Malaya J. Mat. 2(4)(2014) 426–427

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On null sets in measure spaces

Asiyeh Erfanmanesh *

Faculty of Iranian Academic Center for Education Calture and Research, Ahvaz, Iran

Abstract

In this short work, first, we have a review on null sets in measure spaces. Next, we present an interesting example of a null set.

Keywords: Null set, Measure space, Sard's lemma.

2010 MSC: 28C20.

1 Introduction

In the section, we have a brief review on some properties of null sets.
In mathematics, a null set is a set that is negligible in some sense. In measure theory, any set of measure 0 is called a null set (or simply a measure-zero set). More generally, whenever an ideal is taken as understood, then a null set is any element of that ideal.
Null sets play a key role in the definition of the Lebesgue integral: if functions f and g are equal except on a null set, then f is integrable if and only if g is, and their integrals are equal. Indeed, via null sets we give a sufficient and necessary condition for integrability of a bounded real function:

Theorem 1.1. *If $f(x)$ is bounded in $[a,b]$, then a necessary and sufficient condition for the existence of $\int_a^b f(x)dx$ is that the set of discontinuities have measure zero [1].*

A measure in which all subsets of null sets are measurable is complete. Any non-complete measure can be completed to form a complete measure by asserting that subsets of null sets have measure zero. Lebesgue measure is an example of a complete measure; in some constructions, it's defined as the completion of a non-complete Borel measure.
A famous example for a null set is given by Sard's lemma.

Example 1.1 (Sard's lemma). *The set of critical values of a smooth function has measure zero [2].*

In the following, we present some another examples of null sets.

Example 1.2. *Any countable set has zero measure [1].*

Example 1.3. *All the subsets of $\mathbb{R}^n$ whose dimension is smaller than n have null Lebesgue measure in $\mathbb{R}^n$.*

Note that it may possible an uncountable set has zero measure; For instance, the standard construction of the Cantor set is an example of a null uncountable set in $\mathbb{R}$; however other constructions are possible which assign the Cantor set any measure whatsoever.
It is well-known and easy to show that a subset of a set of measure zero also has measure zero and a countable union of sets of measure zero also has measure zero.

*Corresponding author.
E-mail address: erfanmaneshasy@gmail.com (Asiyeh Erfanmanesh).

Remark 1.1. *Isomorphic sets may have different measures; In the other hand, a measure is not preserved by bijections. The most famous example would be the Cantor set* $\mathbf{C}$*. One can show that* $\mathbf{C}$ *has measure zero, yet there exists a bijection between* $\mathbf{C}$ *and* $[0,1]$*, which does not have measure zero.*

Let's end with an interesting example showing that a sum of two measure zero sets may has positive measure.

Example 1.4. *Let* $\mathbf{C}$ *be the Cantor set. Define*

$$\mathbf{C}+\mathbf{C}=\{a+b:\ a,b\in\mathbf{C}\}$$

It can be seen easily that $\mathbf{C}+\mathbf{C}=[0,2]$*. Hence we have a sum of two measure zero sets which has positive measure.*

Another properties of null sets and measurable spaces can be found in [3, 4].

2 An interesting Null Set

In the following theorem, we have presented a null set.

Theorem 2.2. *Let* X *be a nonempty set and* $\mu : 2^X \to [0,\infty)$ *an outer measure. Suppose that* (A_n) *be a sequence of subsets in* 2^X *such that* $\sum_n \mu(A_n) < \infty$*. Consider the set* $F = \{x \in X :\ x \text{ belong to infinitely many of } A_k's\}$*. Then* $\mu(F)=0$*.*

Proof. By Example 1.2, it is enough to prove that F is countable. Evidently, for each $x \in F$, there is $n_x \in \mathbb{N}$ so that $x \in \bigcap_{k=n_x}^{\infty} A_k$. Define the relation $\sim$ on X as follow:

$$x\sim y \Leftrightarrow n_x=n_y$$

It is easy to verify that $\sim$ is an equivalence relation on F. Set $N_F := \{n_x :\ x\in F\}$. Clearly $N_F \subset \mathbb{N}$. Now, define the function $f : E\mathbf{C}(F) \to N_F$ by $f([x]) = n_x$, where $E\mathbf{C}(F)$ denotes the set of all equivalent classes of F. Since equivalence classes partite F, so f is well-defined. Obviously, f is onto. Let $n_x = n_y$. This implies that $x \sim y$, i.e., $x \in [y]$. Also, it follows that $y \in [x]$. Therefore, $x = y$. Thus f is an one to one corresponding. Hence $E\mathbf{C}(F)$ is a countable set. Finally, by defining the function $g : E\mathbf{C}(F) \to F$, $g([x]) = x$, we conclude that F is countable, as desired. □

References

[1] Charalambos D. Aliprantis, *Principles of Real Analysis*, Academic Press, 3 Ed, ISBN: 0120502577, 451 pages, 2008.

[2] A. Sard, The measure of the critical values of differentiable maps, *Bulletin of the American Mathematical Society*, 48(12)(1942), 883–890.

[3] A.N. Kolmogrov and S.V. Fomin, *Measure, Lebesgue Integrals, and Hilbert Space*, Academic Press INC., New York, 1960.

[4] C. Swartz, *Measure, Integration and Function Spaces*, World Scientific Publishing Co.Pvt.Ltd., Singapore, 1994.

Received: March 12, 2014; *Accepted*: July 20, 2014