanishing derivatives at point M_θ, and this parameter will also have a great influence on the asymptotic rates of our estimates. More precisely, we introduce the following additional smoothness condition.

$$(A7)\begin{cases} f^{(l)}(\theta, M_\theta(x), x) = 0, & \text{if;} 1 \leq l < j \\ \text{and } f^{(j)}(\theta, \cdot, x), & \text{is uniformly continuous on } \mathcal{S}_{\mathbb{R}} \\ \text{such that,} & |f^{(j)}(\theta, \cdot, x)| > C > 0 \end{cases}$$

Theorem 5.5. *Under the assumptions of Theorem 4.4 hold together with (A6)-(A7) we have*

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{M}_\theta(x) - M_\theta(x)| = O(h_K^{\frac{b_1}{j}}) + O(h_H^{\frac{b_2}{j}}) + O_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n^{1-\gamma}\phi(h_K)}\right)^{\frac{1}{2j}}\right)$$

Let us now define the application framework of our results to prediction problem by applying the result in the above Theorem, we obtain the following result.

Corollary 5.4. *Under the assumptions of Theorem 5.5, we have as n goes to infinity*

$$\widehat{M}_\theta(x) - M_\theta(x) \longrightarrow 0 \quad a.co.$$

5.2 Conditional quantile in functional single-index model

In this part of paper we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are from a single functional index model and data are independent and identically distributed (i.i.d.)

We will consider the problem of the estimation of the conditional quantiles. Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $< X, \theta >$. Now, let $t_\theta(\alpha)$ be the α-order quantile of the distribution of Y given $< X, \theta >=< x, \theta >$. From the *cond-cdf* $F(\theta, \cdot, x)$, it is easy to give the general definition of the α-order quantile:

$$t_\theta(\alpha) = \inf\{t \in \mathbb{R} : \; F(\theta, t, x) \geq \alpha\}, \qquad \forall \alpha \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our part (the functional feature of $< X, \theta >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $t_\theta(\alpha)$. This is insuring unicity of the conditional quantile $t_\theta(\alpha)$ which is defined by:

$$t_\theta(\alpha) \; = \; F^{-1}(\theta, \alpha, x). \tag{5.17}$$

In what remains, we wish to stay in a free distribution framework. This will lead to assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modelling (see Section 2).
As by-product of (5.17) and (3.1), it is easy to derive an estimator $\widehat{t}_\theta(\alpha)$ of $t_\theta(\alpha)$:

$$\widehat{t}_\theta(\alpha) \; = \; \widehat{F}^{-1}(\theta, \alpha, x). \tag{5.18}$$

As we will see later on, such an estimator is unique as soon as H is an increasing continuous function.
Naturally, we will estimate this quantile by mean of the conditional distribution estimator studied in previous sections. Here also, as far as we know, the literature on (conditional and/or unconditional) quantile estimation is quite important when the explanatory variable X is real (see for instance Samanta, 1989, for previous results and Berlinet *et al.*, 2001, for recent advances and references). In the functional case, the conditional quantiles for scalar response and a scalar/multivariate covariate have received considerable interest in the statistical literature. For completely observed data, several nonparametric approaches have been proposed, for instance, Gannoun et al., (2003) introduced a smoothed estimator based on double kernel and local constant kernel methods and Berlinet et al., (2001) established its asymptotic normality. Under random censoring, Gannoun et al., (2005) introduced a local linear (LL) regression (see Koenker and Bassett (1978) for the definition) and El Ghouch and Van Keilegom (2009) studied the same LL estimator. Ould-Saïd (2006) constructed a kernel estimator of the conditional quantile under independent and identically distributed (i.i.d.) censorship model and established its strong uniform convergence rate. Liang and De Uña-Àlvarez (2011) established the strong uniform convergence (with rate) of the conditional quantile function under α-mixing assumption.

Recently, many authors are interested in the estimation of conditional quantiles for a scalar response and functional covariate. Ferraty *et al.*, (2005) introduced a nonparametric estimator of conditional quantile defined as the inverse of the conditional cumulative distribution function when the sample is considered as an α-mixing sequence. They stated its rate of almost complete consistency and used it to forecast the well-known El Niño time series and to build confidence prediction bands. Ezzahrioui and Ould-Saïd (2008) established the asymptotic normality of the kernel conditional quantile estimator under α-mixing assumption. Recently, and within the same framework, Dabo-Niang and Laksaci (2012) provided the consistency in L^p norm of the conditional quantile estimator for functional dependent data.

So, in this work we propose to estimate $t_\theta(\alpha)$ by the estimate $\widehat{t}_\theta(\alpha)$ defined as (5.18) or as

$$\widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) = \alpha. \tag{5.19}$$

To insure existence and unicity of this quantile, we will assume that

(A8) $F(\theta, \cdot, x)$ is strictly increasing,

Note that, because H is a *cdf* satisfying (H4), such a value $\widehat{t}_\theta(\alpha)$ is always existing. It could be the case that it is not unique, but if this happens all the remaining of the paper will concern any among all the values $\widehat{t}_\theta(\alpha)$ satisfying (5.19).

In order to insure unicity of $\widehat{t}_\theta(\alpha)$ we will make the following, quite unrestrictive, assumption:

(A9) H is is strictly increasing,

As for the mode estimation problem discussed before, the difficulty occur in estimating the conditional quantile $t_\theta(\alpha)$ is linked with the flatness of the curve of the conditional distribution $F(\theta, \cdot, x)$ around $t_\theta(\alpha)$. More precisely, we will suppose that there exists some integer $j > 0$ such that:

$$(A10) \begin{cases} F^{(l)}(\theta, t_\theta(\alpha), x) = 0, & \text{if; } 1 \le l < j \\ \text{and } F^{(j)}(\theta, \cdot, x), & \text{is uniformly continuous on; } \mathcal{S}_\mathbb{R} \\ \text{such that,} & |F^{(j)}(\theta, t_\theta(\alpha), x)| > C > 0 \end{cases}$$

Theorem 5.6. *If the conditions of Theorem 4.4 hold together with (A8)-(A10), we have*

$$\sup_{x \in \mathcal{S}_\mathcal{H}} |\widehat{t}_\theta(\alpha) - t_\theta(\alpha)| = O\left(h_K^{\frac{b_1}{j}} + h_H^{\frac{b_2}{j}}\right) + O\left(\left(\frac{\log d_n^{\mathcal{S}_\mathcal{H}}}{n\ \phi_x(h_K)}\right)^{\frac{1}{2j}}\right), \qquad a.co. \tag{5.20}$$

Proof. Let us write the following Taylor expansion of the function $\widehat{F}(\theta, \cdot, x)$:

$$\begin{aligned} \widehat{F}(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l}{l!} \widehat{F}^{(l)}(\theta, t_\theta(\alpha), x) \\ &+ \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j}{j!} \widehat{F}^{(j)}(\theta, t^*, x), \end{aligned}$$

where t^* is some point between $t_\theta(\alpha)$ and $\widehat{t}_\theta(\alpha)$. It suffices now to use the first part of condition (A10) to be able to rewrite this expression as:

$$\begin{aligned} \widehat{F}(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l}{l!} \left(\widehat{f}^{(l-1)}(\theta, t_\theta(\alpha), x) - f^{(l-1)}(\theta, t_\theta(\alpha), x)\right) \\ &+ \frac{(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j}{j!} \widehat{f}^{(j-1)}(\theta, t^*, x), \end{aligned}$$

As long as we could be able to check that

$$\exists\tau > 0, \sum_{n=1}^{n=\infty} \left(f^{(j-1)}(\theta, t^*, x) < \tau \right) \; < \; \infty, \tag{5.21}$$

we would have

$$\begin{aligned} (t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j \quad &= \quad O\left(\widehat{F}(\theta, t_\theta(\alpha), x) - F(\theta, t_\theta(\alpha), x)\right) \\ &+ \quad O\left(\sum_{l=1}^{j-1}(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^l(\widehat{f}^{(l-1)}(\theta, t_\theta(\alpha), x) - f^{(l-1)}(\theta, t_\theta(\alpha), x)\right), \; a.co. \end{aligned} \tag{5.22}$$

By comparing the rates of convergence given in Theorems 4.3 and 4.4, we see that the leading term in right hand side of equation (5.22) is the first one. So we have

$$(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j \; = \; O_{a.co.}\left(\widehat{F}(\theta, t_\theta(\alpha), x) - F(\theta, t_\theta(\alpha), x)\right),$$

Because of Theorem 4.4, this is enough to get the claimed result, and so (5.21) is the only result that remains to check. This will be done directly by using the uniform continuity of the function $f^{(j-1)}(\theta, \cdot, x)$ given by second part of (A10) together with the third part of (A7) and with the following lemma.

Lemma 5.11. *If the conditions of Theorem 4.3 hold together with (A8) and (A9), then we have:*

$$\widehat{t}_\theta(\alpha) - t_\theta(\alpha) \; \to \; 0, \; a.co. \tag{5.23}$$

□

The next part is devoted to another type of application called the cross-validation method, this application has been already given in [2].

5.3 The cross-validation method

This method is widely applied, it can be used to compare the performances of different predictive modeling procedures. For instance, in optical character recognition; a mechanical or electronic conversion of scanned or photographed images of typewritten or printed text into machine-encoded/computer-readable text, this later is widely used as a form of data entry from some sort of original paper data source, whether passport documents, invoices, bank statement, receipts, business card, mail, or any number of printed records. It can also be used in variable selection; the process of selecting a subset of relevant features for use in model construction.

After this short introduction let's give an application of the method:

1. The regression operator $\widehat{r}_\theta(x)$ depends on the functional parameter θ, So, a crucial question arises: how to choose the functional index θ? The answer is nontrivial and a firstway consists in extending the standard cross-validation procedure to our functional context. For this, one considers various quadratic distances, namely the averaged squared error

$$\mathrm{ASE}(\theta) = n^{-1}\sum_{j=1}^{n}\left(r_{\theta_0}(X_j) - \widehat{r}_\theta(X_j)\right)^2, \tag{5.24}$$

the integrated squared error

$$\mathrm{ISE}(\theta) = \; \left[(r\theta_0(X_0) - \widehat{r}_\theta(X_0))^2 \,|Z_1, \ldots, Z_n\right], \tag{5.25}$$

and the mean integrated squared error

$$\mathrm{MISE}(\theta) = \; [ISE(\theta)]\,. \tag{5.26}$$

The main goal consists in finding a θ which minimizes (in some sense) over Θ_n these quantities. However, because all these quadratic distances depend on the unknown regression operator r_{θ_0}, the criterion used in practice for choosing θ is

$$CV(\theta) = n^{-1} \sum_{j=1}^{n} \left(Y_j - \widehat{r}_{\theta}^{-j}(X_j) \right)^2 \tag{5.27}$$

where $\widehat{r}_{\theta}^{-j}$ is the leave-one-out estimate of $r_\theta(x)$, given by

$$\widehat{r}_{\theta}^{-j}(x) = \frac{(n-1)^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} Y_i K\left(h_K^{-1}(< x - X_i, \theta >) \right)}{(n-1)^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} K\left(h_K^{-1}(< x - X_i, \theta >) \right)}. \tag{5.28}$$

So, the selection rule will be to choose θ_{CV} which minimizes the so-called cross-validation criterion $CV(\theta)$. Clearly, for a given θ, $CV(\theta)$ is a computable quantity. It measures a quadratic distance between $(Y_1, \ldots, Y_n)$ and its prediction $\widehat{r}_{\theta}^{-j}(X_1), \ldots, \widehat{r}_{\theta}^{-j}(X_n)$ when, for each i, $\widehat{r}_{\theta}^{-i}(.)$ is built without the ith data (X_i, Y_i). So, the method of cross-validation consists in choosing among several candidates θ, the one who is the most adapted to our data set (X_i, Y_i) in terms of prediction. This method is inspired by the cross-validation ideas which have been proposed in various standard nonparametric estimation problems (see [17] for the regression problem, [22] for the density and [26] for the hazard function).

From a practical point of view, some questions arise in order to implement this single-functional index model. What about the identifiability of the model given a sample of observed curves $(x_1, \ldots, x_n)$? How to build the set of functional indexes $\Theta_{\mathcal{F}}$? What about the choice of the bandwidth h?

Emphasizes the good behaviour of this simple cross-validated procedure, even in pathological situations. To see that, one focuses on a favourable case (i.e. $\theta_0 \in \Theta_{\mathcal{F}}$).

First of all, one builds a sample of n curves curves as follows:

$$x_i(t_j) = a_i cos(2\pi t_j) + b_i sin(4\pi t_j) + 2c_i(t_j - 0.25)(t_j - 0.5),$$

where $0 = t_1 < t_2 < \ldots < t_{n-1} < t_n = 1$ are equispaced points, the a_i's, b_i's and c_i's being independent observations uniformly distributed on $[0, 1]$. Once the curves are defined, one simulates a single-functional index model as follows:

- Choose one $\theta_0(\cdot)$.
- Choose one link function $r(\cdot)$.
- Compute the inner products $< \theta_0, x_1 >, \ldots, < \theta_0, x_n >$.
- Generate independently $\varepsilon_1, \ldots, \varepsilon_n$, from a centred Gaussian of variance equal to 0.05 times the empirical variance of
  $r(< \theta_0, x_1 >), \ldots, r(< \theta_0, x_n >)$ (i.e. signal-to-noise ratio = 0.05).
- Simulate the corresponding responses: $Y_i = r(< \theta_0, x_i >) + \varepsilon_i$.

Finally, the observations $(x_k, Y_k)_{k=1,\ldots,m}$ are used for the learning step and the others (i.e. $(x_l, Y_l)_{l=m+1,\ldots,n}$ allow the computation of the mean square error of prediction:

$$\text{MSEP} = \frac{1}{n-m} \sum_{j=n-m}^{n} \left(Y_j \widehat{r}\left(< \theta_{CV}, x_j >\right)\right)^2.$$

In order to highlight the specificity of such a single-functional index model, the obtained predictions are compared with those coming from a pure nonparametric functional data analysis (NPFDA) method (see [12] for details and references therein). Actually, the NPFDA regression method uses the following kernel estimator:

$$\forall x \in \mathcal{H}, \quad \widehat{r}(x) = \frac{\sum_{i=1}^{n} Y_i K\left(h^{-1}(d(X_i, x)\right)}{\sum_{i=1}^{n} K\left(h^{-1}(d(X_i, x)\right)} \tag{5.29}$$

for estimating the regression operator m in the nonparametric model $Y_i = r(X_i) + \varepsilon_i$, for all $i = 1, \dots, n$, where $d(.,.)$ is a fixed semi-metric.

If one looks at the NPFDA kernel estimator (5.29), it suffices to replace the fixed semi-metric $d(\cdot,\cdot)$ with $d_{\theta_{CV}}(\cdot,\cdot)$. What does this mean? It means that the functional index model can be seen as one way of building an nonparametric functional data analysis (NPFDA) kernel estimator with a data-driven semi-metric. In particular, in pure nonparametric functional models when one has no idea of the semi-metric, the functional index model appears to be a method for performing an adaptative one. The functional index model makes the NPFDA method more flexible. In this sense, the functional index model is not a competitive statistical technique with respect to the NPFDA method, but rather a complementary one.

2. If we wish to predict a real characteristic denoted Y of X_n knowing the curve X_{n-1}, we have to consider the observations (X_i, y_i) where y_i is the characteristic we want to provide at the instant i. For example:

   - If we want to predict the value of the process at time t_j knowing the curve X_{n-1}, we set $Y_i = X_{i+1}(t_j)$.
   - For the sup, we pose $Y_i = \sup_t X_{i+1}(t)$.
   - If we look for the time where the process reaches maximum, we set $Y_i = \arg\sup_t X_{i+1}(t)$.

   By using the conditional mode as a prediction tool, we can predict Y by $\widehat{M_\theta(X_{n}}_{-1})$.

6 Appendix

Proof of Lemma 4.6 For all $x \in \mathcal{S}_{\mathcal{H}}$ and $\theta \in \Theta_{\mathcal{H}}$, we set

$$k(x) = \arg\min_{k\in\{1\ldots r_n\}} \|x - x_k\| \text{ and } j(\theta) =) = \arg\min_{j\in\{1\ldots l_n\}} \|\theta - t_j\|.$$

Let us consider the following decomposition

$$\begin{aligned}
\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta,x) - \left(\widehat{F}_D(\theta,x)\right)\right| \leq & \underbrace{\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta,x) - (\widehat{F}_D(\theta,x_{k(x)})\right|}_{\Pi_1} \\
& + \underbrace{\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(\theta,x_{k(x)}) - \widehat{F}_D(t_{j(\theta)},x_{k(x)})\right|}_{\Pi_2} \\
& + \underbrace{\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\widehat{F}_D(t_{j(\theta)},x_{k(x)}) - \left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right)\right|}_{\Pi_3} \\
& + \underbrace{\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right) - \left(\widehat{F}_D(\theta,x_{k(x)})\right)\right|}_{\Pi_4} \\
& + \underbrace{\sup_{x\in\mathcal{S}_{\mathcal{H}}} \sup_{\theta\in\Theta_{\mathcal{H}}} \left|\left(\widehat{F}_D(\theta,x_{k(x)})\right) - \left(\widehat{F}_D(\theta,x)\right)\right|}_{\Pi_5}
\end{aligned}$$

For Π_1 and Π_2, we employe the Hölder continuity condition on K, Cauchy Schwartz's and the Bernstein's inequalities, we get

$$\Pi_1 = O\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right), \ \Pi_2 = O\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right) \tag{6.30}$$

Then, by using the fact that $\Pi_4 \leq \Pi_1$ and $\Pi_5 \leq \Pi_2$, we get for n tending to infinity

$$\Pi_4 = O\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right), \ \Pi_5 = O\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right) \tag{6.31}$$

Now, we deal with Π_3, for all $\eta > 0$, we have

$$\left(\Pi_3 > \eta\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right)\right) \leq$$

$$d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \max_{k\in\{1\ldots d_n^{S_\mathcal{H}}\}} \max_{j\in\{1\ldots d_n^{\Theta_\mathcal{H}}\}} \left(\left|\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right)\right| > \eta\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right)\right).$$

Applying Bernstein's exponential inequality to

$$\frac{1}{\phi(h_K)}\left(K_i\left(t_{j(\theta)}, x_{k(x)}\right) - \left(K_i\left(t_{j(\theta)}, x_{k(x)}\right)\right)\right),$$

then under (A7), we get

$$\Pi_3 = O\left(\sqrt{\frac{\log d_n^{S_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

Lastly the result will be easily deduced from the latter together with (6.30) and (6.31).

Proof Corollary 4.3 It is easy to see that,

$\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} |\widehat{F}_D(\theta, x)| \leq 1/2 \implies \exists x \in \mathcal{S}_\mathcal{H}, \ \exists\theta \in \Theta_\mathcal{H}$, such that

$$1 - \widehat{F}_D(\theta, x) \geq 1/2 \implies \sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in\mathcal{S}_\mathcal{H}} |1 - \widehat{F}_D(\theta, x)| \geq 1/2.$$

We deduce from Lemma 4.6 the following inequality

$$\left(\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} |\widehat{F}_D(\theta, x)| \leq 1/2\right) \leq \left(\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in\mathcal{S}_\mathcal{H}} |1 - \widehat{F}_D(\theta, x)| \leq 1/2\right).$$

Consequently,

$$\sum_{n=1}^{\infty}\left(\inf_{\theta\in\Theta_\mathcal{H}} \inf_{x\in\mathcal{S}_\mathcal{H}} \widehat{F}_D(\theta, x) < \frac{1}{2}\right) < \infty$$

□

Proof of Lemma 4.7 One has

$$\begin{aligned}
\widehat{F}_N(\theta, y, x) - F(\theta, y, x) &= \frac{1}{K_1(x,\theta)}\left[\sum_{i=1}^{n} K_i(x,\theta) H_i(y)\right] - F(\theta, y, x) \\
&= \frac{1}{K_1(x,\theta)}\left(K_1(x,\theta)\left[E\left(H_1(y)| < X_1, \theta >\right) - F(\theta, y, x)\right]\right).
\end{aligned} \tag{6.32}$$

Moreover, we have

$$(H_1(y)| < X_1, \theta >) = \int_{\mathbb{R}} H\left(h_H^{-1}(y - z)\right) f(\theta, z, X_1) dz,$$

now, integrating by parts and using the fact that H is a *cdf*, we obtain

$$(H_1(y)| < X_1, \theta >) = \int_{\mathbb{R}} H^{(1)}(t) F(\theta, y - h_H t, X_1) dt.$$

Thus, we have

$$|(H_1(y)| < X_1, \theta >) - F(\theta, y, x)| \leq \int_{\mathbb{R}} H^{(1)}(t) \, |F(\theta, y - h_H t, X_1) - F(\theta, y, x)| \, dt.$$

Finally, the use of (A2) implies that

$$|(H_1(y)|X_1) - F^x(y)| \leq C_{\theta,x} \int_{\mathbb{R}} H^{(1)}(t) \left(h_K^{b_1} + |t|^{b_2} h_H^{b_2} \right) dt. \quad (6.33)$$

Because this inequality is uniform on $(\theta, y, x) \in \Theta_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathbb{R}}$ and because of (H4), (4.14) is a direct consequence of (6.32), (6.33) and of Corollary 4.3.

□

Proof of Lemma 4.8 We keep the notation of the Lemma 4.6 and we use the compact of $\mathcal{S}_{\mathbb{R}}$, we can write that, for some, $t_1, \ldots, t_{z_n} \in \mathcal{S}_{\mathbb{R}}$, $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{z_n} (y_m - l_n, y_m + l_n)$ with $l_n = n^{-1/2b_2}$ and $z_n \leq C n^{-1/2b_2}$. Taking $m(y) = \arg \min_{\{1,2,\ldots,z_n\}} |y - t_m|$.

Thus, we have the following decomposition:

$$\begin{aligned}
\left| \widehat{F}_N(\theta, y, x) - \left(\widehat{F}_N(\theta, y, x) \right) \right| = & \underbrace{\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right|}_{\Gamma_1} \\
& + \underbrace{\left| \widehat{F}_N(\theta, y, x_{k(x)}) - \left(\widehat{F}_N(\theta, y, x_{k(x)}) \right) \right|}_{\Gamma_2} \\
& + 2 \underbrace{\left| \widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) \right|}_{\Gamma_3} \\
& + 2 \underbrace{\left| \left(\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) \right) - \left(\widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) \right) \right|}_{\Gamma_4} \\
& + \underbrace{\left| \left(\widehat{F}_N(\theta, y, x_{k(x)}) \right) - \left(\widehat{F}_N(\theta, y, x) \right) \right|}_{\Gamma_5}
\end{aligned}$$

↪ Concerning Γ_1 we have

$$\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \left| \frac{1}{K_1(\theta, x)} K_i(\theta, x) H_i(y) - \frac{1}{K_1(\theta, x_{k(x)})} K_i(\theta, x_{k(x)}) H_i(y) \right|.$$

We use the Hölder continuity condition on K, the Cauchy-Schwartz inequality, the Bernstein's inequality and the boundness of H (assumption (H4)). This allows us to get:

$$\begin{aligned}
\left| \widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)}) \right| & \leq \frac{C}{\phi(h_K)} \frac{1}{n} \sum_{i=1}^{n} \left| K_i(\theta, x) H_i(y) - K_i(\theta, x_{k(x)}) H_i(y) \right| \\
& \leq \frac{C}{\phi(h_K)} \frac{1}{n} \sum_{i=1}^{n} |H_i(y)| \left| K_i(\theta, x) - K_i(\theta, x_{k(x)}) \right| \\
& \leq \frac{C' r_n}{\phi(h_K)}
\end{aligned}$$

↪ Concerning Γ_2, the monotony of the functions $\widehat{F}_N(\theta, \cdot, x)$ and $\widehat{F}_N(\theta, \cdot, x)$ permits to write, $\forall m \leq z_n$, $\forall x \in \mathcal{S}_{\mathcal{H}}$, $\forall \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned}
\widehat{F}_N(\theta, y_{m(y)} - l_n, x_{k(x)}) \leq \sup_{y \in (y_{m(y)} - l_n, y_{m(y)} + l_n)} \widehat{F}_N(\theta, y, x) \leq \widehat{F}_N(\theta, y_{m(y)} + l_n, x_{k(x)}) \\
\widehat{F}_N(\theta, y_{m(y)} - l_n, x_{k(x)}) \leq \sup_{y \in (y_{m(y)} - l_n, y_{m(y)} + l_n)} \widehat{F}_N(\theta, y, x) \leq \widehat{F}_N(\theta, y_{m(y)} + l_n, x_{k(x)}). \quad (6.34)
\end{aligned}$$

Next, we use the Hölder's condition on $F(\theta, y, x)$ and we show that, for any $y_1,\ y_2 \in \mathcal{S}_{\mathbb{R}}$ and for all $x \in \mathcal{S}_{\mathcal{H}},\ \theta \in \Theta_{\mathcal{H}}$

$$\begin{aligned}\left|\widehat{F}_N(\theta, y_1, x) - \widehat{F}_N(\theta, y_2, x)\right| &= \frac{1}{K_1(x,\theta)} \left| (K_1(x,\theta)F(\theta, y_1, X_1)) - (K_1(x,\theta)F(\theta, y_2, X_1))\right| \\ &\leq C|y_1 - y_2|^{b_2}. \end{aligned} \tag{6.35}$$

Now, we have, for all $\eta > 0$

$$\left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right)$$

$$=$$

$$\left(\max_{j\in\{1\dots d_n^{\Theta_\mathcal{H}}\}} \max_{k\in\{1\dots d_n^{\mathcal{S}_\mathcal{H}}\}} \max_{1\leq m\leq z_n} \left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$z_n d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \max_{j\in\{1\dots d_n^{\Theta_\mathcal{H}}\}} \max_{k\in\{1\dots d_n^{\mathcal{S}_\mathcal{H}}\}} \max_{1\leq m\leq z_n} \left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{F}} d_n^{\Theta_\mathcal{F}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$2z_n d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}} \exp\left(-C\eta^2 \log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\right)$$

choising $z_n = O\left(l_n^{-1}\right) = O\left(n^{\frac{1}{2b_2}}\right)$, we get

$$\left(\left|\widehat{F}_N(\theta, y, x_{k(x)}) - \widehat{F}_N(\theta, y, x_{k(x)})\right| > \eta\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right) \leq C' z_n \left(d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\right)^{1-C\eta^2}$$

putting $C\eta^2 = \beta$ and using (A4), we get

$$\Gamma_2 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning the terms Γ_3 and Γ_4, using Lipschitz's condition on the kernel H, one can write

$$\begin{aligned}\left|\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right| &\leq C\frac{1}{n\phi(h_K)} \sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)}) \left|H_i(y) - H_i(y_{m(y)}\right| \\ &\leq \frac{Cl_n}{nh_H\phi(h_K)} \sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)}).\end{aligned}$$

Once again a standard exponential inequality for a sum of bounded variables allows us to write

$$\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) = O\left(\frac{l_n}{h_H}\right) + O_{a.co}\left(\frac{l_n}{h_H}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right).$$

Now, the fact that $\lim_{n\to\infty} n^\gamma h_H = \infty$ and $l_n = n^{-1/2b_2}$ imply that:

$$\frac{l_n}{h_H\phi(h_K)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right),$$

then

$$\Gamma_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

Hence, for n large enough, we have

$$\Gamma_3 \leq \Gamma_4 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning Γ_5, we have

$$\left(\widehat{F}_N(\theta, y, x_{k(x)})\right) - \left(\widehat{F}_N(\theta, y, x)\right) \leq \sup_{x\in\mathcal{S}_\mathcal{H}} \left|\widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)})\right|,$$

then following similar proof used in the study of Γ_1 and using the same idea as for $\left(\widehat{F}_D(\theta, x_{k(x)})\right) - \left(\widehat{F}_D(\theta, x)\right)$ we get, for n tending to infinity,

$$\Gamma_5 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

□

Proof of Lemma 4.9. Let $H_i^{(j+1)}(y) = H^{(j+1)}\left(h_H^{-1}(y - Y_i)\right)$, note that

$$\widehat{f}_N^{(j)}(\theta, y, x) - f^{(j)}(\theta, y, x) = \frac{1}{h_H^{j+1} K_1(x,\theta)}\left(K_1(x,\theta)\left[\left(H_1^{(j+1)}(y)| < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right]\right). \quad (6.36)$$

Moreover,

$$\begin{aligned}\left(H_1^{(j+1)}(y)| < X, \theta >\right) &= \int_\mathbb{R} H^{(j+1)}\left(h_H^{-1}(y - z)\right) f(\theta, z, X) dz, \\ &= -\sum_{l=1}^{j} h_H^l \left[H^{(j-l+1)}\left(h_H^{-1}(y - z)\right) f^{(l-1)}(\theta, z, X)\right]_{-\infty}^{+\infty} \\ &\quad + h_H^j \int_\mathbb{R} H^{(1)}\left(h_H^{-1}(y - z)\right) f^{(j)}(\theta, z, X) dz. \end{aligned} \quad (6.37)$$

Condition (H8) allows us to cancel the first term in the right side of (6.37) and we can write:

$$\left|\left(H_1^{(j+1)}(y)| < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right| \leq h_H^{j+1} \int_\mathbb{R} H^{(1)}(t) \left|f^{(j)}(\theta, y - h_H t, X) - f^{(j)}(\theta, y, x)\right| dt.$$

Finally, (A5) allows to write

$$\left|\left(H_1^{(j+1)}(y)| < X, \theta >\right) - h_H^{j+1} f^{(j)}(\theta, y, x)\right| \leq C_{\theta,x} h_H^{j+1} \int_\mathbb{R} H^{(1)}(t) \left(h_K^{b_1} + |t|^{b_2} h_H^{b_2}\right) dt. \quad (6.38)$$

This inequality is uniform on $(\theta, y, x) \in \Theta_\mathcal{F} \times \mathcal{S}_\mathcal{F} \times \mathcal{S}_\mathbb{R}$, now to finish the proof it is sufficient to use (H4).

□

Proof of Lemma 4.10. Let $l_n = n^{-\frac{3}{2}\gamma - \frac{1}{2}}$ and $z_n \leq Cn^{-\frac{3}{2}\gamma - \frac{1}{2}}$.

Consider the following decomposition

$$\begin{aligned}\left|\widehat{f}_N^{(j)}(\theta, y, x) - \left(\widehat{f}_N^{(j)}(\theta, y, x)\right)\right| = &\underbrace{\left|\widehat{f}_N^{(j)}(\theta, y, x) - \widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right|}_{\Delta_1} \\ &+ \underbrace{\left|\widehat{f}_N^{(j)}(\theta, y, x_{k(x)}) - \left(\widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right)\right|}_{\Delta_2} \\ &+ 2\underbrace{\left|\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right|}_{\Delta_3} \\ &+ 2\underbrace{\left|\left(\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)})\right) - \left(\widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right)\right|}_{\Delta_4} \\ &+ \underbrace{\left|\left(\widehat{f}_N^{(j)}(\theta, y, x_{k(x)})\right) - \left(\widehat{f}_N^{(j)}(\theta, y, x)\right)\right|}_{\Delta_5}\end{aligned}$$

$\rightsquigarrow$ Concerning Δ_1, we use the Hölder continuity condition on K, the Cauchy-Schwartz's inequality and the Bernstein's inequality. With theses arguments we get

$$\Delta_1 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right).$$

Then using the fact that $\Delta_5 \leq \Delta_1$, we obtain

$$\Delta_5 \leq \Delta_1 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right). \quad (6.39)$$

$\rightsquigarrow$ For Δ_2, we follow the same idea given for Γ_2, we get

$$\Delta_2 = O\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right)$$

$\rightsquigarrow$ Concerning Δ_3 and Δ_4, Using Lipschitz's condition on the kernel H,

$$\left|\widehat{f}_N^{(j)}(t_{j(\theta)}, y, x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)}, y_{m(y)}, x_{k(x)})\right| \leq \frac{l_n}{h_H^{j+2}\phi(h_k)},$$

using the fact that $\lim_{n\to\infty} n^{\gamma} h_H = \infty$ and choosing $l_n = n^{-\frac{3}{2}\gamma - \frac{1}{2}}$ implies

$$\frac{l_n}{h_H^{j+2}\phi(h_k)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right)$$

So, for n large enough, we have

$$\Delta_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right).$$

And as $\Delta_4 \leq \Delta_3$, we obtain

$$\Delta_4 \leq \Delta_3 = O_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n h_H^{2j+1}\phi(h_K)}}\right). \quad (6.40)$$

Finally, the lemma can be easily deduced from (6.39) and (6.40)

□

Proof of Lemma 5.11. Because of (H4) and (A9) the function $\widehat{F}(\theta, \cdot, x)$ is uniformly continuous and strictly increasing. So, we have:

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, \forall y, |\widehat{F}(\theta, y, x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \leq \delta(\epsilon) \Rightarrow |y - t_\theta(\alpha)| \leq \epsilon.$$

This leads directly to

$$\begin{aligned}\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0,\ \ \left(|\widehat{t}_\theta(\alpha) - t_\theta(\alpha)| > \epsilon\right) &\leq \left(|\widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \geq \delta(\epsilon)\right) \\ &= \left(|F(\theta, t_\theta(\alpha), x) - \widehat{F}(\theta, t_\theta(\alpha), x)| \geq \delta(\epsilon)\right).\end{aligned}$$

Finally, It suffices to use the result of Theorem 4.3 to get the claimed result.

□

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