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Mild solution for fractional functional integro-differential equation with not instantaneous impulse

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Abstract

In this investigation, we prove the existence uniqueness and continuous dependence results of mild solution for nonlocal fractional differential equations with state dependent delay subject to not instantaneous impulse. We illustrate the existence result by an example involving partial derivatives.

Keywords: Fractional order differential equation, Functional differential equations, Impulsive conditions, Fixed point theorem.

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1 Introduction

The generalization of the theory of ordinary differential equation is a differential equation with an arbitrary non integer order. Fractional differential equations are widely used in modeling of several fields such as Science, Physics, Engineering and Economy due to this reason differential equations with fractional order have received increasing attentions in recent years, see [1, 2, 3, 4, 5, 6, 7]. Fractional equations with delay properties arise in serval fields such as biological and physical with state dependent delay or non constant delay. Presently, existence results of mild solutions for such problems became very attractive and several researchers are working on it. Many number of papers have been written on the fractional order problems with state dependent delay [13, 14, 17, 18, 22, 25, 27] and references therein.

Impulsive differential equations with fractional order have gained much attention, since it is much richer in terms of its applications. Impulsive effect exist widely in many phenomena in which their states are changed abruptly at certain time of moments. Recently, the results of existence and uniqueness of impulsive evolution equations in infinite dimensional spaces have been investigated by several authors [8, 9, 10, 11, 12, 15, 19, 20, 21, 23, 24, 26, 29].

Araya et al. [6] study the following problem:

$$D_t^{\alpha} u(t) = Au(t) + t^n f(t, u(t), u'(t)),\ t \in \mathbb{R}, n \in Z_+, 1 \le \alpha \le 2,$$

and introduce the concept of α-resolvent families and then proved the existence and uniqueness results of almost automorphism mild solution. Mophou et al. [7] established the existence and uniqueness of mild solution of the following Cauchy problem

$$D_t^{\alpha} x(t) = Ax(t) + t^n f(t, x(t), Bx(t)),\ t \in [0, T], n \in Z_+,\ x(0) = x_0 + g(x).$$

Recently, Hernndez et al. [9] have introduced a new class of abstract impulsive differential equations for which the impulses are not instantaneous

$$u'(t) = Au(t) + f(t, u(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N,\ u(0) = x_0, \tag{1.2}$$

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and established the existence and uniqueness results of mild and classical solutions by using classical fixed point theorems. In the model equation (1.1)-(1.2), the impulses start abruptly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. As pointed in [9], there are many different motivations for the study of this type of problem. For example as in [9], we note the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorbtion for the body are gradual and continuous processes, we can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval.

Further, Pierri et al. [10] have extended the results of [9] by using the theory of analytic semigroup and fractional power of closed operators and established the existence results of solutions for a class of semi-linear abstract impulsive differential equations with not instantaneous impulses. Further, Wang et al. [12] study the problem (1.1)-(1.2) for the cases if $\alpha \in (0,1]$ and $\alpha = 1$ with $A = 0$ and with periodic boundary condition $u(0) = u(T)$.

Kumar et al. [11] have studied the the following fractional order problem with not instantaneous impulse

$$^{C}D_t^{\beta} u(t) + Au(t) = f(t, u(t), g(u(t))),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.3}$$
$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N,\ u(0) = u_0 \in H, \tag{1.4}$$

by using the Banach fixed point theorem with condensing map established the existence and uniqueness results.

Motivated by the above mention works [6, 7, 8, 9], we consider the following fractional differential equation with not instantaneous impulses of the form

$$D_t^{\alpha} u(t) = Au(t) + t^n f(t, u_{\rho(t,u_t)}) + \int_0^t q(t-s) h(s, u_{\rho(s,u_s)}) ds,\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{1.5}$$
$$u(t) + l(u) = \phi(t),\ t \in (-\infty, 0], \tag{1.6}$$
$$u(t) = g_i(t, y(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{1.7}$$

where D_t^{α} is Caputo's fractional derivative of order $\alpha \in (0,1], n \in Z^+$ and $J = [0, T]$ is operational interval. The map $A : D(A) \subset X \to X$ is the a closed linear sectorial type operator defined on a Banach space $(X, \|\cdot\|)$. Here $f, h : J \times \mathfrak{B}_h \to X, q : J \to X$, and $\rho : J \times \mathfrak{B}_h \to (-\infty, T]$ are appropriate functions and satisfied some axioms. The functions $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \cdots, N$, is stand for impulsive conditions and $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$, are pre-fixed numbers. The nonlocal condition $l : X \to X$, defined as $l(u) = \sum_{k=1}^{r} c_k u(t_k)$, where $c_k, k = 1, \cdots, r$, are given constants and $0 < t_1 < t_2 < \cdots < t_r < T$. The history function $u_t : (-\infty, 0] \to X$ is element of $\mathfrak{B}_h$ and defined by $u_t(\theta) = u(t+\theta),\ \theta \in (-\infty, 0]$ respectively. The nonlocal condition [28], $u(0) + l(u)$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local condition $u(0) = u_0$. The Problem (1.5)-(1.7) appears in mathematical models of viscoelasticity and other fields of science which gives the better result using nonlocal condition.

Equation (1.5) is very important due to its appearance in mathematical modeling of viscoelasticity and other fields of science and engineering. This fact motivate us to study the existence results of the equation (1.5) with not instantaneous impulses and nonlocal condition. To the best of our knowledge the existence results for the considered problem (1.5)-(1.7) in the present paper are new. This paper has four sections, in which second section provides some basic definitions, theorems, notations and lemma. Third section is equipped with existence results of the mild solution of the considered problem and fourth section contained an example to verify the results.

2 Preliminaries and Definitions

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|u\|_X = \sup_{t \in J}\{|u(t)| : u \in X\}$ and $L(X)$ denotes the Banach space of bounded linear operators from X into X equipped with its natural topology. Due to infinite delay, we use abstract phase space $\mathfrak{B}_h$ as defined in [15] details are as follow:

Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with $l = \int_{-\infty}^{0} h(s) ds < \infty, t \in (-\infty, 0]$. For any $a > 0$, we define

$$\mathfrak{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

equipped the space $\mathfrak{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \forall\ \psi \in \mathfrak{B}$. Let us define

$$\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X, \text{ s.t. for any } a \geq c > 0, \psi\mid_{[-c,0]} \in \mathfrak{B}\ \&\ \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < \infty\}.$$

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi \in \mathfrak{B}_h$, then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a complete Banach space. We consider the space

$$\mathfrak{B}_h' := PC((-\infty, T]; X),\ T < \infty,$$

be a Banach space of all such functions $u : (-\infty, T] \to X$, which are continuous every where except for a finite number of points $t_i,\ i = 1, 2, \ldots, N$, at which $u(t_i^+)$ and $u(t_i^-)$ exists and endowed with the norm

$$\|u\|_{\mathfrak{B}_h'} = \sup\{\|u(s)\|_X : s \in [0, T]\} + \|\phi\|_{\mathfrak{B}_h}, u \in \mathfrak{B}_h',$$

where $\|\cdot\|_{B_h'}$ to be a semi-norm in $\mathfrak{B}_h'$.

For a function $u \in \mathfrak{B}_h'$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\bar{u}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } \ t \in (t_i, t_{i+1}], \\ u(t_i^+), \text{ for } \ t = t_i. \end{cases}$$

If $u : (-\infty, T] \to X$ s.t. $u \in \mathfrak{B}_h'$ then for all $t \in J$, the following conditions hold:

(C_1) $u_t \in \mathfrak{B}_h$.

(C_2) $\|u(t)\|_X \leq H\|u_t\|_{\mathfrak{B}_h}$.

(C_3) $\|u_t\|_{\mathfrak{B}_h} \leq K(t)\sup\{\|u(s)\|_X : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_h}$, where $H > 0$ is constant; $K, M : [0, \infty) \to [0, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $u(t)$.

(C_{4_ϕ}) The function t$\to \phi_t$ is well defined and continuous from the set

$$\Re(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in [0, T] \times \mathfrak{B}_h\}$$

into $\mathfrak{B}_h$ and there exists a continuous and bounded function $J^\phi : \Re(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}_h} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \Re(\rho^-)$.

Lemma 2.1. *([14])Let $u : (-\infty, T] \to X$ be function such that $u_0 = \phi, u\mid_{J_k} \in C(J_k, X)$ and if (C_{4_ϕ}) hold, then*

$$\|u_s\|_{\mathfrak{B}_h} \leq (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b \sup\{\|u(\theta)\|_X;\ \theta \in [0, \max\{0, s\}]\},\ s \in \Re(\rho^-) \cup J,$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t),\ M_b = \sup_{s\in[0,T]} M(s)$ and $K_b = \sup_{s\in[0,T]} K(s)$.

Example 2.1. *[27] Let $g : (-\infty, 0) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi, \theta) \leq \gamma(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, 0) \setminus N_\xi$ where $N_\xi \subseteq (-\infty, 0)$ is a set with Lebesgue measure zero. The space $\mathfrak{B}_h = C_0 \times L(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous at zero, Lebesgue measurable and $g\|\varphi\|$ is Lebesgue integrable on $(-\infty, 0)$. The seminorm in $C_0 \times L(g; X)$ is defined by*

$$\|\varphi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} g(\theta)\|\varphi(\theta)\|d\theta.$$

It is clear that $C_0 \times L(g; X)$ is complete Banach space.

Definition 2.1 ([5]). Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f : [0, \infty) \to \mathbb{R}$ such that $f \in C^n(\mathbb{R}_+, X)$ is defined as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = {}_a J_t^{n-\alpha} f^{(n)}(t),$$

where $a \geq 0,\ n \in N$. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{{}_0D_t^\alpha f(t); \lambda\} = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0); \quad n-1 < \alpha \leq n.$$

Definition 2.2 ([5]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is defined by

$$ {}_aJ_t^0 f(t) = f(t),\ {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t > 0, $$

where $a \geq 0,\ \ n \in N$ and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. A two parameter function of the Mittag-Lefler type is defined by the series expansion

$$ E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi\iota} \int_c \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - y} d\mu,\ \alpha,\ \beta > 0,\ y \in C, $$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |y|^{\frac{1}{\alpha}}$ counter clockwise. The Laplace integral of this function given by

$$ \int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0. $$

For more details on the above definition one can see the monographs of I. Podlubny [5].

Definition 2.4. *([16])* A closed and linear operator A is said to be sectorial if there are constants $\omega \in R,\ \theta \in [\frac{\pi}{2}, \pi],\ M > 0$, such that the following two conditions are satisfied:

(1) $\sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega,\ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$,

(2) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda-\omega|},\ \lambda \in \sum_{(\theta,\omega)}$,

where X is the complex Banach space with norm denoted $\|.\|_{L(X)}$.

Definition 2.5. *([6])* Let A be a closed and linear operator with domain D(A) defined on a Banach space X. Let $\rho(A)$ be the resolvent set of A, we call A is the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $T_\alpha : R^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$ (\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt,\ Re\lambda > \omega,\ x \in X. $$

In this case, $T_\alpha(t)$ is called α-resolvent family generated by A.

Definition 2.6. *([13])* Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$ \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} u = \int_0^\infty e^{-\lambda t} S_\alpha(t) u dt,\ Re\lambda > \omega,\ u \in X. $$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Lemma 2.2. Consider the following Cauchy problem of order $0 < \alpha \leq 1$

$$ {}_aD_t^\alpha u(t) \quad = \quad Au(t) + f(t),\ t \in J = [a, T],\ a \geq 0,\ u(a) = u_0, \tag{2.8} $$

then a function $u(t) \in C([a, T], \mathbb{R})$ is called the solution of the equation (2.8) if f satisfies the uniform Holder condition with exponent $\beta \in (0, 1]$ and A is a sectorial operator and also satisfy the following integral equation

$$ u(t) = S_\alpha(t-a) u_0 + \int_a^t T_\alpha(t-s) f(s) ds, \tag{2.9} $$

where $S_\alpha(t), T_\alpha(t)$ are analytic solution operator and α-resolvent family generated by A and defined as

$$ S_\alpha(t) \quad = \quad \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} d\lambda, $$

$$ T_\alpha(t) \quad = \quad \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^\alpha I - A)^{-1} d\lambda, $$

where Γ is a suitable path lying on $\sum_{\theta,\omega}$.

Proof. Let $t = w + a$, then the problem (2.8) translated into the form

$$ {}_0D_w^\alpha \tilde{u}(w) = A\tilde{u}(w) + \tilde{f}(w), \ \tilde{u}(0) = u_0. $$

Now, applying the Laplace transform, we have

$$ \begin{aligned} \lambda^\alpha L\{\tilde{u}(w)\} - \lambda^{\alpha-1}\tilde{u}(0) &= AL\{\tilde{u}(w)\} + L\{\tilde{f}(w)\} \\ L\{\tilde{u}(w)\}[\lambda^\alpha - A] &= \lambda^{\alpha-1}\tilde{u}(0) + L\{\tilde{f}(w)\}. \end{aligned} \tag{2.10} $$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from (2.10), we obtain

$$ L\{\tilde{u}(w)\} = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\tilde{u}(0) + (\lambda^\alpha I - A)^{-1}L\{\tilde{f}(w)\}. $$

Therefore, by taking the inverse Laplace transformation, we have

$$ \tilde{u}(w) = E_{\alpha,1}(Aw^\alpha)\tilde{u}(0) + \int_0^w E_{\alpha,\alpha}(A(w-\tau)^\alpha)\tilde{f}(\tau)d\tau. \tag{2.11} $$

Putting $w = t - a$, in equation (2.11) then we obtain

$$ u(t) = E_{\alpha,1}(A(t-a)^\alpha)u_0 + \int_0^{t-a} (t-a-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-a-\tau)^\alpha)f(\tau)d\tau. $$

This is the same as

$$ u(t) = E_{\alpha,1}(A(t-a)^\alpha)u_0 + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds. \tag{2.12} $$

Let $S_\alpha(t) = E_{\alpha,1}(At^\alpha)$, and $T_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$, then equation (2.12) we have

$$ u(t) = S_\alpha(t-a)u_0 + \int_a^t T_\alpha(t-s)f(s)ds. $$

This completes the proof of the lemma. □

Now, we state the definition of mild solution based on definition 2.1 in [9].

Definition 2.7. A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_h$ is called a mild solution of the problem (1.5)-(1.7) if $u(0) = \phi(0),\ u(t) = g_j(t, u(t))$ for $t \in (t_j, s_j]$ and each $j = 1, 2, \cdots, N$, satisfies the following integral equation

$$ u(t) = \begin{cases} S_\alpha(t)(\phi(0) - l(u)) + \int_0^t T_\alpha(t-s)s^n f(s, u_{\rho(s,u_s)})ds \\ + \int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, u_{\rho(\xi,u_\xi)})d\xi ds, \ \text{ for all } \ t \in [0, t_1], \\ S_\alpha(t-s_i)g_i(s_i, u(s_i)) + \int_{s_i}^t T_\alpha(t-s)s^n f(s, u_{\rho(s,u_s)})ds \\ + \int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi, u_{\rho(\xi,u_\xi)})d\xi ds, \ \text{ for all } \ t \in [s_i, t_{i+1}], \end{cases} $$

for every $i = 1, 2, \cdots, N$. It can be verified easily from the lemma (2.2).

3 Existence and Uniqueness Result

In this section, we prove the existence results of mild solutions for the impulsive system (1.5)-(1.7). If $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then $S_\alpha(t) \le Me^{\omega t}$ and $T_\alpha(t) \le Ce^{\omega t}(1 + t^{\alpha-1})$. Let $\widetilde{M_S} := \sup_{0\le t\le T}\|S_\alpha(t)\|_{L(X)},\ \ \widetilde{M_T} := \sup_{0\le t\le T} Ce^{\omega t}(1 + t^{1-\alpha})$. So we have $\|S_\alpha(t)\|_{L(X)} \le \widetilde{M_S},\ \ \|T_\alpha(t)\|_{L(X)} \le t^{\alpha-1}\widetilde{M_T}$.

To prove our results we shall assume the function $\rho : [0, T] \times \mathfrak{B}_h \to (-\infty, T]$ is continuous and $\phi \in \mathfrak{B}_h$. If $y \in \mathfrak{B}_h$ we defined $\bar{y} : (-\infty, T) \to X$ as the extension of y to $(-\infty, T]$ such that $\bar{y(t)} = \phi$. We defined $\widetilde{y}$: $(-\infty, T) \to X$ such that $\widetilde{y} = y + x$ where $x : (-\infty, T) \to X$ is the extension of $\phi \in \mathfrak{B}_h$ such that $x(t) = S_\alpha(t)\phi(0)$ for $t \in J$. In the sequel we introduce the following axioms:

(H_1) There exists positive constants L_f, L_h, L_{g_i}, L_l such that

$$\|f(t,\varphi)-f(t,\psi)\|_X \le L_f\|\varphi-\psi\|_{\mathfrak{B}_h},\ \|h(t,\varphi)-h(t,\psi)\|_X \le L_h\|\varphi-\psi\|_{\mathfrak{B}_h},$$
$$\|g_i(t,u)-g_i(t,v)\|_X \le L_{g_i}\|u-v\|_X,\ \|l(u)-l(v)\|_X \le L_l\|u-v\|_X,$$

$t\in J, u,v\in X,\ \varphi,\psi\in\mathfrak{B}_h$ and each $i=1,2,\cdots,N$.

Theorem 3.1. Let the assumption (H_1) hold and the constant

$$\Delta = c^* + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b < 1,$$

where $c^* = \max\{\widetilde{M_S}L_{g_i}, \widetilde{M_S}L_l\}$ and for $i=1,2,\cdots,N$. Then there exists a unique mild solution $u(t)$ on J for the system (1.5)-(1.7).

Proof. Let $\bar{\phi}:(-\infty,T)\to X$ be the extension of ϕ to $(-\infty,T]$ such that $\bar{\phi(t)}=\phi(0)$ on J. Consider the space $\mathfrak{B}''_h = \{y\in\mathfrak{B}'_h : y(0)=\phi(0)\}$ and $y(t)=\phi(t)$, for $t\in(-\infty,0]$ endowed with the uniform convergence topology. Let us consider a operator $P:\mathfrak{B}''_h\to\mathfrak{B}''_h$ defined as $Pu(0)=\phi(0),\ Pu(t)=g_i(t,\bar{u}(t))$ for $t\in(t_i,s_i]$ and

$$Pu(t) = \begin{cases} S_\alpha(t)(\phi(0)-l(\bar{u})) + \int_0^t T_\alpha(t-s)s^n f(s,\bar{u}_{\rho(s,\bar{u}_s)})ds \\ +\int_0^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,\bar{u}_{\rho(\xi,\bar{u}_\xi)})d\xi ds, \ \text{ for all } \ t\in[0,t_1], \\ S_\alpha(t-s_i)g_i(s_i,\bar{u}(s_i)) + \int_{s_i}^t T_\alpha(t-s)s^n f(s,\bar{u}_{\rho(s,\bar{u}_s)})ds \\ +\int_{s_i}^t T_\alpha(t-s)\int_0^s q(s-\xi)h(\xi,\bar{u}_{\rho(\xi,\bar{u}_\xi)})d\xi ds, \ \text{ for all } \ t\in[s_i,t_{i+1}], \end{cases}$$

where $\bar{u}:(-\infty,T]\to X$ is such that $\bar{u}(0)=\phi$ and $\bar{u}=u$ on J. It is obvious that P is well defined. We will show that the operator $P:\mathfrak{B}''_h\to\mathfrak{B}''_h$ has a fixed point. So let $u(t),u^*(t)\in\mathfrak{B}''_h$ and $t\in[0,t_1]$, we get

$$\begin{aligned}\|Pu(t)-Pu^*(t)\|_X \ &\le\ \|S_\alpha(t)\|_{L(X)}\|l(\bar{u}))-l(\bar{u}^*))\|_X + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ &\quad \times s^n\|f(s,\bar{u}_{\rho(s,\bar{u}_s)})-f(s,\bar{u}^*_{\rho(s,\bar{u}^*_\xi)})\|_X ds + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ &\quad \times\int_0^s q(s-\xi)\|h(\xi,\bar{u}_{\rho(\xi,\bar{u}_\xi)})-h(\xi,\bar{u}^*_{\rho(\xi,\bar{u}^*_\xi)})\|_X d\xi ds \\ &\le\ \widetilde{M_S}L_l\|u-u^*\|_{\mathfrak{B}''_h} + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b\|u-u^*\|_{\mathfrak{B}''_h} \\ &\quad +\frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b\|u-u^*\|_{\mathfrak{B}''_h}.\end{aligned}$$

For $t\in[s_i,t_{i+1}]$, we have

$$\begin{aligned}\|Pu(t)-u^*(t)\|_X \ &\le\ \|S_\alpha(t-s_i)\|_{L(X}\|g_i(s_i,\bar{u}(s_i))-g_i(s_i,\bar{u^*}(s_i))\|_X \\ &\quad +\int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)}s^n\|f(s,\bar{u}_{\rho(s,\bar{u}_s)})-g_i(s_i,\bar{u^*}(s_i))\|_X ds \\ &\quad +\int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)}\int_0^s q(s-\xi)\|h(\xi,\bar{u}_{\rho(\xi,\bar{u}_\xi)})-h(\xi,\bar{u^*}_{\rho(\xi,\bar{u^*}_\xi)})\|_X d\xi ds \\ &\le\ \widetilde{M_S}L_{g_i}\|u-u^*\|_{\mathfrak{B}''_h} + \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_fK_b\|u-u^*\|_{\mathfrak{B}''_h} \\ &\quad +\frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_hK_b\|u-u^*\|_{\mathfrak{B}''_h}.\end{aligned}$$

For $t\in(t_j,s_j]$, we get $\|Pu(t)-u^*(t)\|_X \le L_{g_j}\|u-u^*\|_{\mathfrak{B}''_h},\ j=1,2,\cdots,N$, gathering above results, we obtain

$$\|Pu(t)-u^*(t)\|_X \le \Delta\|u-u^*\|_{\mathfrak{B}''_h}.$$

Since $\Delta<1$, which implies that P is a contraction map and there exists a unique fixed point which is the mild solution of problem (1.5)-(1.7). This completes the proof of the theorem. □

4 Continuous Dependence of Mild Solutions

Theorem 4.2. *Suppose that the assumptions (H_1) are satisfied and the following inequalities hold:*

$$\widetilde{M_S}L_{g_i} + C'K_b < 1.$$

Then for each ϕ, ϕ^*, let u, u^* be the corresponding mild solutions of the system (1.5)-(1.7), then the following inequalities hold:

$$\begin{aligned} \|u - u^*\|_X &\leq \frac{\widetilde{M_S} + C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_l + C'K_b]}\|\phi - \phi^*\|, t \in [0, t_1], \\ \|u - u^*\|_X &\leq \frac{C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_{g_i} + C'K_b]}\|\phi - \phi^*\|, t \in [s_i, t_{i+1}], \end{aligned}$$

for $i = 1, 2, \cdots, N$.

Proof. Estimating for $t \in [0, t_1]$, we have

$$\begin{aligned} \|u - u^*\|_X &\leq \|S_\alpha(t)\|_{L(X)}(\|\phi(0) - \phi^*(0)\|_{\mathfrak{B}_h} + \|l(\bar{u}) - l(\bar{u}^*)\|_X) \\ &\quad + \int_0^t \|T_\alpha(t-s)\|_{L(X)} s^n \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - f(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds \\ &\quad + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u}^*_{\rho(\xi,\bar{u}^*_\xi)})\|_X d\xi ds \\ &\leq \widetilde{M_S}(\|\phi - \phi^*\| + L_l\|u - u^*\|_X) + (\frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f \\ &\quad + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h) \times ((M_b + J^\phi)\|\phi - \phi^*\|_{\mathfrak{B}_h} + K_b\|u - u^*\|), \\ \|u - u^*\|_X &\leq \frac{\widetilde{M_S} + C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_l + C'K_b]}\|\phi - \phi^*\|, \end{aligned}$$

where

$$C' = \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h. \tag{4.13}$$

Similar way, when $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|u - u^*\|_X &\leq \|S_\alpha(t - s_i)\|_{L(X}\|g_i(s_i, \bar{u}(s_i)) - g_i(s_i, \bar{u^*}(s_i))\|_X \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)} s^n \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}) - g_i(s_i, \bar{u^*}(s_i))\|_X ds \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)} \int_0^s q(s-\xi)\|h(\xi, \bar{u}_{\rho(\xi,\bar{u}_\xi)}) - h(\xi, \bar{u^*}_{\rho(\xi,\bar{u^*}_\xi)})\|_X d\xi ds \\ &\leq \widetilde{M_S}L_{g_i}\|u - u^*\|_X + (\frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)}T^{\alpha+n}\widetilde{M_T}L_f \\ &\quad + \frac{T^\alpha}{\alpha}\widetilde{M_T}q^*L_h) \times ((M_b + J^\phi)\|\phi - \phi^*\|_{\mathfrak{B}_h} + K_b\|u - u^*\|), \\ \|u - u^*\|_X &\leq \frac{C'(M_b + J^\phi)}{1 - [\widetilde{M_S}L_{g_i} + C'K_b]}\|\phi - \phi^*\|, \end{aligned}$$

where C' is given in equation (4.13). This completes the proof of the theorem. □

5 Example

Consider the following nonlocal impulsive fractional partial differential equation:

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{t}{9}\int_{-\infty}^{t} e^{2(s-t)}\frac{u(s-\sigma_1(s)\sigma_2(\|u\|),x)}{16}ds \\
&\quad +\int_0^t \sin(t-s)\int_{-\infty}^{\xi} e^{2(\nu-\xi)}\frac{u(\nu-\sigma_1(\nu)\sigma_2(\|u\|),x)}{25}d\nu ds, \\
&\quad (t,y)\in \cup_{i=1}^{N}[s_i,t_{i+1}]\times[0,\pi], \qquad (5.14)\\
u(t,0) &= u(t,\pi)=0,\ \ t\geq 0, \qquad (5.15)\\
u(t,x) &+ \sum_{k=1}^{r} c_k u(s_k,x)=\phi(t,x), t\in(-\infty,0], x\in[0,\pi], \qquad (5.16)\\
u(t,x) &= G_i(t,u(t,x)),\ x\in[0,\pi],\ t\in(t_i,s_i], \qquad (5.17)
\end{aligned}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(0,1]$, $0=t_0=s_0<t_1\leq s_1<\cdots<t_N\leq s_N<t_{N+1}=1$ are fixed real numbers, $\phi\in\mathfrak{B}_h$, and r is a positive integer, $0<t_0<t_1,\cdots,<t_r<1$. Let $X=L^2[0,\pi]$ and define the operator $A:D(A)\subset X\to X$ by $Aw=w''$ with the domain $D(A):=\{w\in X:\ w,w'$ are absolutely continuous, $w''\in X, w(0)=0=w(\pi)\}$. Then

$$Aw=\sum_{n=1}^{\infty} n^2(w,w_n)w_n,\ w\in D(A),$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx),\ n\in\mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in X and is given by

$$T(t)\omega=\sum_{n=1}^{\infty} e^{-n^2 t}(\omega,\omega_n)\omega_n,\ \text{for all } \omega\in X,\ \text{and every } t>0.$$

The subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$, s.t. $\|S_\alpha(t)\|_{L(X)}\leq \widetilde{M}_S$ for $t\in[0,1]$.

Let $h(s)=e^{2s},\ s<0$ then $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}<\infty$, for $t\in(-\infty,0]$ and define

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{\theta\in[s,0]}\|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,1]\times\mathfrak{B}_h$, where $\phi(\theta)(x)=\phi(\theta,x),\ (\theta,x)\in(-\infty,0]\times[0,\pi]$.

Set $u(t)(x)=u(t,x)$, and $\rho(t,\phi)=\rho_1(t)\rho_2(\|\phi(0)\|)$, we have

$$\begin{aligned}
f(t,\phi)(x) &= \frac{1}{9}\int_{-\infty}^{0} e^{2(s)}\frac{\phi}{16}ds;\ g(t,\phi)(x)=\int_{-\infty}^{0} e^{2(s)}\frac{\phi}{25}ds,\\
g_i(t,u)(x) &= G_i(t,u(t,x)),\ l(u)=\sum_{k=1}^{r}c_k u(s_k,x),
\end{aligned}$$

then with these settings the equations (5.14)-(5.17) can be written in the abstract form of equations (1.5)-(1.7). We assume that $\rho_i:[0,\infty)\to[0,\infty),\ i=1,2$, are continuous functions. Now, we can see that for

$(t,\phi),(t,\psi)\in J\times\mathfrak{B}_h$, we have

$$\begin{aligned}
&\|f(t,\phi)-f(t,\psi)\|_{L^2}\\
&=\left[\int_0^\pi\left\{\|\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\frac{\phi}{16}ds-\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\frac{\psi}{16}ds\|\right\}^2 dy\right]^{1/2}\\
&\leq\left[\int_0^\pi\left\{\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\|\frac{\phi}{16}-\frac{\psi}{16}\|ds\right\}^2 dy\right]^{1/2}\\
&\leq\left[\int_0^\pi\left\{\frac{1}{144}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\right\}^2 dy\right]^{1/2}\\
&\leq\frac{\sqrt{\pi}}{144}\|\phi-\psi\|_{\mathfrak{B}_h}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|h(t,\phi)-h(t,\psi)\|_{L^2}\leq\frac{\sqrt{\pi}}{25}\|\phi-\psi\|_{\mathfrak{B}_h},\\
&\|l(u)-l(v)\|_{L^2}\leq\sum_{k=1}^{r}c_k\|u-v\|_{L^2},\\
&\|g_i(t,u)-g_i(t,v)\|_{L^2}\leq L_{G_i}\|u-v\|_{L^2}.
\end{aligned}$$

Hence all the function $f, g_i,\ h$ and l satisfy assumptions of (H_1). We deduced that the system (5.14)-(5.17) has a unique mild solution on $[0,1]$.

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References

[1] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives theory and applications*, Gordon and Breach, Yverdon, 1993.

[2] K. S. Miller, B. Ross, *An introduction to the fractional calculus and differential equations*, John Wiley, New York, 1993.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204, Elsevier Science B.V., Amsterdam, 2006.

[4] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[5] I. Podlubny, *Fractional differential equations*, Acadmic Press, New York, 1999.

[6] D. Araya, C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal.*, 69(2008), 3692–3705.

[7] G.M. Mophou, G.M. N'Gurkata, Existence of the mild solution for some fractional differential equations with nonlocal conditions, *Semigroup Forum*, 79(2009), 315–322.

[8] J. Wang, W. Wei, Y. Yang, On some impulsive fractional differential equations in Banach spaces, *Opuscula Mathematica*, 30(4)(2010).

[9] E.Hernandez, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, 141(5)(2013), 1641–1649.

[10] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comp.*, 219(2013), 6743–6749.

[11] P. Kumara, D. N. Pandey, D. Bahuguna, On a new class of abstract impulsive functional differential equations of fractional order, *J. Nonlinear Sci. Appl.*, 7(2014), 102–114.

[12] J. Wang, X. Li, Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses, *J. Appl. Math. Comput.*, 2014 DOI 10.1007/s12190-013-0751-4.

[13] R. P. Agarwal, B. D. Andrade, On fractional integro-differential equations with state-dependent delay, *Comp. Math. App.*, 62(2011), 1143–1149.

[14] M. Benchohra, F. Berhoun, Impulsive fractional differential equations with state dependent delay, *Commun. Appl. Anal.*, 14(2)(2010), 213–224.

[15] J. Dabas, A. Chauhan, M. Kumar, Existence of mild solution for impulsive fractional equation with infinity delay, *Int. J. Diff. Equ.*, Art ID 793023, (2011).

[16] M. Haase, *The functional calculus for sectorial operators*, Oper. theory Adv. and Appl., vol. 169 Birkhauser-Verlag Basel , 2006.

[17] S. Abbas, M. Benchohra, Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay, *An Int. J. Theory Appl.*, (2010).

[18] J. P. Carvalho dos Santos, M. Mallika Arjunan, Existence results for fractional neutral integro-differential equations with state-dependent delay, *Comp. Math. Appl.*, 62(2011), 1275–1283.

[19] A. Chauhan, J. Dabas, Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition, *Commun. Nonlinear Sci. Num. Sim.*, 19 (2014), 821–829.

[20] A. Chauhan, J. Dabas, Existence of mild solutions for impulsive fractional order semilinear evolution equations with nonlocal conditions, *Electron. J. Diff. Equ.*, Vol. 2011 (2011),, No. 107 pp. 1–10.

[21] J. Dabas, A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equations with infinity delay, *Math. Comp. Modell.*, 57(2013), 754-763.

[22] J. Dabas, G.R. Gautam, Impulsive neutral fractional integro-differential equation with state dependent delay and integral boundary condition, *Elect. J. Diff. Equ.*, Vol. 2013 (2013), No. 273, pp. 1–13.

[23] A. Chauhan, J. Dabas, M. Kumar, Integral boundary-value problem for impulsive fractional functional integro-differential equation with infinite delay, *Electron J. Diff. Equ.*, 2012(2012) No.229, pp. 1–13.

[24] N. K. Tomar, J. Dabas, Controllability of impulsive fractional order semilinear evolution equations with nonlocal conditions, *J. Nonlinear Evolution Equ. Appl.*, 5(2012)(2012), 57–67.

[25] X. Fu, R. Huang, Existence of solutions for neutral integro-differential equations with state-dependent delay, *App. Math. Comp.*, 224(2013), 743–759.

[26] Y. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, *Chaos, Solitons and Fractals*, 33(2007), 1601–1609.

[27] Y. Hino, S. Murakami, T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, vol. 1473, Springer, Berlin, 1991.

[28] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.*, 40(1991), 11–19.

[29] G. R. Gautam, J. Dabas, Existence result of fractional functional integro-differential equation with not instantaneous impulse, *Int. J. Adv. Appl. Math. Mech.*, 1(3)(2014), 11–21.

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