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Some new Ostrowski type inequalities for functions whose second derivative are h-convexe via Riemann-Liouville fractionnal

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Abstract

A new identity similar to an identity proved in Erhan Set. (2012) [16] for fractional integrals is established.By making use of the established identity, some new Ostrowski type inequalities for Riemann–Liouville fractional integral are obtained.

Keywords: Ostrowski type inequalities, Riemann-Liouville integrals, (s, m) −convex function.

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1 Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1. *[13] Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mappingin the interior $I^\circ$of I, and $a, b \in I^\circ$, with $a < b$.*

If $|f'| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b] \tag{1.1}$$

This is well-know Ostrowski inequality.In recent years, a number of authors have written about generalizations,extensions and variants of such inequalities (see [1, 2, 3]).

Let us recall definitions of some kinds of convexity as follows.

Definition 1.1. *We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is convex function if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.2}$$

holds for all $x, y \in I$, and $t \in [0, 1]$.

Definition 1.2. *[7] We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is P- function if f is non-negative and the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{1.3}$$

holds for all $x, y \in I$, and $t \in [0, 1]$.

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Definition 1.3. *[8] We say that $f : [0,\infty) \to \mathbb{R}$ is s-convex function in the second sense, if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.4}$$

holds for all $x, y \in (0,b]$, $t \in [0,1]$ and for fixed $s \in (0,1]$

Definition 1.4. *[15] Let $h : J \subset \mathbb{R} \to \mathbb{R}$,be a positive function.We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ $(I \neq \phi)$ is h-convex function,*

if f is non-negative and

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.5}$$

holds for all $x, y \in I$,and $t \in [0,1]$.

Definition 1.5. *[17] We say that $f : [0,b] \to \mathbb{R}$ $(0 < b)$ is said to be $m-$convex, where $m \in (0,1]$ and $b > 0$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)\, f(y) \tag{1.6}$$

Definition 1.6. *[12] We say that $f : [0,b] \to \mathbb{R}$ $(0 < b)$ is said to be $(s,m)-$convex, where $(s,m) \in (0,1]^2$ and $b > 0$ if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y) \tag{1.7}$$

Definition 1.7. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f(x), J^{\alpha}_{b^-}f(x)$ of order $\alpha > 0$,with $a > 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.8}$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b \tag{1.9}$$

and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$
where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \tag{1.10}$$

noting also

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.11}$$

Motivated by the recent results given in [1, 2, 6, 9], in the present paper, we provide some companions of Ostrowski type inequalities involving Riemann – Liouville fractional integrals for functions whose second derivatives absolute value are h-convex.

2 OSTROWSKI TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

In order to prove our main results we need the following identity.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$.If $f'' \in L_1[a,b]$,then the following equaliy for fractional integrals holds for any $x \in [a,b]$*

$$L_{\alpha}(x) = (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt + (x-b)\int_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right] \tag{2.12}$$

where

$$L_\alpha(x) = (\alpha+1)(b-x)^\alpha (x-a)^\alpha (b-a) f(x) - \Gamma(\alpha+2)\left[(b-x)^{\alpha+1} J^\alpha_{x^-} f(a) + (x-a)^{\alpha+1} J^\alpha_{x^+} f(b)\right] \tag{2.13}$$

Proof. We have

$$\begin{aligned} J^\alpha_{x^-} f(a) &= \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)dt \\ &= \frac{1}{\Gamma(\alpha)} \left[\frac{(x-a)^\alpha}{\alpha} f(x) - \int_a^x \frac{(t-a)^\alpha}{\alpha} f'(t)dt\right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^\alpha f(x) - \left[\frac{(x-a)^{\alpha+1}}{\alpha+1} f'(x) - \int_a^x \frac{(t-a)^{\alpha+1}}{\alpha+1} f''(t)dt\right]\right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(x-a)^\alpha f(x) - (x-a)^{\alpha+1} f'(x) + \int_a^x (t-a)^{\alpha+1} f''(t)dt\right], \end{aligned} \tag{2.14}$$

multiplying both side of (2.14) by $\Gamma(\alpha+2)(b-x)^{\alpha+1}$, we get

$$\Gamma(\alpha+2)(b-x)^{\alpha+1} J^\alpha_{x^-} f(a) = \left[\begin{array}{c} (\alpha+1)(b-x)^{\alpha+1}(x-a)^\alpha f(x) - (b-x)^{\alpha+1}(x-a)^{\alpha+1} f'(x) + \\ (b-x)^{\alpha+1}(x-a)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt \end{array}\right]. \tag{2.15}$$

And

$$\begin{aligned} J^\alpha_{x^+} f(b) &= \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)dt \\ &= \frac{1}{\Gamma(\alpha)} \left[\frac{(b-x)^\alpha}{\alpha} f(x) + \int_x^b \frac{(b-t)^\alpha}{\alpha} f'(t)dt\right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(b-x)^\alpha f(x) + \left[\frac{(b-x)^{\alpha+1}}{\alpha+1} f'(x) - \int_x^b \frac{(b-t)^{\alpha+1}}{\alpha+1} f''(t)dt\right]\right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(b-x)^\alpha f(x) + (b-x)^{\alpha+1} f'(x) + \int_x^b (b-t)^{\alpha+1} f''(t)dt\right], \\ &= \frac{1}{\Gamma(\alpha+2)} \left[(\alpha+1)(b-x)^\alpha f(x) + (b-x)^{\alpha+1} f'(x) + (b-x)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right] \end{aligned} \tag{2.16}$$

multiplying both side of (2.16) by $\Gamma(\alpha+2)(x-a)^{\alpha+1}$, we get

$$\Gamma(\alpha+2)(x-a)^{\alpha+1} J^\alpha_{x^+} f(b) = \left[\begin{array}{c} (\alpha+1)(x-a)^{\alpha+1}(b-x)^\alpha f(x) + (x-a)^{\alpha+1}(b-x)^{\alpha+1} f'(x) + \\ (x-a)^{\alpha+1}(b-x)^{\alpha+2} \int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt \end{array}\right]. \tag{2.17}$$

Summing (2.15) and (2.17), we obtain

$$\Gamma(\alpha+2)(b-x)^{\alpha+1} J^\alpha_{x^-} f(a) + \Gamma(\alpha+2)(x-a)^{\alpha+1} J^\alpha_{x^+} f(b) =$$

$$(\alpha+1)(b-a)(x-a)^\alpha (b-x)^\alpha f(x) +$$

$$(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(x-a)\int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt + (b-x)\int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right] \tag{2.18}$$

we can rewrite (2.18) as follows

$$(\alpha+1)(b-a)(x-a)^{\alpha}(b-x)^{\alpha} f(x) - \left[\Gamma(\alpha+2)(b-x)^{\alpha+1} J_{x^-}^{\alpha} f(a) + \Gamma(\alpha+2)(x-a)^{\alpha+1} J_{x^+}^{\alpha} f(b)\right]$$

$$= (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt + (x-b)\int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right] \tag{2.19}$$

thus (2.19) implies (2.12). □

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a, b \in I$, with $a < b$.*

If $|f''|$ is convex function on $[a,b]$, and f'' is bounded,i.e., $\|f''\|_{\infty} = \sup_{x\in[a,b]} |f''| < \infty$,for any $x \in [a,b]$,then the following inequality holds

$$|L_{\alpha}(x)| \leq \frac{(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)} \|f''\|_{\infty} \tag{2.20}$$

Proof. By lemma 2.1, and Under the given assumptions on f'' we have

$$|L_{\alpha}(x)| = \left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1} f''(tx+(1-t)a)\,dt + (x-b)\int_0^1 t^{\alpha+1} f''(tx+(1-t)b)\,dt\right]\right|$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1} \times \left[(x-a)\int_0^1 t^{\alpha+1}(t|f''(x)| + (1-t)|f''(a)|)dt + (b-x)\int_0^1 t^{\alpha+1}(t|f''(x)| + (1-t)|f''(b)|)dt\right]$$

$$\leq \|f''\|_{\infty}(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-x+x-a)\int_0^1 t^{\alpha+1}dt$$

$$= \frac{(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\|f''\|_{\infty}$$

□

Remark 2.1. *Under the same hypotheses of Theorem 2.1 at the exception of the convexity of $f''$the inequality* (2.20) *remains valid.*

Corollary 2.1. *With the assumptions in Theorem 2.1, in the case where $\alpha = 1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{24}\|f''\|_{\infty} \tag{2.21}$$

Proof. Choose $x = \frac{a+b}{2}$ and $\alpha = 1$ in (2.9), we get

$$\frac{(b-a)^3}{2}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)5}{48}\|f''\|_{\infty} \tag{2.22}$$

dividing both side of (2.22) by $\frac{(b-a)^3}{2}$ we obtain (2.21). □

Remark 2.2. *The inequality* (2.21) *is obtained in [9], choose $x = \frac{a+b}{2}$ in theorem 2.2.*

Corollary 2.2. *With the assumptions in Theorem 2.1, in the case where $\alpha = 1$,one has the inequality.*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(b-a)^2}{96}\ \|f''\|_\infty \tag{2.23}$$

Proof. Apply Theorem 1.1, a faith on the interval $\left[a,\frac{a+b}{2}\right]$,taking $\alpha = 1$ in (2.9) , and replace x by $\frac{3a+b}{4}$, we get

$$\frac{(b-a)^3}{16}\left[\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right|\right] \le \frac{(b-a)^5}{1536}\|f''\|_\infty \tag{2.24}$$

(2.24) implies

$$\left|f(\frac{3a+b}{4})-\frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right| \le \frac{(b-a)^2}{96}\|f''\|_\infty \tag{2.25}$$

Apply Theorem 1.1 another faith on the interval $\left[\frac{a+b}{2},b\right]$, taking $\alpha = 1$ in (2.9) , and replace x by $\frac{a+3b}{4}$, we get

$$\left|f(\frac{a+3b}{4})-\frac{2}{b-a}\int_{\frac{b+a}{2}}^{b} f(t)dt\right| \le \frac{(b-a)^2}{96}\|f''\|_\infty \tag{2.26}$$

summing (2.25) and (2.26), dividing the result by 2 we obtain (2.23). □

Remark 2.3. *The inequality* (2.23) *is obtained in [9] corollary 2.3*

Corollary 2.3. *Let $f : I \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a < b$.*

If $|f''|^q$ is convex function on $[a,b]$, $p,q \ge 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$,

then the following inequality holds

$$|L_\alpha(x)| \le\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\ \|f''\|_\infty \tag{2.27}$$

Proof. under the assumptions given on $f''$and using the well-known Hölder's inequality for lemma 2.1, we get

$$|L_\alpha(x)| = \left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt+(x-b)\int_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right]\right|$$

$$\le (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(x-a)\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)a)\right|dt+(b-x)\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)b)\right|dt\right]$$

$$\le (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}+\\ (b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(t\left|f''(x)\right|^q+(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}+\\(b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left\|f''\right\|_\infty$$

□

Theorem 2.2. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $|f''|\in L_1[a,b]$, where $a,b\in I$, with $a<b$.*

If $|f''|$ is P-convex on $[a,b]$, and and f'' is bounded,i.e., $\|f''\|_\infty=\sup\limits_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$, then the following inequality holds

$$|L_\alpha(x)|\leq\frac{2\,(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\left\|f''\right\|_\infty \tag{2.28}$$

Proof. by lemma 2.1, and Under the given assumptions on f'', we have

$$|L_\alpha(x)|=\left|(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(a-x)\int\limits_0^1 t^{\alpha+1}f''(tx+(1-t)a)\,dt+(x-b)\int\limits_0^1 t^{\alpha+1}f''(tx+(1-t)b)\,dt\right]\right|$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[(x-a)\int\limits_0^1 t^{\alpha+1}(\left|f''(x)\right|+\left|f''(a)\right|)dt+(b-x)\int\limits_0^1 t^{\alpha+1}(\left|f''(x)\right|+\left|f''(b)\right|)dt\right]$$

$$=2\left\|f''\right\|_\infty(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)\int\limits_0^1 t^{\alpha+1}dt=\frac{2\,(x-a)^{\alpha+1}(b-x)^{\alpha+1}(b-a)}{(\alpha+2)}\left\|f''\right\|_\infty$$

□

Corollary 2.4. *With the assumptions in Theorem 2.2, in the case where $\alpha=1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int\limits_a^b f(t)\,dt\right|\leq\frac{(b-a)^2}{12}\left\|f''\right\|_\infty \tag{2.29}$$

Proof. just take in (2.28), $\alpha=1$, $x=\frac{a+b}{2}$ and dividing both side of the result by $\frac{(b-a)^3}{2}$ we obtain (2.29). □

Corollary 2.5. *With the assumptions in Theorem 2.2, in the case where $\alpha=1$,one has the inequality*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\frac{(b-a)^2}{48}\left\|f''\right\|_\infty \tag{2.30}$$

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.2 a faith on the interval $\left[a,\frac{a+b}{2}\right]$,taking $\alpha=1$ and $x=\frac{3a+b}{4}$, and a second time on the interval $[\frac{a+b}{2},b]$ for $\alpha=1$ and $x=\frac{a+3b}{4}$, make the sum and dividing the results by 2, we obtain (2.30). □

Corollary 2.6. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $|f''|\in L_1[a,b]$, where $a,b\in I$, with $a<b$.*

If $|f''|^q$ is P-convex on $[a,b]$, $p,q \geq 1, \frac{1}{p}+\frac{1}{q}=1$ and and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)| \leq 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\|f''\|_{\infty} \tag{2.31}$$

Proof. by lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_{\alpha}(x)| \leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\int\limits_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int\limits_0^1|f''(tx+(1-t)a)|^q\ dt\right)^{\frac{1}{q}}+\\(b-x)\left(\int\limits_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int\limits_0^1|f''(tx+(1-t)a)|^q\ dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(|f''(x)|^q+|f''(a)|^q\right)\ dt\right)^{\frac{1}{q}}+\\(b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left(|f''(x)|^q+\ |f''(b)|^q\right)\ dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\|f''\|_{\infty}$$

□

Theorem 2.3. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$, with $a<b$.*

If $|f''|$ is s-convex on $[a,b]$ with $s\in(0,1)$, and and f'' is bounded,i.e., $\|f''\|_{\infty}=\sup_{x\in[a,b]}|f''|<\infty$, for any $x\in[a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)| \leq\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\frac{1}{\alpha+s+2}+\beta(\alpha+2,s+1)\right]\|f''\|_{\infty} \tag{2.32}$$

Proof. by lemma 2.1, and since $|f''|$ is s-convex and $|f''|\leq M$, then we have

$$|L_{\alpha}(x)| \leq\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\ dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{array}\right]$$

$$\begin{aligned}\leq\quad&(x-a)^{\alpha+1}(b-x)^{\alpha+1}\\&\times\left[\begin{array}{c}(x-a)\int\limits_0^1 t^{\alpha+1}\left(t^s|f''(x)|\ +(1-t)^s|f''(a)|\right)\ dt+\\(b-x)\int\limits_0^1 t^{\alpha+1}\left(t^s|f''(x)|\ +(1-t)^s|f''(b)|\right)\ dt\end{array}\right]\end{aligned}$$

$$\begin{aligned}\leq\quad&\|f''\|_{\infty}\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}\\&\times\left[(x-a)\ \left(\int\limits_0^1 t^{\alpha+s+1}dt+\int\limits_0^1 t^{\alpha+1}\ (1-t)^s\,dt\right)+(b-x)\left(\int\limits_0^1 t^{\alpha+s+1}dt+\int\limits_0^1 t^{\alpha+1}\ (1-t)^s\,dt\right)\right]\end{aligned}$$

$$= (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\frac{1}{\alpha+s+2}+\beta(\alpha+2,s+1)\right]\|f''\|_\infty$$

□

Corollary 2.7. *With the assumptions in Theorem 2.3, in the case where $\alpha = 1$,one has the inequality*

$$\left|f(\frac{a+b}{2})-\frac{1}{(b-a)}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^2}{8}\left[\frac{s^2+3s+4}{(s+3)(s+2)(s+1)}\right]\|f''\|_\infty \tag{2.33}$$

Proof. The proof is similar to that of Corollary 2.1 □

Corollary 2.8. *With the assumptions in Theorem 2.3, in the case where $\alpha = 1$,one has the inequality*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(b-a)^2}{32}\left[\frac{s^2+3s+4}{(s+3)(s+2)(s+1)}\right]\|f''\|_\infty \tag{2.34}$$

Proof. The proof is similar to that of Corollary 2.2 □

Corollary 2.9. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$, with $a < b$.*

If $|f''|^q$ is s-convex on $[a,b]$ with $s \in (0,1)$, $p, q \ge 1$, $\frac{1}{p}+\frac{1}{q}=1$ and and f'' is bounded,i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$,

then the following inequality holds

$$|L_\alpha(x)| \le 2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\|f''\|_\infty \tag{2.35}$$

Proof. by lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_\alpha(x)| \le (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\ dt+\\ (b-x)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{array}\right]$$

$$\le (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\ dt\right)^{\frac{1}{q}}+\\ (b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\ dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le (x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\begin{array}{c}(x-a)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s|f''(x)|^q+(1-t)^s|f''(a)|^q\right)\ dt\right)^{\frac{1}{q}}+\\ (b-x)\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s|f''(x)|^q+(1-t)^s\ |f''(b)|^q\right)\ dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(t^s\ +(1-t)^s\ \right)\ dt\right)^{\frac{1}{q}}\|f''\|_\infty$$

$$2^{\frac{1}{q}}\ (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\|f''\|_\infty$$

□

Theorem 2.4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|$ is h-convex on $[a,b]$, and f'' is bounded,i.e., $\|f''\|_{\infty} = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$,

then the following inequality holds

$$|L_{\alpha}(x)| \leq \|f''\|_{\infty} (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 (t^{\alpha+1}+(1-t)^{\alpha+1})h(t)dt. \tag{2.36}$$

Proof. by lemma 2.1, and since $|f''|$ is h-convex and $|f''| \leq M$, then we have

$$|L_{\alpha}(x)| \leq (x-a)^{\alpha+1}(b-x)^{\alpha+1} \left[\begin{array}{c} (x-a)\int_0^1 t^{\alpha+1} |f''(tx+(1-t)a)| \; dt+ \\ (b-x)\int_0^1 t^{\alpha+1} |f''(tx+(1-t)b)| \, dt \end{array} \right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1} \times \left[\begin{array}{c} (x-a) \int_0^1 t^{\alpha+1}(h(t)|f''(x)|+h(1-t)|f''(a)|)dt+ \\ (b-x)\int_0^1 t^{\alpha+1}(h(t)|f''(x)|+h(1-t)|f''(b)|)dt \end{array} \right]$$

$$\leq \|f''\|_{\infty} (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 t^{\alpha+1}(h(t)+h(1-t))dt$$
$$= \|f''\|_{\infty} (b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1} \int_0^1 (t^{\alpha+1}+(1-t)^{\alpha+1})h(t)dt.$$

□

Corollary 2.10. *With the assumptions in Theorem 2.4, in the case where $\alpha = 1$,one has the inequality*

$$\left| f(\frac{a+b}{2}) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{\|f''\|_{\infty}\,(b-a)^2}{8} \int_0^1 (2t^2-2t+1)h(t)dt. \tag{2.37}$$

Proof. The proof is similar to that of Corollary 2.1 □

Corollary 2.11. *With the assumptions in Theorem 2.4, in the case where $\alpha = 1$,one has the inequality*

$$\left| \frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{\|f''\|_{\infty}(b-a)^2}{32} \int_0^1 (2t^2-2t+1)h(t)dt. \tag{2.38}$$

Proof. The proof is similar to that of Corollary 2.2 □

Corollary 2.12. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a,b \in I$, with $a<b$.*

If $|f''|^q$ is h-convex on $[a,b]$, $p,q \geq 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and and f'' is bounded,i.e., $\|f''\|_{\infty} = \sup_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_{\alpha}(x)| \leq 2^{\frac{1}{q}}(b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}} \|f''\|_{\infty} \tag{2.39}$$

Proof. By lemma 2.1, the assumptions given on $f''$and using the well-known Hölder's inequality, we have

$$|L_\alpha(x)| \leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix} (x-a)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)a)|\,dt+ \\ (b-x)\int_0^1 t^{\alpha+1}|f''(tx+(1-t)b)|\,dt\end{bmatrix}$$

$$\leq \frac{1}{(\alpha+1)(b-a)}$$

$$\times\left[(b-x)(x-a)^2\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)a)\right|\,dt+(a-x)(b-x)^2\int_0^1 t^{\alpha+1}\left|f''(tx+(1-t)b)\right|dt\right]$$

$$\leq (x-a)^{\alpha+1}(b-x)^{\alpha+1}\begin{bmatrix} (x-a)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}+ \\ (b-x)\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\end{bmatrix}$$

$$=(b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left[\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(h(t)\left|f''(x)\right|^q+h(1-t)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$\leq 2^{\frac{1}{q}}(b-a)(x-a)^{\alpha+1}(b-x)^{\alpha+1}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}\|f''\|_\infty.$$

□

Now, using the above reasoning we can obtain some new Ostrowski Type inequalities involving Riemann-Liouville fractional integrals for functions whose derivatives are m-convex.

Theorem 2.5. *Let $f: I\to\mathbb{R}$ be a twice differentiable mapping on I° such that $|f''|\in L_1[a,b]$, where $a,b\in I$, with $0<a<b$.*

If $|f''|$ is $m-$convex function on $[a,b]$, $m\in(0,1]$ and f'' is bounded,i.e., $\|f''\|_\infty=\sup\limits_{x\in[a,b]}|f''|<\infty$,for any $x\in[a,b]$,then the following inequality holds

$$|L_\alpha(x)|\leq(1-m)\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(\Upsilon_1+\Upsilon_2)\left\|f''\right\|_\infty \tag{2.40}$$

where $\Upsilon_1=(x-a)\left[\frac{1}{\alpha+3}\left(\frac{x-a}{x-ma}\right)+\frac{1}{\alpha+2}\left(\frac{(1-m)a}{x-ma}+\frac{m}{(1-m)}\right)\right]$,
and $\Upsilon_2=(b-x)\left[-\frac{1}{\alpha+3}\left(\frac{b-x}{b-mx}\right)+\frac{1}{\alpha+2}\left(\frac{1}{1-m}\right)\right]$.

Proof. By lemma 2.1, and Under the given assumptions on f'' we have

$$|L_\alpha(x)|=\left|(b-x)^{\alpha+1}\int_a^x(y-a)^{\alpha+1}f''(y)\,dy+(x-a)^{\alpha+1}\int_x^b(b-y)^{\alpha+1}f''(y)\,dy\right|$$

$$=\left|\begin{array}{l}(b-x)^{\alpha+1}(x-ma)\int_{\frac{(1-m)a}{x-ma}}^1(tx+m(1-t)a-a)^{\alpha+1}f''(tx+m(1-t)a)\,dt+ \\ (x-a)^{\alpha+1}(b-mx)\int_{\frac{(1-m)x}{b-mx}}^1(b-(tb+m(1-t)x))^{\alpha+1}f''(tb+m(1-t)x)\,dt\end{array}\right|$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2} \int_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} \left|f''(tx+m(1-t)a)\right| dt + \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2} \int_{\frac{(1-m)x}{b-mx}}^{1} (1-t)^{\alpha+1} \left|f''(tb+m(1-t)x)\right| dt \end{array} \right]$$

$$\leq (1-m)\left\|f''\right\|_{\infty} \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2} \int_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} \left(t+\frac{m}{1-m}\right) dt + \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2} \int_{\frac{(1-m)x}{b-mx}}^{1} (1-t)^{\alpha+1} \left(t+\frac{m}{1-m}\right) dt \end{array} \right]$$

$$\leq \left\|f''\right\|_{\infty} \left[\begin{array}{c} (1-m)(b-x)^{\alpha+1}(x-ma)^{\alpha+2} \left[\begin{array}{c} \int_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+2} dt + \\ \left(\frac{(1-m)a}{x-ma}+\frac{m}{(1-m)}\right) \int_{\frac{(1-m)a}{x-ma}}^{1} \left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} dt \end{array} \right] + \\ (1-m)(x-a)^{\alpha+1}(b-mx)^{\alpha+2} \left[\begin{array}{c} \int_{\frac{(1-m)x}{b-mx}}^{1} -(1-t)^{\alpha+2} dt + \\ \left(\frac{1}{1-m}\right) \int_{\frac{(1-m)x}{b-mx}}^{1} (1-t)^{\alpha+1} dt \end{array} \right] \end{array} \right]$$

$$\leq (1-m)\ (x-a)^{\alpha+1}(b-x)^{\alpha+1}(\Upsilon_1+\Upsilon_2)\left\|f''\right\|_{\infty}$$

□

Corollary 2.13. *With the assumptions in Theorem 2.5, in the case where $\alpha=1$, one has the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \frac{(1-m)(b-a)^2}{16} \varphi \left\|f''\right\|_{\infty} \tag{2.41}$$

where $\varphi = \frac{(1-m)(b-a)^2}{4(b+(1-2m)a)(2b-m(b+a))} + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{2(1-m)a}{b+(1-2m)a}\right]$.

Proof. Choose $x=\frac{a+b}{2}$ and $\alpha=1$ in (2.40), we get

$$\frac{(b-a)^3}{2} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t)\,dt \right| \leq \varphi \frac{(1-m)(b-a)^5}{32} \left\|f''\right\|_{\infty} \tag{2.42}$$

dividing both side of (2.42) by $\frac{(b-a)^3}{2}$ we obtain (2.41). □

Corollary 2.14. *With the assumptions in Theorem 2.5, in the case where $\alpha=1$,one has the inequality.*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq (\Psi_1+\Psi_2) \frac{(1-m)(b-a)^2}{128} \left\|f''\right\|_{\infty} \tag{2.43}$$

where $\Psi_1 = \frac{b-a}{4}\left[\frac{1}{b+(3-4m)a} - \frac{1}{(2-m)b+(2-3m)a}\right] + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{4(1-m)a}{b+(3-4m)a}\right]$,
and $\Psi_2 = \frac{b-a}{4}\left[\frac{1}{(3-2m)b+(1-2m)a} - \frac{1}{(4-3m)b-ma}\right] + \frac{1}{3}\left[\frac{1+m}{1-m} + \frac{2(1-m)(a+b)}{(3-2m)b+(1-2m)a}\right]$.

Proof. Apply Theorem 2.5, a faith on the interval $\left[a, \frac{a+b}{2}\right]$,taking $\alpha = 1$ in (2.40) , and replace x by $\frac{3a+b}{4}$, we get

$$\frac{(b-a)^3}{16}\left[\left|f(\frac{3a+b}{4}) - \frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right|\right] \leq \frac{(1-m\ )\,(b-a)^5}{1024}\left\|f''\right\|_\infty \Psi_1 \tag{2.44}$$

(2.44) implies

$$\left|f(\frac{3a+b}{4}) - \frac{2}{b-a}\int_a^{\frac{b+a}{2}} f(t)dt\right| \leq \frac{(1-m\ )\,(b-a)^2}{64}\left\|f''\right\|_\infty \Psi_1 \tag{2.45}$$

Apply Theorem 2.5 another faith on the interval $\left[\frac{a+b}{2}, b\right]$, taking $\alpha = 1$ in (2.40) , and replace x by $\frac{a+3b}{4}$, we get

$$\left|f(\frac{a+3b}{4}) - \frac{2}{b-a}\int_{\frac{b+a}{2}}^{b} f(t)dt\right| \leq \frac{(1-m\ )\,(b-a)^2}{64}\left\|f''\right\|_\infty \Psi_2 \tag{2.46}$$

summing (2.45) and (2.46), dividing the result by 2 we obtain (2.43). □

Theorem 2.6. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $|f''| \in L_1[a,b]$, where $a, b \in I$, with $a < b$.*

If $|f''|$ is (s,m) $-$convex on $[a,b]$, where $(s,m) \in (0,1]^2$, and and f'' is bounded,i.e., $\|f''\|_\infty = \sup\limits_{x\in[a,b]} |f''| < \infty$, for any $x \in [a,b]$, then the following inequality holds

$$|L_\alpha(x)| \leq (1-m)\ \left\|f''\right\|_\infty \left[\frac{m}{(1-m)\,(\alpha+2)}\ (b-x)^{\alpha+1}(x-\ a)^{\alpha+1}\ \ (b-a) + \chi_1 + \chi_2\right] \tag{2.47}$$

where $\chi_1 = (b-x)^{\alpha+1}(x-ma)^{-s}\left((1-m)\,a\right)^{\alpha+s+2}\beta(\alpha+2, -s-\alpha-2)$,
and $\chi_2 = (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\beta(\alpha+2, s+1)$.

Proof. by lemma 2.1, and Under the given assumptions on f'', we have

$$|L_\alpha(x)| = \left|(b-x)^{\alpha+1}\int_a^x (y-a)^{\alpha+1} f''(y)\,dy + (x-a)^{\alpha+1}\int_x^b (b-y)^{\alpha+1} f''(y)\,dy\right|$$

$$= \left|\begin{array}{l}(b-x)^{\alpha+1}(x-ma)\displaystyle\int_{\frac{(1-m)a}{x-ma}}^{1} (tx+m(1-t)a-a)^{\alpha+1} f''(tx+m(1-t)a)\,dt+ \\ (x-a)^{\alpha+1}(b-mx)\displaystyle\int_{\frac{(1-m)x}{b-mx}}^{1} (b-(tb+m(1-t)x))^{\alpha+1} f''(tb+m(1-t)x)\,dt\end{array}\right|$$

$$\leq \left[\begin{array}{l}\left|(b-x)^{\alpha+1}(x-ma)^{\alpha+2}\displaystyle\int_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1} f''(tx+m(1-t)a)\,dt\right| + \\ \left|(x-a)^{\alpha+1}(b-mx)^{\alpha+2}\displaystyle\int_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1} f''(tb+m(1-t)x)\,dt\right|\end{array}\right]$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}\left|f''(tx+m(1-t)a)\right|dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}\left|f''(tb+m(1-t)x)\right|dt \end{array}\right]$$

$$\leq \left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}\left(t^s\left|f''(x)\right|+m(1-t^s)\left|f''(a)\right|\right)dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}\left(t^s\left|f''(b)\right|+m(1-t^s)\left|f''(x)\right|\right)dt \end{array}\right]$$

$$\leq (1-m)\left\|f''\right\|_\infty\left[\begin{array}{c} (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}\left(t-\frac{(1-m)a}{x-ma}\right)^{\alpha+1}\left(t^s+\frac{m}{1-m}\right)dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}\left(t^s+\frac{m}{1-m}\right)dt \end{array}\right]$$

$$\leq (1-m)\left\|f''\right\|_\infty\left[\begin{array}{c} \frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+2}+\frac{m}{(1-m)(\alpha+2)}(x-a)^{\alpha+1}(b-x)^{\alpha+2}+ \\ (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\left(\frac{(1-m)a}{x-ma}\right)^{\alpha+s+2}\int\limits_{\frac{(1-m)a}{x-ma}}^{1}(1-t)^{\alpha+1}t^{-(s+\alpha+3)}dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_{\frac{(1-m)x}{b-mx}}^{1}(1-t)^{\alpha+1}t^s dt \end{array}\right]$$

$$\leq (1-m)\left\|f''\right\|_\infty\left[\begin{array}{c} \frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+2}+\frac{m}{(1-m)(\alpha+2)}(x-a)^{\alpha+1}(b-x)^{\alpha+2}+ \\ (b-x)^{\alpha+1}(x-ma)^{\alpha+2}\left(\frac{(1-m)a}{x-ma}\right)^{\alpha+s+2}\int\limits_0^1(1-t)^{\alpha+1}t^{-(s+\alpha+3)}dt+ \\ (x-a)^{\alpha+1}(b-mx)^{\alpha+2}\int\limits_0^1(1-t)^{\alpha+1}t^s dt \end{array}\right]$$

$$\leq (1-m)\left\|f''\right\|_\infty\left[\frac{m}{(1-m)(\alpha+2)}(b-x)^{\alpha+1}(x-a)^{\alpha+1}(b-a)+\chi_1+\chi_2\right]$$

□

Corollary 2.15. *With the assumptions in Theorem 2.6, in the case where $\alpha=1$,one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int\limits_a^b f(t)\,dt\right|\leq\Gamma\left(\frac{1-m}{b-a}\right)\left\|f''\right\|_\infty \tag{2.48}$$

where $\Gamma=\frac{m(b-a)^3}{24m}+\frac{(b+(1-2m)a)^{-s}((1-m)a)^{s+3}}{2^{1-s}}\beta(3,-s-3)+\frac{((2-m)b-ma)^3}{2^4}\beta(3,s+1)$.

Proof. just take in (2.47), $\alpha=1$, $x=\frac{a+b}{2}$ and dividing both side of the result by $\frac{(b-a)^3}{2}$ we obtain (2.48). □

Corollary 2.16. *With the assumptions in Theorem 2.6, in the case where $\alpha=1$,one has the inequality*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq(\zeta_1+\zeta_2)\frac{1-m}{2(b-a)}\left\|f''\right\|_\infty, \tag{2.49}$$

where $\zeta_1 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{b+(3-4m)a}{4}\right)^{-s} ((1-m)\,a)^{s+3}\, \beta\,(3, -s-3) + \frac{((2-m)b-ma)^3}{64}\beta\,(3,\ s+1)$,
and $\zeta_2 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{(3-2m)b+(1-2m)a}{4}\right)^{-s} \left(\frac{(1-m)(b+a)}{2}\right)^{s+3} \beta\,(3, -s-3) + \frac{((4-3m)b+(2-3m)a)^3}{64}\beta\,(3,\ s+1)$.

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.6 a faith on the interval $\left[a, \frac{a+b}{2}\right]$,taking $\alpha = 1$ and $x = \frac{3a+b}{4}$, and a second time on the interval $[\frac{a+b}{2}, b]$ for $\alpha = 1$ and $x = \frac{a+3b}{4}$, make the sum and dividing the results by $\frac{(b-a)^3}{32}$, we obtain (2.49). □

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