Malaya J. Mat. 2(4)(2014) 419–425

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Fractional differintegral operators of the generalized Mittag-Leffler type function

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Abstract

In the present paper we study a new function called as R-function [6], which is an extension of the generalized Mittag-Leffler functions. We derive the relations that exist between the R-function and Saigo-Maeda fractional calculus operators. Some results derived by Kumar and Kumar [6], Kilbas [4], Kilbas and Saigo [5]; and Sharma and Jain [23] are special cases of the main results derived in this paper.

Keywords: Fractional calculus, fractional differintegral operators, generalized Mittag-Leffler function, R-function.

2010 MSC: 26A33, 33C45, 33E12, 33E20.

1 Introduction and preliminaries

The Mittag-Leffler function has gained importance and popularity during the last one and a half decades due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differintegral equations.

In 1903, the Swedish mathematician Gosta Mittag-Leffler [9, 10] introduced studied the function $E_\alpha(z)$, defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \ (\alpha \in C, Re(\alpha) > 0). \tag{1.1}$$

A generalization of this series given by Wiman [27] who defined the function $E_{\alpha,\beta}(z)$ as follows

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0). \tag{1.2}$$

The function $E_{\alpha,\beta}(z)$ is now known as Wiman function, which was later studied by Agarwal [1] and others. The generalization of (1.2) was introduced by Prabhakar [11] in terms of the series representation as given following:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)\, n!}, \ (\alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0), \tag{1.3}$$

Shukla and Prajapati [24] defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)\, n!}, \ (\alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0), \tag{1.4}$$

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where $q \in (0,1) \cup N$ and $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol which in particular reduces to

$$q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q} \right)_n , \quad q \in N .$$

Srivastava and Tomovski [26] introduced and investigated a further generalization of (1.3), which is defined in the following way:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} z^n}{\Gamma(\alpha n+\beta)\, n!}, \ (z,\beta,\gamma \in C; Re\,(\alpha) > max\,\{0, Re(k)-1\}\,;\, Re(k) > 0)\,, \tag{1.5}$$

which, in the special case when $k = q$ $(q \in (0,1) \cup N)$ and $min\,\{Re(\beta),\, Re(\gamma)\} > 0$, is given by (1.4).
It is an entire function of order $\rho = [Re\,(\alpha)]^{-1}$. Some special cases of (1.3) are

$$E_\alpha(z) = E_{\alpha,1}^{1}(z)\,, E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{1}(z)\,, \phi(\beta,\gamma;z) = {}_1F_1(\beta,\gamma;z) = \Gamma(\gamma)\, E_{1,\gamma}^{\beta}(z)\,, \tag{1.6}$$

An interesting generalization of (1.2) is recently introduced by Kilbas and Saigo [5] in terms of a special entire function as given below

$$E_{\alpha,m,r}(z) = \sum_{n=0}^{\infty} c_n\, z^n, \tag{1.7}$$

where $c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(im+r)+1]}{\Gamma[\alpha(im+r+1)+1]}$ and an empty product is to be interpreted as unity.

In order to prove our main results we only provide here the basic definitions of left-sided fractional calculus operators. The readers can refer for detailed account of fractional calculus operators in several papers [15, 16, 17] and many more

Let $\alpha, \alpha', \beta, \beta', \gamma \in C,\ x > 0$, then the left-sided $(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma})$ generalized fractional integral operators of a function $f(x)$ for $Re(\gamma) > 0$ is defined by Saigo and Maeda [16], in the following form:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\, t^{-\alpha'}\, F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\mathrm{d}t, \tag{1.8}$$

This operator reduce to the left-sided Saigo fractional integral operator [15] due to the following relation:

$$I_{0+}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0+}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C), \tag{1.9}$$

Further, if we set $\beta = -\alpha$, then operator (1.9) reduces to left-sided Riemann-Liouville fractional integral operator

$$I_{0+}^{\alpha,-\alpha,\gamma} f(x) = I_{0+}^{\alpha} f(x), \tag{1.10}$$

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, and $x \in R_+$, then the left-sided generalized fractional differentiation operator [16] involving the Appell function F_3 as a kernel are defined by the following equation:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{1.11}$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n \left(x^{\alpha'}\right) \int_0^x (x-t)^{n-\gamma-1}\, t^{\alpha}$$
$$\times\ F_3\left(-\alpha', -\alpha, n-\beta', -\beta, n-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\ dt, \tag{1.12}$$

The above operator reduce to the left-sided Saigo fractional derivative operator [15, 18] as

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x)\,, \quad (Re(\gamma) > 0); \tag{1.13}$$

If we set $\beta = -\alpha$, then operator (1.13) reduces to left-sided Riemann-Liouville fractional derivative operator

$$D_{0+}^{\alpha,-\alpha,\gamma} f(x) = D_{0+}^{\alpha} f(x). \tag{1.14}$$

Under various fractional calculus operators, the computations of image formulas for special functions are very important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations. Therefore, in the literature we found several papers on the subject, see for instance [12], [13], [19]-[21] and [2] and references cited therein.

2 The generalized Mittag-Leffler type function (R-function)

The R-function is introduced and studied by Kumar and Kumar [6] as follows:

$$ {}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(z) = {}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}\left(a_1,...,a_p;b_1,...,b_q;z\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\left(a_j\right)_n}{\prod_{j=1}^{q}\left(b_j\right)_n} \frac{(\gamma)_{kn}\, z^n}{n!\Gamma(\alpha n+\beta)}, \tag{2.1} $$

where $\alpha,\beta,\gamma \in C, Re(\alpha) > max\{0, Re(k)-1\}; Re(k) > 0; \left(a_j\right)_n$ and $\left(b_j\right)_n$ are the Pochhammer symbols. The series (2.1) is defined when none of the parameters b_j's, $j = \overline{1,q}$ is a negative integer or zero. If any parameter a_j is a negative integer or zero, then the series (2.1) terminates to a polynomial in z , and the series is convergent for all z if $p < q+1$. It can also converge in some cases if we have $p = q+1$ and $|z| = 1$. Let $\gamma = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$, it can be shown that if $Re(\gamma) > 0$ and $p = q+1$ the series is absolutely convergent for $|z| = 1$, in order convergent for $z = -1$ when $0 \le Re(\gamma) < 1$ and divergent for $|z| = 1$ when $1 \le Re(\gamma)$.

Special Cases of the R-function:

(i) If we set $a_j = b_j = 1$, we have

$$ {}_{0}^{k}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}\, z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma,k}(z), \tag{2.2} $$

where $E_{\alpha,\beta}^{\gamma,k}(z)$ is the generalized Mittag-Leffler function which introduced by Srivastava and Tomovski [26].

(ii) In the special case of (2.2), when $k = q \ \ (q \in (0,1) \cup N)$ and $min\{Re(\beta),\ Re(\gamma)\} > \ 0$, we have the following:

$$ {}_{0}^{q}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma,q}(z), \tag{2.3} $$

where $E_{\alpha,\beta}^{\gamma,q}(z)$ was considered earlier by Shukla and Prajapati [24].

(iii) If we set $a_j = b_j = k = 1$ in (2.1), we have

$$ {}_{0}^{1}R_{0}^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}\, z^n}{n!\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{\gamma}(z), \tag{2.4} $$

where $E_{\alpha,\beta}^{\gamma}(z)$ is generalization of the Mittag-Leffler function which introduced by Prabhakar [11], and studied by Haubold et al. [3] and others.

(iv) If we put $\gamma = 1$ in (2.4), we have

$$ {}_{0}^{1}R_{0}^{\alpha,\beta;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z), \tag{2.5} $$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function [27] (also known as Wiman function), which was later studied by Agarwal [1] and others.

(v) If we take $\beta = \gamma = 1$ in (2.4), we have

$$ {}_{0}^{1}R_{0}^{\alpha,1;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha,1}^{1}(z) = E_{\alpha}(z), \tag{2.6} $$

where $E_{\alpha}(z)$ is the Mittag-Leffler function [9, 10], compare (1.1).

(vi) If we take $\alpha = \beta = \gamma = 1$ in (2.4), we obtain

$$ {}_{0}^{1}R_{0}^{1,1;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = E_{1,1}^{1}(z) = E_{1}(z) = e^{x}, \tag{2.7} $$

where e^x is the Exponential function [14].

(vii) If we set $\gamma = k = 1$ in (2.1), then the R-function can be represented in the Wright generalized hypergeometric function [28] ${}_p\psi_q(z)$ and the H-function [4, 8] as given below

$$
\begin{aligned}
{}_p^1R_q^{\alpha,\beta;1}(z) = {}_p^1R_q^{\alpha,\beta;1}\left(a_1,...,a_p;b_1,...,b_q;z\right) &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)}\ {}_{p+1}\psi_{q+1}\left[z\Big|_{(b_1,1),\cdots,(b_q,1),(\beta,\alpha)}^{(a_1,1),\cdots,(a_p,1),(1,1)}\right] \\
&= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} H_{p+1,q+2}^{1,p+1}\left[-z\Big|_{(0,1),(1-b_j,1)_{1,q},(1-\beta,\alpha)}^{(1-a_j,1)_{1,p},(0,1)}\right],
\end{aligned}
\tag{2.8}
$$

where H-function is as defined in the monograph by Mathai et al. [8].

(viii) If we set $p = q = 0$, and $\gamma = k = 1$ in (2.1), then we obtain another special case of R-function in terms of the Wright generalized hypergeometric function as given below:

$$
{}_0^1R_0^{\alpha,\beta;1}(z) = {}_0^1R_0^{\alpha,\beta;1}(-;1;z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\, z^n}{\Gamma(\alpha n+\beta)\, n!} = \frac{(1)_n\, z^n}{\Gamma(\alpha n+\beta)\, n!} = {}_1\psi_1\left[z\Big|_{(\beta,\alpha)}^{(1,1)}\right],
\tag{2.9}
$$

(ix) If we set $\alpha = \beta = \gamma = k = 1$ in (2.1), then the R-function reduces to the generalized hypergeometric function ${}_pF_q$ (see for detail [7, 14, 17]) as given

$$
{}_p^1R_q^{1,1;1}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \left(a_j\right)_n}{\prod_{j=1}^q \left(b_j\right)_n} \frac{z^n}{n!} = {}_pF_q\left(\left(a_j\right)_{1,p};\left(b_j\right)_{1,q};z\right).
\tag{2.10}
$$

3 Main results

This section deals with results, which established well defined ralations for generalized fractional differintegrals (fractional integral and differential operators) and generalized Mittag-Leffler type function (R-function), defined by (2.1).

Theorem 3.1. *Let $\vartheta, \vartheta', \eta, \eta', \delta, \alpha, \beta, \gamma \in C$, $Re(\delta) > 0$, $Re(\alpha) > 0$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= x^{-\vartheta-\vartheta'+\delta} \frac{\Gamma\left(1+\delta-\vartheta-\vartheta'-\eta\right)\Gamma\left(1+\eta'-\vartheta'\right)}{\Gamma\left(1+\delta-\vartheta-\vartheta'\right)\Gamma\left(1+\delta-\vartheta'-\eta\right)\Gamma\left(1+\eta'\right)} \\
&\times\ {}_{p+3}^{\ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\delta-\vartheta-\vartheta'-\eta,1+\eta'-\vartheta';\right. \\
&\qquad \left. b_1,\ldots,b_q,1+\delta-\vartheta-\vartheta',1+\delta-\vartheta'-\eta,1+\eta';x\right).
\end{aligned}
\tag{3.1}
$$

Proof. Following the definition of Saigo-Maeda fractional integral [16] as given in (1.8), we have the following relation:

$$
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{-\vartheta}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1}\, t^{-\vartheta'} F_3\left(\vartheta,\vartheta',\eta,\eta',\delta;1-\frac{t}{x},1-\frac{x}{t}\right) {}_p^kR_q^{\alpha,\beta;\gamma}(t)\, dt
$$

by virtue of (2.1), we obtain

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= \frac{x^{-\vartheta}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1}\, t^{-\vartheta'} F_3\left(\vartheta,\vartheta',\eta,\eta',\delta;1-\frac{t}{x},1-\frac{x}{t}\right) \\
&\times \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \left(a_j\right)_n}{\prod_{j=1}^q \left(b_j\right)_n} \frac{(\gamma)_{kn}\, t^n}{n!\Gamma\left(\alpha n+\beta\right)}\, dt.
\end{aligned}
\tag{3.2}
$$

Interchanging the order of integration and evaluating the inner integral with the help of Beta function, we arrive at

$$
\begin{aligned}
I_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_p^kR_q^{\alpha,\beta;\gamma}(x)\right) &= x^{-\vartheta-\vartheta'+\delta} \frac{\Gamma\left(1+\delta-\vartheta-\vartheta'-\eta\right)\Gamma\left(1+\eta'-\vartheta'\right)}{\Gamma\left(1+\delta-\vartheta-\vartheta'\right)\Gamma\left(1+\delta-\vartheta'-\eta\right)\Gamma\left(1+\eta'\right)} \\
&\times \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \left(a_j\right)_n (1)_n \left(1+\delta-\vartheta-\vartheta'-\eta\right)_n \left(1+\eta'-\vartheta'\right)_n}{\prod_{j=1}^q \left(b_j\right)_n \left(1+\delta-\vartheta-\vartheta'\right)_n \left(1+\delta-\vartheta'-\eta\right)_n \left(1+\eta'\right)_n} \frac{(\gamma)_{kn}\, x^n}{n!\Gamma\left(\alpha n+\beta\right)}
\end{aligned}
$$

$$= x^{-\vartheta-\vartheta'+\delta} \frac{\Gamma(1+\delta-\vartheta-\vartheta'-\eta)\,\Gamma(1+\eta'-\vartheta')}{\Gamma(1+\delta-\vartheta-\vartheta')\,\Gamma(1+\delta-\vartheta'-\eta)\,\Gamma(1+\eta')}$$
$$\times\ {}_{p+3}^{\ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\delta-\vartheta-\vartheta'-\eta,1+\eta'-\vartheta'\,;\right.$$
$$\left. b_1,\ldots,b_q,1+\delta-\vartheta-\vartheta',1+\delta-\vartheta'-\eta,1+\eta';x\right).$$

The interchange of the order of summation is permissible under the conditions stated along with the theorem. This shows that a Saigo-Maeda fractional integral of the R-function is again the R-function with increased order $(p+3,q+3)$.

This completes the proof of the Theorem 1. □

In view of the relation (1.9), we obtain the result given by Kumar and Kumar [6] concerning Saigo fractional integral operator asserted by the following corollary.

Corollary 3.1. *Let $\vartheta,\eta,\delta,\alpha,\beta,\gamma\in C$, $Re(\vartheta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$I_{0+}^{\vartheta,\eta,\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{-\eta}\Gamma(1+\delta-\eta)}{\Gamma(1+\vartheta+\delta)\Gamma(1-\eta)}$$
$$\times\ {}_{p+2}^{\ k}R_{q+2}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\delta-\eta;b_1,\ldots,b_q,1+\vartheta+\delta,1-\eta;x\right). \tag{3.3}$$

Further, if we put $\eta=-\vartheta$ in (3.3) then we obtain following Corollary concerning Riemann-Liouville fractional integral operator [17]:

Corollary 3.2. *Let $\vartheta,\alpha,\beta,\gamma\in C, Re(\vartheta)>0, Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$I_{0+}^{\vartheta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{\vartheta}}{\Gamma(1+\vartheta)}\ {}_{p+1}^{\ k}R_{q+1}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1+\vartheta;x\right). \tag{3.4}$$

Theorem 3.2. *Let $\vartheta,\vartheta',\eta,\eta',\delta,\alpha,\beta,\gamma\in C$, $Re(\delta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = x^{\vartheta+\vartheta'-\delta}\frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$
$$\times\ {}_{p+3}^{\ k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\vartheta'+\eta'-\delta,1+\vartheta-\eta\,;\right.$$
$$\left. b_1,\ldots,b_q,1+\vartheta+\vartheta'-\delta,1+\vartheta+\eta'-\delta,1-\eta;x\right). \tag{3.5}$$

Proof. Following the definition of Saigo-Maeda fractional derivative [16] as given in (1.12), we have the following relation:

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{\vartheta'}}{\Gamma(r-\delta)}\left(\frac{d}{dx}\right)^{r}\int_0^x (x-t)^{r-\delta-1}\,t^{\vartheta}$$
$$\times F_3\left(-\vartheta',-\vartheta,r-\eta',-\eta,r-\delta;1-\frac{t}{x},1-\frac{x}{t}\right)\ {}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(t)\,dt$$

where $r=[-Re(\delta)]+1$ by virtue of (2.1), we obtain

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = \frac{x^{\vartheta'}}{\Gamma(r-\delta)}\left(\frac{d}{dx}\right)^{r}\int_0^x (x-t)^{r-\delta-1}\,t^{\vartheta}$$
$$\times\ F_3\left(-\vartheta',-\vartheta,r-\eta',-\eta,r-\delta;1-\frac{t}{x},1-\frac{x}{t}\right)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{kn}\,t^n}{n!\Gamma(\alpha n+\beta)}\,dt. \tag{3.6}$$

By using $\frac{d^r}{dx^r}x^m = \frac{\Gamma(m+1)}{\Gamma(m-r+1)}x^{m-r}$ $(m,r\in N_0;\ m\geq r)$ in (3.6) and interchanging the order of integration and evaluating the inner integral with the help of Beta function, we arrive at

$$D_{0+}^{\vartheta,\vartheta',\eta,\eta',\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right) = x^{\vartheta+\vartheta'-\delta}\frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$
$$\times\ \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,(1)_n\,(1+\vartheta+\vartheta'+\eta'-\delta)_n\,(1+\vartheta-\eta)_n}{\prod_{j=1}^{q}(b_j)_n\,(1+\vartheta+\vartheta'-\delta)_n\,(1+\vartheta+\eta'-\delta)_n\,(1-\eta)_n}\frac{(\gamma)_{kn}\,x^n}{n!\Gamma(\alpha n+\beta)}$$

$$= x^{\vartheta+\vartheta'-\delta} \frac{\Gamma(1+\vartheta+\vartheta'+\eta'-\delta)\,\Gamma(1+\vartheta-\eta)}{\Gamma(1+\vartheta+\vartheta'-\delta)\,\Gamma(1+\vartheta+\eta'-\delta)\,\Gamma(1-\eta)}$$

$$\times\ {}_{p+3}^{\;\;k}R_{q+3}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\vartheta'+\eta'-\delta,1+\vartheta-\eta\ ;\right.$$

$$\left.b_1,\ldots,b_q,1+\vartheta+\vartheta'-\delta,1+\vartheta+\eta'-\delta,1-\eta;x\right).$$

This shows that a Saigo-Maeda fractional derivative of the R-function is again the R-function with increased order $(p+3,q+3)$.

This completes the proof of the Theorem 2. □

Now, on making use the relation (1.13), we obtain the result concerning Saigo fractional derivative operator given by [6] asserted by the following corollary.

Corollary 3.3. *Let $\vartheta,\eta,\delta,\alpha,\beta,\gamma \in C$, $Re(\vartheta)>0$, $Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta,\eta,\delta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{\eta}\,\Gamma(1+\vartheta+\eta+\delta)}{\Gamma(1+\delta)\Gamma(1+\eta)}$$

$$\times\ {}_{p+2}^{\;\;k}R_{q+2}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1,1+\vartheta+\eta+\delta;b_1,\ldots,b_q,1+\delta,1+\eta;x\right). \tag{3.7}$$

Again, if we further put $\eta=-\vartheta$ in (3.7), then we obtain following corollary concerning Riemann-Liouville fractional derivative operator [17]:

Corollary 3.4. *Let $\vartheta,\alpha,\beta,\gamma\in C, Re(\vartheta)>0, Re(\alpha)>0$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$, then there holds the relation*

$$D_{0+}^{\vartheta}\left({}_{p}^{k}R_{q}^{\alpha,\beta;\gamma}(x)\right)=\frac{x^{-\vartheta}}{\Gamma(1-\vartheta)}\,{}_{p+1}^{\;\;k}R_{q+1}^{\alpha,\beta;\gamma}\left(a_1,\ldots,a_p,1;b_1,\ldots,b_q,1-\vartheta;x\right) \tag{3.8}$$

It is remarked in passing that a number of known and new results can be obtained as special cases of the Theorems 3.1 and 3.2.

4 Conclusion

In this paper we derive a new generalization of Mittag-Leffler function and obtain the relations between the R-function and Saigo-Maeda fractional calculus operators. The results are also extension of work done by Kumar and Kumar [6] and Sharma [22]. The provided results are new and have uniqueness identity in the literature. A number of known results can easily be found as special cases of our main results.

References

[1] R.P. Agarwal, A propos d'une note de M. Pierre Humbert, *C.R. Acad. Sci. Paris* 236(1953), 2031-2032.

[2] D. Baleanu, P. Agarwal and S. D. Purohit, Certain fractional integral formulas involving the product of generalized Bessel functions, *The Scientific World Journal*, 2014(2014), Article ID 567132, 9 pp.

[3] H.J. Haubold, A.M. Mathai, and R.K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.*, Article ID 298628, (2011), 1-51.

[4] A.A. Kilbas, Fractional calculus of the generalized Wright function, *Fract. Calc. Appl. Anal.*, **8**(2) (2005), 113-126.

[5] A.A. Kilbas and M. Saigo, Fractional integrals and derivatives of Mittag-Leffler type function, *Doklady Akad. Nauk Belarusi*, 39(4) (1995), 22-26.

[6] D. Kumar and S. Kumar, Fractional calculus of the generalized Mittag-Leffler type function, *International Scholarly Research Notices*, 2014(2014), Article ID 907432, 6 pages.

[7] C.F. Lorenzo, and T.T. Hartley, Generalized function for the fractional calculus, NASA/TP-1999-209424, (1999).

[8] A.M. Mathai and R.K. Saxena,The H-function with Applications in Statistics and other Disciplines, John Wiley and Sons, Inc., New York, (1978).

[9] G.M. Mittag-Leffler, Sur la nouvelle fonction $E_{\alpha}(x)$, *C.R. Acad. Sci. Paris* 137 (1903), 554-558.

[10] G.M. Mittag-Leffler, Sur la representation analytique d'une branche uniforme d'une function monogene, *Acta Math.* 29(1905), 101-181.

[11] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the Kernel, *Yokohama Math. J.*, 19(1971), 7-15.

[12] S. D. Purohit, S. L. Kalla and D. L. Suthar, Fractional integral operators and the multiindex Mittag-Leffler functions, *SCIENTIA Series A: Mathematical Sciences*, 21 (2011), 87-96.

[13] S.D. Purohit, D.L. Suthar and S.L. Kalla, Marichev-Saigo-Maeda fractional integration operators of the Bessel function, *Le Matematiche*, LXVII (2012), 21-32.

[14] E.D. Rainville, Special Functions, Chelsea Publishing Company, Bronx, New York, (1960).

[15] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, *Math. Rep., College General Ed. Kyushu Univ.*, 11(1978), 135-143.

[16] M. Saigo and N. Maeda, More generalization of fractional calculus Transform Methods and Special Functions, Varna, Bulgaria, (1996), 386-400.

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon et alibi, (1993).

[18] R.K. Saxena, J. Ram and D. Kumar, Generalized fractional differentiation for Saigo operators involving Aleph-function, *J. Indian Acad. Math.*, 34(1) (2012), 109-115.

[19] R.K. Saxena, J. Ram and D. Kumar, On the Two-Dimensional Saigo-Maeda fractional calculus associated with Two-Dimensional Aleph Transform, *Le Matematiche*, 68 (2013), 267-281.

[20] R.K. Saxena, J. Ram and D. Kumar, Generalized Fractional Integral of the Product of Two Aleph-functions, *Applications and Applied Mathematics*, 8(2),(2013), 631-646.

[21] R.K. Saxena, J. Ram and D. Kumar, Generalized Fractional Integration of the Product of two $\aleph$-Functions Associated with the Appell Function F_3, *ROMAI Journal*, 9(1) (2013), 147-158.

[22] K. Sharma, Application of Fractional Calculus Operators to Related Areas, Gen. Math. Notes, **7** (1) (2011), 33-40.

[23] M. Sharma and R. Jain, A note on a generalized M-Series as a special function of fractional calulus, *Fract. Calc. Appl. Anal.*, 12(4) (2009), 449-452.

[24] A.K. Shukla and J.C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.*, 336 (2007), 797-811.

[25] H.M. Srivastava and R.K. Saxena, Operators of fractional integration and their applications, *Appl. Math. Comput.* 118 (2001), 1-52.

[26] H.M. Srivastava and Z. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *Appl. Math. Comput.*, 211 (2009), 198-210.

[27] A. Wiman, Uber de fundamental satz in der theorie der funktionen $E_{\alpha}(x)$, *Acta Math.* 29 (1905), 191-201.

[28] E.M. Wright, The asymptotic expansion of generalized hypergeometric function, *J. London Math. Soc.*, 10(1935), 286-293.

Received: April 3, 2014; *Accepted*: July 10, 2014

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