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On solutions for classes of fractional differential equations

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Abstract

We provide a new solution of diffusion fractional differential equation using fractal index method. Also we shall impose a new solution for Riccati equation of arbitrary order. The fractional operators are taken in sense of the Riemann-Liouville operators.

Keywords: Fractional calculus; fractional differential equations; fractal index.

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1 Introduction

Fractional differential equations are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and tools not only in mathematics but also in physics, dynamical systems, control systems and engineering to create the mathematical modeling of many physical phenomena. Furthermore, they employed in social science such as food supplement, climate and economics. Fractional differential equations concerning the Riemann-Liouville fractional operators or Caputo derivative have been recommended by many authors (see [1-5]).

Transform is a significant technique to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature such as wave transformation, Laplace transform, the Fourier transform, the Bücklund transformation, the integral transform, the local fractional integral transforms and the fractional complex transform and Mellin transform (see [6-10]).

One of the most tools in the theory of fractional calculus is viewed by the RiemannLiouville operators. It imposes advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms. In this note, we shall deal with scalar linear time-space fractional differential equations. The time and the space are taken in sense of the Riemann-Liouville fractional operators. Also, This type of differential equation arises in many interesting applications [11-17].

Several researchers have studied fractional dynamic equations generalizing the diffusion or wave equations in terms of R-L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the Mittag-Leffler (M-L) functions and their generalizations. In this work we shall provide a new solution of diffusion fractional differential equation using fractal index method. Also we shall impose a new solution for Riccati equation of arbitrary order. The fractional operators are taken in sense of the Riemann-Liouville operators.

2 Preliminaries

The idea of the fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Abel in 1823 investigated the generalized tautochrone prob-

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lem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Since that time the fractional calculus has haggard the attention of many researchers in all area of sciences (see [1-5]).

Definition 2.1. *The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product, $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0,\ t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. *The fractional (arbitrary) order derivative of the function f of order $0 < \alpha \le 1$ is defined by*

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

In the sequel, we use the notation $\frac{\partial^\alpha}{\partial t^\alpha}$.

Remark 2.1. *From Definition 2.1 and Definition 2.2, $a = 0$, we have*

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha},\ \ \mu > -1;\ 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha},\ \ \mu > -1;\ \alpha > 0.$$

The Leibniz rule is

$$D_a^\alpha[f(t)g(t)] = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D_a^{\alpha-k} f(t) D_a^k g(t) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D_a^{\alpha-k} g(t) D_a^k f(t),$$

where

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}.$$

Definition 2.3. *The Caputo fractional derivative of order $\mu > 0$ is defined, for a smooth function $f(t)$ by*

$${}^cD^\mu f(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu] + 1$, (the notation $[\mu]$ stands for the largest integer not greater than μ).

Note that there is a relationship between Riemann-Liouville differential operator and the Caputo operator

$$D_a^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{f(a)}{(t-a)^\mu} + {}^cD_a^\mu f(t);$$

and they are equivalent in a physical problem (i.e., a problem which specifies the initial conditions) [16].

3 The fractal index

We consider the equation

$$t^\alpha D_t^\alpha u(t,x) + \Big(a(x^\beta - t) + b(ux^\beta - t^2) + c(u^2 x^\beta - t^3)\Big) + eD_x^{3\beta} u(t,x) = 0, \tag{3.1}$$

$$\Big(t \in J := [0,T],\ x \in \mathbb{R}\Big)$$

where $u(0,x) = 0$, $e \neq 0$, $a, b, c \in \mathbb{R}$ and $\alpha, \beta \in (0,1]$.

Eq.(1) involves well known time-space fractional diffusion equations. Several researchers have studied fractional dynamic equations generalizing the diffusion or wave equations in terms of R-L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the MittagLeffler (M-L) functions and their generalizations. The mathematical study of fractional diffusion equations began with the work of Kochubei [18,19]. Later this study followed by the work of Metzler and Klafter [20] and Zaslavsky [21]. Recently, Mainardi et all obtained the time fractional diffusion equation from the standard diffusion equation [22,23].

Let $X = x^\alpha$ and $f = X^n$, $n \neq 0$ then we obtain

$$\begin{aligned}\frac{\partial^\alpha f}{\partial x^\alpha} &= \frac{\partial f}{\partial X}\frac{\partial^\alpha X}{\partial x^\alpha}\\ &= \frac{\Gamma(1+n\alpha)x^{n\alpha-\alpha}}{\Gamma(1+n\alpha-\alpha)} := \frac{\partial f}{\partial X}\theta_\alpha\\ &= n\theta_\alpha x^{n\alpha-\alpha}\end{aligned}$$

we receive

$$\theta_\alpha = \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}.$$

Consequently

$$\begin{aligned}\frac{\partial^{2\alpha} f}{\partial x^{2\alpha}} &= \frac{\partial^2 f}{\partial X^2}\theta_{\alpha\alpha}\\ &:= n(n-1)\theta_{\alpha\alpha}x^{n\alpha-2\alpha}\end{aligned}$$

where

$$\theta_{\alpha\alpha} = \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}\frac{\Gamma(1+n\alpha-\alpha)}{(n-1)\Gamma(1+n\alpha-2\alpha)}.$$

And

$$\begin{aligned}\frac{\partial^{3\alpha} f}{\partial x^{3\alpha}} &= \frac{\partial^3 f}{\partial X^3}\theta_{\alpha\alpha\alpha}\\ &:= n(n-1)(n-2)\theta_{\alpha\alpha\alpha}x^{n\alpha-3\alpha}\end{aligned}$$

where

$$\begin{aligned}\theta_{\alpha\alpha\alpha} &= \frac{\Gamma(1+n\alpha)}{n\Gamma(1+n\alpha-\alpha)}\frac{\Gamma(1+n\alpha-\alpha)}{(n-1)\Gamma(1+n\alpha-2\alpha)}\frac{\Gamma(1+n\alpha-2\alpha)}{(n-2)\Gamma(1+n\alpha-3\alpha)}\\ &= \frac{\Gamma(1+n\alpha)}{n(n-1)(n-2)\Gamma(1+n\alpha-3\alpha)}.\end{aligned}$$

Now we proceed to impose a solution for the Eq. (1) using the fractal method.

Let the solution takes the form

$$u(t,x) = \sum_{n=1}^{\infty}\mu_n(x)t^n, \tag{3.2}$$

where u is analytic in J. By balancing the first two terms of the equation (1) (w.r.t t), we have $n = 2$. Therefore,

$$u(t,x) = \mu_1(x)t + \mu_2(x)t^2, \quad \mu_i(0) = 1,\ i = 1,2. \tag{3.3}$$

By using some properties of the fractional calculus, we obtain

$$\begin{aligned}D_t^\alpha u(t,x) &= \mu_1(x)D_t^\alpha t + \mu_2(x)D_t^\alpha t^2\\ &= \frac{\mu_1(x)}{\Gamma(2-\alpha)}t^{1-\alpha} + \frac{\mu_2(x)}{\Gamma(3-\alpha)}t^{2-\alpha}.\end{aligned}$$

This implies

$$t^\alpha D_t^\alpha u(t,x) = \frac{\mu_1(x)}{\Gamma(2-\alpha)}t + \frac{\mu_2(x)}{\Gamma(3-\alpha)}t^2. \tag{3.4}$$

Moreover,

$$eD_x^{3\beta}u(t,x) = etD_x^{3\beta}\mu_1(x) + et^2D_x^{3\beta}\mu_2(x) \tag{3.5}$$

and

$$\begin{aligned}(a+bu+cu^2)x^\beta - \left(at+bt^2+ct^3\right) = &\Big(a+b[\mu_1(x)t+\mu_2(x)t^2+\mu_3(x)t^3] \\ &+c[\mu_1^2t^2+2\mu_1\mu_2t^3+...]\Big)x^\beta \\ &-\left(at+bt^2+ct^3\right)\end{aligned} \tag{3.6}$$

Next we shall calculate the functions $\mu_i(x)$, $i = 1, 2$. Comparing the coefficients of Eq. (4-6) with respect to t, t^2, yields

$$D_x^{3\beta}\mu_1(x) + (\phi_1+\psi_1)\mu_1(x) - \frac{a}{e} = 0, \tag{3.7}$$

where

$$\phi_1 := \frac{1}{e\Gamma(2-\alpha)}, \quad \psi_1(x^\beta) := \frac{b}{e}x^\beta,\ e \neq 0.$$

To calculate the fractal index for the equation (7), we assume the transform $X = x^\beta$ and the solution can be expressed in a series of the form

$$\mu_1(X) = \sum_{m=0}^{\infty}\gamma_m X^m, \quad \mu_1(0) = 1 \tag{3.8}$$

where γ_m are constants. Substitute (8) in (7) and by using the fractal index we receive

$$\begin{aligned}&\frac{\partial^3}{\partial X^3}\sum_{m=0}^{\infty}\theta_{\beta\beta\beta m}\gamma_m X^m + \phi_1\sum_{m=0}^{\infty}\gamma_m X^m + \frac{b}{e}\sum_{m=0}^{\infty}\gamma_m X^{m+1} = 0 \\ &\sum_{m=3}^{\infty}\frac{\Gamma(1+m\beta)}{\Gamma(1+m\beta-3\beta)}\gamma_m X^{m-3} + \phi_1\sum_{m=0}^{\infty}\gamma_m X^m + \frac{b}{e}\sum_{m=0}^{\infty}\gamma_m X^{m+1} - \frac{a}{e} = 0\end{aligned} \tag{3.9}$$

where

$$\theta_{\beta\beta\beta m} = \frac{\Gamma(1+m\beta)}{m(m-1)(m-2)\Gamma(1+m\beta-3\beta)}.$$

Comparing the coefficients of X^0, we have

$$\Gamma(1+3\beta)\gamma_3 + \phi_1\gamma_0 + \frac{b}{e}\gamma_{-1} = \frac{a}{e};$$

but $\gamma_0 = 1$ and $\gamma_{-1} = 0$, in general we obtain

$$\gamma_m = \frac{(\frac{a}{e}-\phi_1)^m}{\Gamma(1+m\beta)}, \quad m \geq 3.$$

Hence

$$\mu_1(x) = E_\beta((\frac{a}{e}-\phi_1)x^\beta),$$

where E_β is a Mittag-Leffler function.

Similarly for μ_2; if we let

$$\mu_2(X) = \sum_{m=0}^{\infty}\delta_m X^m, \quad \mu_2(0) = 1 \tag{3.10}$$

we can receive

$$\mu_2(x) = E_\beta((\frac{b}{e}-\phi_2)x^\beta), \quad \phi_2 := \frac{1}{e\Gamma(3-\alpha)}.$$

Thus we have the following solution of the Eq.(1):

$$u(t,x) = tE_\beta((\frac{a}{e}-\phi_1)x^\beta) + t^2E_\beta((\frac{b}{e}-\phi_2)x^\beta). \tag{3.11}$$

4 Fractional Riccati equation

$$D_x^\alpha \psi(x) = \sigma + \psi^2(x), \tag{4.12}$$

where $\sigma \in \mathbb{R}$. To calculate the fractal index for the equation (12), we assume the transform $X = x^\alpha$ and the solution can be expressed in a series of the form

$$\psi(X) = \sum_{m=0}^{\infty} \psi_m X^m, \quad \psi(0) = 1 \tag{4.13}$$

where ψ_m are constants. Substitute (13) in (12) and by applying the fractal index we impose

$$\begin{aligned} \frac{\partial}{\partial X} \sum_{m=0}^{\infty} \theta_{\alpha m} \psi_m X^m &= \sigma + \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big) \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big), \\ \sum_{m=1}^{\infty} \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)} \psi_m X^{m-1} &= \sigma + \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big) \Big(\sum_{m=0}^{\infty} \psi_m X^m \Big), \end{aligned} \tag{4.14}$$

where

$$\theta_{\alpha m} = \frac{\Gamma(1+m\alpha)}{m\Gamma(1+m\alpha-\alpha)}.$$

Comparing the coefficients of X^0, we have

$$\Gamma(1+\alpha)\psi_1 = \sigma + \psi_0^2;$$

but $\psi_0 = 1$ so in general we obtain

$$\psi_m = \frac{(\sigma+1)^m}{\Gamma(1+m\alpha)}, \quad m \geq 1.$$

Hence

$$\psi(x) = E_\alpha((\sigma+1)x^\alpha). \tag{4.15}$$

Note that Zhang and Zhang [17] derived some exact solutions to Eq.(15) take the form

$$\psi(x) = \begin{cases} -\sqrt{-\sigma} \tanh_\alpha(\sqrt{-\sigma} x) & \text{for } \sigma < 0 \\ -\sqrt{-\sigma} \coth_\alpha(\sqrt{-\sigma} x) & \text{for } \sigma < 0 \\ \sqrt{\sigma} \tan_\alpha(\sqrt{\sigma} x) & \text{for } \sigma > 0 \\ -\sqrt{\sigma} \cot_\alpha(\sqrt{\sigma} x) & \text{for } \sigma > 0 \\ -\frac{\Gamma(1+\alpha)}{x^\alpha + \upsilon} & \text{for } \sigma = 0, \end{cases} \tag{4.16}$$

where υ is a constant. In the next section, we shall use (15) to locate exact solution of fractional differential equation using Bäcklund transformation of fractional Riccati equation.

5 Bäcklund transformation method

In this section, based on the Bäcklund transformation method and the known seed solutions, we will impose a technique for solving fractional partial differential equations. It will be shown that the use of the Bäcklund transformation permits us to get new exact solutions from the known seed solutions. The Bäcklund transformation for the fractional Riccati equation is determined by

$$\phi(\eta) = \frac{-\sigma B + D\psi(\eta)}{D + B\psi(\eta)}, \tag{5.17}$$

where $\phi(\eta)$ satisfies the fractional Riccati equation (12) and $B \neq 0, D$ are arbitrary parameters, and ψ are the known solutions of Eq. (12).

Our method can be summarized as follows:

Step 1: Using the wave transform

$$u(t,x_1,...,x_j) = u(\eta),$$
$$\eta = \eta_0 + \lambda t + \lambda_1 x_1 + ... + \lambda_j x_j,$$

where $\lambda, \lambda_i (i = 1,...,j)$ are constants. Hence the equation

$$F\Big(u, u_t, u_{x_1}, ..., u_{x_j}, u_{x_1 x_1}, ..., D^{\alpha}_{x_1} u, ..., D^{\alpha}_{x_j} u, ...\Big) = 0, \tag{5.18}$$

becomes

$$\Phi\Big(u(\eta), u'(\eta), u''(\eta), ..., D^{\alpha}_{\eta} u\Big) = 0, \tag{5.19}$$

where $(') = \frac{d}{d\eta}$.

Step 2: Assuming a solution of the form

$$u(\eta) = \sum_{m=0}^{n} a_m \phi^m(\eta), \tag{5.20}$$

where a_m $(m = 0,...,n)$ are constants to be calculated and ϕ computes from the Bäcklund transform.

Step 3: Substituting (20) in (19) and setting the coefficients of the powers of ϕ to be zero, we impose a nonlinear algebraic system in a_m and λ.

Step 4: Solving the system to obtain these values and substituting them into Eq.(20) we receive the exact solutions of (18).

6 Applications

In this section we shall illustrate two examples to examine our method.

6.1 Example

Water as a liquid moves through the vadose region in response to gravity and gradients of pressure. Recall that the vadose region has hole spaces filled with both air and liquid water. The water pressure depends on the water saturation and related capillary forces. Because the soil is only partially saturated the pressure is negative due to capillarity. If the soil is uniform in its properties such as composition, capillary pressures are most negative where the soil is dry, and most positive where it is wet. As a FDE it can be represented as

$$D^{\alpha}_t u - \kappa u D^{\alpha}_x u - \delta D^{2\alpha}_x u = 0, \tag{6.21}$$

where x is the position in this model and u is the so-called volumetric water content. It denotes the proportion of the space filled by water. δ is the so-called soil moisture diffusivity and κ is the saturation dependent hydraulic conductivity. Equation (21) describes the infiltration in the vadose region. The advection is due the gravity and the diffusion is due to capillary wicking.

Using the wave transform

$$u(t,z) = u(\eta), \quad \eta = \lambda t + x,$$

we receive

$$\lambda^{\alpha} D^{\alpha}_{\eta} u - \kappa u D^{\alpha}_{\eta} u - \delta D^{2\alpha}_{\eta} u = 0. \tag{6.22}$$

By applying the above method yields

$$u(\eta) = a_0 + a_1 \frac{-\sigma B + D\psi(\eta)}{D + B\psi(\eta)},$$

where ψ defined in (15) and

$$a_0 = \frac{\lambda^{\alpha}}{\kappa}, \quad a_1 = \frac{-2\delta}{\kappa}, \kappa \neq 0.$$

Thus we have exact solutions of (21) as follows:

$$u(\eta) = \frac{\lambda^{\alpha}}{\kappa} - \frac{2\delta}{\kappa}\Big(\frac{-\sigma B + DE_{\alpha}((\sigma+1)\eta^{\alpha})}{D + BE_{\alpha}((\sigma+1)\eta^{\alpha})}\Big).$$

6.2 Example

In 1973, Fischer Black and Myron Scholes [24] suggested the famous theoretical valuation formula for options. The main fictional idea of Black and Scholes excites in the texture of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such an process strengthens the use of the no-arbitrage principle as well. The Black-Scholes model for the value of an option can be described by the fractional equation

$$D_t^\alpha u + \rho(t,x) D_x^{2\alpha} u + u D_x^\alpha u - r(t,x)u = 0, \quad t \in (0,T) \tag{6.23}$$

where $u(t,x)$ is the European call option price at asset price x(positive real number) and at time t; $r(t,x)$ is the risk free interest rate, and $\rho(t,x)$ represents the volatility function of underlying asset. By employing the

wave transform

$$u(t,x) = u(\eta), \quad \eta = \lambda t + \lambda_1 x,$$

we extradite

$$\lambda^\alpha D_\eta^\alpha u + \lambda_1^{2\alpha} \rho(\eta) D_\eta^{2\alpha} u + \lambda_1^\alpha u D_\eta^\alpha u - r(\eta)u = 0. \tag{6.24}$$

Now in virtue of the above method, we have

$$a_0 = \frac{-\lambda^\alpha \sigma}{\lambda_1^\alpha \sigma - r}, \quad a_1 = \frac{-2\lambda_1^{2\alpha} \rho\sigma}{\lambda_1^\alpha \sigma - r},$$

where $\lambda_1^\alpha \sigma \neq r$. Hence some exact solutions of (23) can be expressed as follows:

$$u(\eta) = \frac{-\lambda^\alpha \sigma}{\lambda_1^\alpha \sigma - r} - \frac{2\lambda_1^{2\alpha} \rho\sigma}{\lambda_1^\alpha \sigma - r} \Big(\frac{-\sigma B + D E_\alpha((\sigma+1)\eta^\alpha)}{D + B E_\alpha((\sigma+1)\eta^\alpha)} \Big).$$

7 Conclusion

From above we conclude that the transform method of fractional differential equation affected on the exact solutions of fractional differential equations. This method has more advantages: it is direct and concise. Thus, we realize that the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations. Moreover, these solutions are analytic in their domains. The applications are taken for liquid move equation (Eq.(21)) and the Black-Scholes model (Eq.(23)).

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