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Oscillation theorems for second-order half-linear neutral difference equations

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Abstract

In this article, some new oscillation criteria are established for the second order neutral difference equation of the form

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) = 0,\ n \geq n_0,$$

where $z(n) = x(n) + p(n)x(\tau(n))$. Our results improve and extend some known results in the literature. Some examples are also provided to show the importance of these results.

Keywords: Second order, half-linear, neutral, oscillation, difference equations.

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1 Introduction

This article deals with the oscillation of all solutions of the second order neutral difference equation of the form

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) = 0,\ n \geq n_0, \tag{1.1}$$

where $z(n) = x(n) + p(n)x(\tau(n))$. Throughout this article, we assume the following hypotheses:

(H_1) α is a ratio of odd positive integers;

(H_2) $\{a(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers;

(H_3) $\{\sigma(n)\}$ and $\{\tau(n)\}$ are sequences of nonnegative integers with $\tau \circ \sigma = \sigma \circ \tau$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined and satisfying the equation (1.1) for all $n \geq n_0$. A nontrivial solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

It is well-known that second order neutral difference equations find applications in so many problems in the field of population dynamics, economics, biology etc. Therefore, there has been much interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different types of second order difference equations, see for example [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Here, we recall some of the previous works that motivate our study.

In[1, 4, 9], the authors discussed the oscillatory behavior of all solutions of equation

$$\Delta^2(x(n) + p(n)x(n-\tau)) + q(n)x(n-\sigma) = 0$$

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under different conditions on the sequence $\{p_n\}$ and $\{q_n\}$. In [8], the authors studied the oscillation of nonlinear difference equation

$$\Delta(a(n)\Delta(x(n)+p(n)x(n-\tau))+q(n)f(x(n-\sigma))=0,$$

under the assumptions

$$\frac{f(u)}{u}\leq M>0,\ \sum_{n=n_0}^{\infty}\frac{1}{a(n)}=\infty \text{ and } 0\leq p(n)<1.$$

In [10, 11], the authors established several oscillation results for the equation

$$\Delta(a(n)(\Delta(x(n)+p(n)x(n-\tau)))^{\gamma})+f(n,x(n-\tau))=0$$

under the assumptions

$$f(n,u)sgn(u)\geq q(n)u^{\gamma},\ \sum_{n=n_0}^{\infty}\frac{1}{(a(n))^{1/\alpha}}=\infty \text{ and}$$

In [7, 8, 10, 11], the authors studied the oscillatory properties of the nonlinear neutral difference equation of the form

$$\Delta(a(n)(\Delta(x(n)+p(n)x(\tau(n))))^{\alpha})+q(n)x^{\beta}(\sigma(n))=0$$

with the condition $0\leq p(n)\leq p<\infty$ and $\tau\circ\sigma=\sigma\circ\tau$. Following this trend, in this paper we establish some new oscillation criteria for the equation (1.1) with the following conditions:

(i)

$$\sum_{n=n_0}^{\infty}\frac{1}{[a(n)]^{1/\alpha}}=\infty \tag{1.2}$$

and

(ii)

$$\sum_{n=n_0}^{\infty}\frac{1}{[a(n)]^{1/\alpha}}<\infty. \tag{1.3}$$

In Sections 2 and 3, we use the following notations for our convenience:

$$Q(n)=\min\{q(n),q[\tau(n)]\} \text{ and } \delta(n)=\sum_{s=\eta(n)}^{\infty}\frac{1}{[a(s)]^{1/\alpha}}.$$

2 Oscillation Results

In this section, we present the following lemma, which will be useful in proving the main results.

Lemma 2.1. *Let $A\geq 0$, $B\geq 0$ and $\alpha\geq 1$. Then*

$$(A+B)^{\alpha}\leq 2^{\alpha-1}(A^{\alpha}+B^{\alpha}). \tag{2.4}$$

Proof. The proof can be found in [16, Lemma 2.1]. □

Theorem 2.1. *Suppose that condition* (1.2) *holds, $\Delta\sigma(n)>0$, $\sigma(n)\leq n$ and $\sigma(n)\leq\tau(n)$ for all $n\geq n_0$. If there exists a positive real sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\rho(s)\left[\frac{Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(\frac{\Delta\rho(s)}{\rho(s)}\right)^{\alpha+1}a[\sigma(s)]\left[1+\frac{1}{(p^{\alpha}[\sigma(s)])^{2}}\right]\right]=\infty, \tag{2.5}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Then by the definition of $z(n)$, we have $z(n) > 0$. From equation (1.1), for all sufficiently large n, we have

$$\begin{aligned}\Delta(a(n)\Delta(z(n))^{\alpha}) + q(n)x^{\alpha}(\sigma(n)) + q(\tau(n))p^{\alpha}(\sigma(n))x^{\alpha}(\sigma(\tau(n)))\\ + p^{\alpha}(\sigma(n))\Delta(a(\tau(n)))(\Delta z(\tau(n)))^{\alpha} = 0,\end{aligned} \tag{2.6}$$

Using (2.4), $\tau \circ \sigma = \sigma \circ \tau$ and the definition of $\{z(n)\}$ in (2.6), we conclude that

$$\Delta(a(n)\Delta(z(n))^{\alpha}) + \frac{1}{2^{\alpha-1}}Q(n)z^{\alpha}(\sigma(n)) + p^{\alpha}(\sigma(n))\Delta(a(\tau(n)))(\Delta z(\tau(n)))^{\alpha} \leq 0. \tag{2.7}$$

From the equation (1.1), we have

$$\Delta(a(n)\Delta(z(n))^{\alpha}) = -q(n)x^{\alpha}(\sigma(n)) < 0,\ n \geq n_1. \tag{2.8}$$

Thus $\{a(n)(\Delta z(n))^{\alpha}\}$ is a decreasing sequence. Here, we have two possible cases for $\Delta z(n)$, namely, (i) $\Delta z(n) < 0$ eventually or (ii) $\Delta z(n) > 0$ eventually.

Case (i): Suppose that $\Delta z(n) < 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Then, from (2.8), we have

$$a(n)\Delta(z(n))^{\alpha} \leq a(n_2)\Delta(z(n_2))^{\alpha} < 0,\ n \geq n_2 \tag{2.9}$$

which implies that

$$z(n) \leq z(n_2) + a^{1/\alpha}(n_2)\Delta z(n_2)\sum_{s=n_2}^{n-1}\frac{1}{a^{1/\alpha(s)}}. \tag{2.10}$$

Letting $n \to \infty$, by (1.2) we see that $z(n) \to -\infty$, which is a contradiction for the positivity of $z(n)$.

Case (ii): Suppose that $\Delta z(n) > 0$ for all $n \geq n_2 \geq n_1 \geq n_0$. Define

$$w(n) = \rho(n)\frac{a(n)(\Delta z(n))^{\alpha}}{(z(\sigma(n))^{\alpha}},\ n \geq n_2, \tag{2.11}$$

then $w(n) > 0$ for all $n \geq n_2$. By (2.8), we have

$$\Delta(z(\sigma(n)) \geq \Delta z(n)\left(\frac{a(n)}{a(\sigma(n))}\right)^{1/\alpha}. \tag{2.12}$$

From (2.11), we obtain

$$\begin{aligned}\Delta w(n) &= \Delta\rho(n)\frac{a(n+1)(\Delta z(n+1))^{\alpha}}{(z[\sigma(n+1)])^{\alpha}} + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}}\\ &\quad -\frac{\rho(n)a(n)(\Delta z(n))^{\alpha}}{(z[\sigma(n)])^{\alpha}(z[\sigma(n+1)])^{\alpha}}\Delta(z[\sigma(n)])^{\alpha}\\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{\rho(n)a(n+1)(\Delta z(n+1))^{\alpha}}{(z[\sigma(n)])^{\alpha}(z[\sigma(n+1)])^{\alpha}}\Delta(z[\sigma(n)])^{\alpha}\\ &\quad +\rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}}\\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}}\\ &\quad -\frac{\rho(n)}{\rho(n+1)}w(n+1)\frac{\Delta(z[\sigma(n)])^{\alpha}}{(z[\sigma(n)])^{\alpha}}.\end{aligned} \tag{2.13}$$

By using Mean Value Theorem, we have

$$\begin{aligned}\Delta w(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}}\\ &\quad -\frac{\rho(n)}{\rho(n+1)}w(n+1)\alpha\frac{w^{1/\alpha}(n+1)}{\rho^{1/\alpha}(n+1)a^{1/\alpha}(\sigma(n))}\\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n+1)])^{\alpha}}\\ &\quad -\alpha\frac{\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{w^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}(\sigma(n))}.\end{aligned} \tag{2.14}$$

Define

$$v(n) = \rho(n)\frac{a[\tau(n)](\Delta z[\tau(n)])^{\alpha}}{(z[\sigma(n)])^{\alpha}},\ n \geq n_2, \tag{2.15}$$

then, we have $v(n) > 0$ and

$$\begin{aligned}\Delta v(n) \quad \leq \quad & \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) + \frac{\rho(n)\Delta(a[\tau(n)](\Delta z[\tau(n)])^{\alpha})}{(z[\sigma(n+1)])^{\alpha}} \\ & -\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{v^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\sigma(n)]}.\end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned}\Delta w(n) + p^{\alpha}[\sigma(n)]\Delta v(n) \leq & \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \frac{\Delta\rho(n)}{\rho(n+1)}p^{\alpha}[\sigma(n)]v(n+1) \\ & + \rho(n)\left[\frac{\Delta(a(n)(\Delta z(n))^{\alpha})}{(z[\sigma(n)])^{\alpha}} + p^{\alpha}[\sigma(n)]\frac{\Delta(a[\tau(n)](\Delta z[\tau(n)]))^{\alpha}}{(z[\sigma(n)])^{\alpha}}\right] \\ & - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right].\end{aligned} \tag{2.17}$$

Using (2.7) in (2.17), we have

$$\begin{aligned}\Delta w(n) + p^{\alpha}[\sigma(n)]\Delta v(n) \leq & \frac{\Delta\rho(n)}{\rho(n+1)}\left[w(n+1) + p^{\alpha}[\sigma(n)]v(n+1)\right] - \frac{\rho(n)Q(n)}{2^{\alpha-1}} \\ & - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right].\end{aligned} \tag{2.18}$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\begin{aligned}w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq & -\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) \\ & + \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}w(s+1) - \frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}w^{\frac{\alpha+1}{\alpha}}(s+1)\right] \\ & + \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha p(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\right].\end{aligned} \tag{2.19}$$

$$\begin{aligned}\text{Let } A(n) \quad = \quad & \left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\right)^{\frac{\alpha}{\alpha+1}} w(n+1) \text{ and} \\ B(n) \quad = \quad & \left(\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]}\right)^{\frac{-\alpha}{\alpha+1}}\right)^{\alpha}.\end{aligned} \tag{2.20}$$

Now, using the inequality

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha} - A^{\frac{\alpha+1}{\alpha}} \leq \frac{1}{\alpha}B^{\frac{\alpha+1}{\alpha}}, \tag{2.21}$$

by taking $A = A(n)$ and $B = B(n)$ in the second part of the right hand side of the inequality (2.19), we have

$$\begin{aligned}w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq & \frac{-1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s) \\ & + \sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\sigma(s)] \\ & + \sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1) - \frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\sigma(s)]}v^{\frac{\alpha+1}{\alpha}}v(s+1)\right].\end{aligned} \tag{2.22}$$

Now, let

$$C(n) = \left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]} \right)^{\frac{\alpha}{\alpha+1}} v(n+1)$$

and

$$D(n) = \left[\frac{\alpha}{\alpha+1} \frac{\Delta\rho(n)}{\rho(n+1)} \left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\sigma(n)]} \right)^{\frac{-\alpha}{\alpha+1}} \right]^{\alpha}. \tag{2.23}$$

Now, using the inequality (2.21) by taking $A = C(n)$ and $B = D(n)$ in the third part of the right hand side of (2.32), we get

$$\begin{aligned} w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq \frac{-1}{2^{\alpha-1}} \sum_{s=n_2}^{n-1} \rho(s)Q(s) \\ + \sum_{s=n_2}^{n-1} \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}} a[\sigma(s)] + \sum_{s=n_2}^{n-1} \frac{(\Delta\rho(s))^{\alpha+1}}{((p^{\alpha}\sigma(s)))^2}. \end{aligned} \tag{2.24}$$

Now,

$$\begin{aligned} w(n) - w(n_2) + p^{\alpha}[\sigma(n)]v(n) - p^{\alpha}[\sigma(n_2)]v(n_2) \leq -\sum_{s=n_2}^{n-1} \rho(s) \left[\frac{1}{2^{\alpha-1}} Q(s) \right. \\ \left. - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{(\Delta\rho(s))}{(\rho(s))} \right)^{\alpha+1} a[\sigma(s)] \left[1 + \frac{1}{(p^{\alpha}[\sigma(s)])^2} \right] \right]. \end{aligned} \tag{2.25}$$

Letting $n \to \infty$ in the last inequality and using (2.5), we see that $w(n) + p^{\alpha}[\sigma(n)]v(n) \to -\infty$, which contradicts the positivity of $w(n) + p^{\alpha}[\sigma(n)]v(n)$. This completes the proof. □

Theorem 2.2. *Assume that condition* (1.2) *holds,* $p(n) \leq p_0 < \infty$, $\Delta\sigma(n) > 0$, $\sigma(n) \leq n$ *and* $\sigma(n) \leq \tau(n)$ *for all* $n \geq n_0$. *Further, suppose that there exists a sequence* $\{\rho(n)\}$ *of positive real numbers such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \rho(s) \left[\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\sigma(s)] \left(1 + \frac{1}{(p_0^{\alpha}[\sigma(s)])^2} \right) \right] = \infty. \tag{2.26}$$

Then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Using the equation (1.1), for all sufficiently large n, we have

$$\begin{aligned} (a(n)(\Delta z(n))^{\alpha}) + q(n)x^{\alpha}[\sigma(n)] + p_0^{\alpha} q[\tau(n)]x^{\alpha}[\sigma(\tau(n))] \\ + p_0^{\alpha}(a[\tau(n)])(\Delta z[\tau(n)]^{\alpha}) \leq 0. \end{aligned} \tag{2.27}$$

By applying (2.4) and the definition of $z(n)$, we conclude that

$$\Delta(a(n)(\Delta z(n))^{\alpha}) + \frac{1}{2^{\alpha-1}} Q(n)z^{\alpha}[\sigma(n)] + p_0^{\alpha}(a[\tau(n)](\Delta z[\tau(n)])^{\alpha}) \leq 0. \tag{2.28}$$

The remainder of the proof is similar to that of Theorem 2.1 and hence it is omitted.

□

Theorem 2.3. *Assume that conditions* (1.2) *holds,* $\tau(n) \leq n$ *and* $\sigma(n) \geq \tau(n)$ *for all* $n \geq n_0$. *Furthermore assume that there exists a positive real sequence* $\{\rho(n)\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \rho(s) \left[\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\Delta\rho(s)}{\rho(s)} \right)^{\alpha+1} a[\tau(s)] \left(1 + \frac{1}{(p^{\alpha}[\sigma(s)])^2} \right) \right] = \infty. \tag{2.29}$$

Then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $x[\tau(n)] > 0$ and $x[\sigma(n)] > 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.1, we have (2.7). From (1.1), we have $\{a(n)(\Delta z(n))^\alpha\}$ is a decreasing sequence. Then, we have two possible cases for $\Delta z(n)$, namely, (i) $\Delta z(n) < 0$ eventually or (ii) $\Delta z(n) > 0$ eventually.

Case (i): If $\Delta z(n) < 0$ for all $n \geq n_2 \geq n_1$, then by the similar proof of case (i) of Theorem 2.1, we get a contradiction.

Case (ii): If $\Delta z(n) > 0$ for $n \geq n_2 \geq n_1$, then we define

$$w(n) = \rho(n)\frac{a(n)(\Delta z(n))^\alpha}{z[\tau(n)])^\alpha},\ n \geq n_2, \tag{2.30}$$

and $w(n) > 0$. for all $n \geq n_2$. By (2.8), we have

$$\Delta z[\tau(n)] \leq \left(\frac{a(n)}{a[\tau(n)]}\right)^{1/\alpha} \Delta z(n),\ n \geq n_2. \tag{2.31}$$

From (2.30), we have

$$\begin{aligned}
\Delta w(n) &= \Delta\rho(n)\frac{a(n+1)(\Delta z(n+1))^\alpha}{(z[\tau(n+1)])^\alpha} + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\quad - \frac{\rho(n)a(n)(\Delta z(n))^\alpha}{(z[\tau(n)])^\alpha(z[\tau(n+1)])^\alpha}\Delta(z[\tau(n)])^\alpha \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{\rho(n)a(n+1)(\Delta z(n+1))^\alpha}{(z[\tau(n)])^\alpha(z[\tau(n+1)])^\alpha}\Delta(z[\tau(n)])^\alpha \\
&\quad + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} - \frac{\rho(n)w(n+1)}{\rho(n+1)}\frac{\Delta(z[\tau(n)])^\alpha}{(z[\tau(n)])^\alpha}.
\end{aligned} \tag{2.32}$$

By using Mean Value Theorem, we have

$$\begin{aligned}
\Delta w(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} \\
&\quad - \frac{\rho(n)w(n+1)}{\rho(n+1)}\frac{\alpha w^{1/\alpha}(n+1)}{\rho^{1/\alpha}(n+1)a^{1/\alpha}[\tau(n)]} \\
&\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} - \alpha\frac{\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{w^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\tau(n)]}.
\end{aligned} \tag{2.33}$$

Define

$$v(n) = \rho(n)\frac{a[\tau(n)](\Delta z[\tau(n)])^\alpha}{(z[\tau(n)])^\alpha},\ n \geq n_2, \tag{2.34}$$

then we get $v(n) > 0$ and

$$\Delta v(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1) + \frac{\rho(n)(a[\tau(n)](\Delta z[\tau(n)])^\alpha)}{(z[\tau(n)])^\alpha} - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\frac{v^{\frac{\alpha+1}{\alpha}}(n+1)}{a^{1/\alpha}[\tau(n)]}. \tag{2.35}$$

From (2.33) and (2.35), we have

$$\begin{aligned}
\Delta w(n) + p^\alpha[\sigma(n)]\Delta v(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \frac{\Delta\rho(n)}{\rho(n+1)}p^\alpha[\sigma(n)]v(n+1) \\
&\quad + \rho(n)\left[\frac{\Delta(a(n)(\Delta z(n))^\alpha)}{(z[\tau(n+1)])^\alpha} + \frac{p^\alpha[\sigma(n)]\Delta(a(\tau(n))(\Delta z(\tau(n)))^\alpha)}{(z[\tau(n+1)])^\alpha}\right] \\
&\quad - \frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{\frac{1}{\alpha}}[\tau(n)]}\left[w^{\frac{\alpha+1}{\alpha}}(n+1) + p^\alpha[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\right].
\end{aligned} \tag{2.36}$$

Using (2.7) in (2.36), we have

$$\Delta w(n)+p^{\alpha}[\sigma(n)]\Delta v(n)\leq\frac{\Delta\rho(n)}{\rho(n+1)}\Big[w(n+1)+p^{\alpha}[\sigma(n)]v(n+1)\Big]$$
$$-\frac{\rho(n)Q(n)}{2^{\alpha-1}}-\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{\frac{1}{\alpha}}[\tau(n)]}\Big[w^{\frac{\alpha+1}{\alpha}}(n+1)+p^{\alpha}[\sigma(n)]v^{\frac{\alpha+1}{\alpha}}(n+1)\Big]. \tag{2.37}$$

Summing the last inequality from n_2 to $n-1$, we have

$$w(n)-w(n_2)+p^{\alpha}[\sigma(n)]v(n)-p^{\alpha}[\sigma(n_2)]v(n_2)\leq-\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s)$$
$$+\sum_{s=n_2}^{n-1}\left[\frac{\Delta\rho(s)}{\rho(s+1)}w(s+1)-\frac{\alpha\rho(s)}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{\frac{1}{\alpha}}[\tau(s)]}w^{\frac{\alpha+1}{\alpha}}(s+1)\right]$$
$$+\sum_{s=n_2}^{n-1}\Big[\frac{p^{\alpha}(\sigma(s))\Delta\rho(s)}{\rho(s+1)}v(s+1)-\frac{\alpha\rho(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\tau(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\Big]. \tag{2.38}$$

Let

$$A(n)=\left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{\frac{\alpha+1}{\alpha}}w(n+1)$$

and

$$B(n)=\left(\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{-\frac{\alpha}{\alpha+1}}\right)^{\alpha}.$$

Now, using the inequality

$$\frac{\alpha+1}{\alpha}AB^{\frac{1}{\alpha}}-A^{\frac{\alpha+1}{\alpha}}\leq\frac{1}{\alpha}B^{\frac{\alpha+1}{\alpha}} \tag{2.39}$$

by taking $A=A(n)$ and $B=B(n)$ on the second part of the right hand side of the inequality (2.38), we have

$$w(n)-w(n_2)+p^{\alpha}[\sigma(n)]v(n)-p^{\alpha}[\sigma(n_2)]v(n_2)\leq-\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s)$$
$$+\sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\tau(s)]$$
$$+\sum_{s=n_2}^{n-1}\Big[\frac{\Delta\rho(s)}{\rho(s+1)}v(s+1)-\frac{\alpha\rho(s)p^{\alpha}[\sigma(s)]}{\rho^{\frac{\alpha+1}{\alpha}}(s+1)a^{1/\alpha}[\tau(s)]}v^{\frac{\alpha+1}{\alpha}}(s+1)\Big]. \tag{2.40}$$

Now, let

$$C(n)=\left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{\frac{\alpha}{\alpha+1}}v(n+1)$$

and

$$D(n)=\left[\frac{\alpha}{\alpha+1}\frac{\Delta\rho(n)}{\rho(n+1)}\left(\frac{\alpha\rho(n)p^{\alpha}[\sigma(n)]}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)a^{1/\alpha}[\tau(n)]}\right)^{-\frac{\alpha}{\alpha+1}}\right]^{\alpha}.$$

Now, using the inequality (2.39) by taking $A=C(n)$ and $B=D(n)$ in the third part of right hand side of (2.40), we have

$$w(n)-w(n_2)+p^{\alpha}[\sigma(n)]v(n)-p^{\alpha}[\sigma(n_2)]v(n_2)\leq-\frac{1}{2^{\alpha-1}}\sum_{s=n_2}^{n-1}\rho(s)Q(s)$$
$$+\sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}a[\tau(s)]+\sum_{s=n_2}^{n-1}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1}}{(\rho(s))^{\alpha}}\frac{a[\tau(s)]}{(p^{\alpha}[\sigma(s)])^{2}}. \tag{2.41}$$

Now,

$$w(n)-w(n_2)+p^\alpha[\sigma(n)]v(n)-p^\alpha[\sigma(n_2)]v(n_2)\le -\sum_{s=n_2}^{n-1}\rho(s)\Bigg[\frac{1}{2^{\alpha-1}}Q(s)$$
$$-\frac{1}{(\alpha+1)^{\alpha+1}}\left(\frac{\Delta\rho(s)}{\rho(s)}\right)^{\alpha+1}a[\tau(s)]\left[1+\frac{1}{(p^\alpha[\sigma(s)])^2}\right]\Bigg]. \tag{2.42}$$

Letting $n\to\infty$ in the last inequality and using (2.29), we see that

$$w(n)+p^\alpha[\sigma(n)]v(n)\to-\infty,$$

which contradicts the positivity of $w(n)+p^\alpha[\sigma(n)]v(n)$. This completes the proof. □

Theorem 2.4. *Assume that condition* (1.2) *holds, $p(n)\le p_0<\infty$, $\tau(n)\le n$ and $\sigma(n)\ge\tau(n)$ for all $n\ge n_0$. Furthermore, if there exists a positive real sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\rho(s)\left[\frac{Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(\frac{\Delta\rho(s)}{\rho(s)}\right)^{\alpha+1}a[\tau(s)]\left(1+\frac{1}{(p_0^\alpha[\sigma(s)])^2}\right)\right]=\infty, \tag{2.43}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $n_1\ge n_0$ such that $x(n)>0, x[\tau(n)]>0$ and $x[\sigma(n)]>0$ for all $n\ge n_1$. Using equation (1.1) and the definition of $z(n)$, we get (2.28) for all sufficiently large n. The remainder of the proof is similar to that of Theorem 2.3 and hence it is omitted. □

Theorem 2.5. *Assume that condition* (1.3) *holds, $p(n)\le p_0<\infty$, $\Delta\tau(n)>0$, $\Delta\sigma(n)>0$, $\sigma(n)\le n$ and $\sigma(n)\le\tau(n)$ for all $n\ge n_0$. Further assume that there exists a positive real sequence $\{\rho(n)\}$ such that* (2.26) *holds. If there exists a sequence $\{\eta(n)\}$ of positive real numbers with $\eta(n)\ge n,\Delta\eta(n)>0$ for all $n\ge n_0$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[\frac{Q(s)\delta^\alpha(s)}{2^{\alpha-1}}+(1+p_0^\alpha)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(s)a^{\frac{1}{\alpha}}[\eta(s)]}\right]=\infty, \tag{2.44}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a positive solution of equation (1.1). Then there exists $n_1\ge n_0$ such that $x(n)>0, x[\tau(n)]>0$ and $x[\sigma(n)]>0$ for all $n\ge n_1$. Proceeding as in Theorem 2.1, we get

$$\Delta(a(n))(\Delta z(n))^\alpha+p_0^\alpha\Delta(a[\tau(n)])(\Delta z[\tau(n)])^\alpha+\frac{1}{2^{\alpha-1}}Q(n)z^\alpha[\sigma(n)]\le 0 \tag{2.45}$$

for all $n\ge n_1$. Also from equation (1.1), we have $a(n)(\Delta z(n))^\alpha$ is decreasing. Then we have two cases for $\Delta z(n)$, namely, (i) $\Delta z(n)<0$ or (ii) $\Delta z(n)>0$ for all $n\ge n_2\ge n_1$.

Case(i): Suppose that $\Delta z(n)>0$ for all $n\ge n_2\ge n_1\ge n_0$. Then the proof is similar to that of Theorem 2.2.

Case(ii): Suppose that $\Delta z(n)<0$ for all $n\ge n_2\ge n_1\ge n_0$. Now define

$$u(n)=\frac{-a(n)(-\Delta z(n))^\alpha}{z^\alpha[\eta(n)]}\text{ for all } n\ge n_2. \tag{2.46}$$

Then $u(n)<0$ for all $n\ge n_2$. Since $a(n)(\Delta z(n))^\alpha$ is decreasing, we have $a(n)(-\Delta z(n))^\alpha$ is increasing and we get

$$a^{\frac{1}{\alpha}}(s)\Delta z(s)\le a^{\frac{1}{\alpha}}(n)\Delta z(n)\text{ for all } s\ge n\ge n_2. \tag{2.47}$$

Dividing the last inequality by $a^{\frac{1}{\alpha}}(s)$ and then summing from $\eta(n)$ to $n-1$, we have

$$z(n)\le z[\eta(n)]+a^{\frac{1}{\alpha}}(n)\Delta z(n)\sum_{s=\eta(n)}^{n-1}\frac{1}{a^{\frac{1}{\alpha}}(s)}. \tag{2.48}$$

Letting $n \to \infty$ in the last inequality, we see that

$$0 \le z[\eta(n)] + a^{\frac{1}{\alpha}}(n)\Delta z(n)\delta(n), \tag{2.49}$$

that is,

$$-\frac{a^{\frac{1}{\alpha}}(n)\Delta z(n)\delta(n)}{z[\eta(n)]} \le 1. \tag{2.50}$$

Hence by (2.46), we have

$$-u(n)\delta^{\alpha}(n) \le 1. \tag{2.51}$$

Now, define

$$v(n) = \frac{-a[\tau(n)](-\Delta z[\tau(n)])^{\alpha}}{z^{\alpha}[\eta(n)]},\ n \ge n_2. \tag{2.52}$$

Then, we have $v(n) < 0$. By using the monotonicity of $a(n)(-z(n))^{\alpha}$ and using $\tau(n) \le n$, we get

$$a(n)(-\Delta z(n))^{\alpha} \ge a[\tau(n)](-\Delta z[\tau(n)])^{\alpha} \text{ for all } n \ge n_2. \tag{2.53}$$

Thus

$$0 < -v(n) \le -u(n). \tag{2.54}$$

From (2.50) and (2.54), we have

$$-\delta^{\alpha}(n)v(n) \le 1. \tag{2.55}$$

Now, from (2.46), we have

$$\begin{aligned}\Delta u(n) &= \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + \frac{a(n+1)\Delta(-z(n+1))^{\alpha}}{z^{\alpha}[\eta(n)]z^{\alpha}[\eta(n+1)]}\Delta^{\alpha} z[\eta(n)] \\ &\le \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + \frac{(-u(n+1))}{z^{\alpha}[\eta(n)]}\Delta^{\alpha} z[\eta(n)].\end{aligned} \tag{2.56}$$

By Mean value Theorem, we have

$$\Delta z^{\alpha}[\eta(n)] \le \alpha z^{\alpha-1}[\eta(n)]\Delta z[\eta(n)]. \tag{2.57}$$

Using (2.57) in (2.56), we get

$$\Delta u(n) \le \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - u(n+1)\alpha\Delta z[\eta(n)]. \tag{2.58}$$

Using monotonicity of $a(n)(\Delta z(n))^{\alpha}$, we have

$$\Delta z[\eta(n)] \le \left(\frac{a(n)}{a[\eta(n)]}\right)^{1/\alpha} \Delta z(n). \tag{2.59}$$

Using (2.59) in (2.58), we get

$$\begin{aligned}\Delta u(n) &\le \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - (-u(n+1))\frac{a^{1/\alpha}(n)}{a^{1/\alpha}[\eta(n)]}\Delta z(n) \\ &\le \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} - \frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]}.\end{aligned} \tag{2.60}$$

Similarly, we have

$$\Delta v(n) \le \frac{\Delta(-a(\tau(n))(-\Delta z(\tau(n)))^{\alpha})}{z^{\alpha}[\eta(\tau(n))]} - \frac{\alpha[-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}. \tag{2.61}$$

From (2.60) and (2.61)

$$\begin{aligned}\Delta u(n) + p_0^{\alpha}\Delta v(n) \le& \frac{\Delta(-a(n)(-\Delta z(n))^{\alpha})}{z^{\alpha}[\eta(n)]} + p_0^{\alpha}\frac{\Delta(-a(\tau(n))(-\Delta z(\tau(n)))^{\alpha})}{z^{\alpha}[\eta(\tau(n))]} \\ &- \frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]} - \frac{\alpha p_0^{\alpha}[-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}.\end{aligned} \tag{2.62}$$

Using (2.7) in (2.62), we have

$$\Delta u(n) + p_0^\alpha \Delta v(n) \leq \frac{-1}{2^{\alpha-1}} Q(n) - \frac{\alpha[-u(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(n)]} - \frac{\alpha p_0^\alpha [-v(n+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(n))]}. \tag{2.63}$$

Multiplying (2.63) by $\delta^\alpha(n)$ and summing the resulting inequality from n_2 to $n-1$, we get

$$\sum_{s=n_2}^{n-1} \Delta u(s)\delta^\alpha(s) + \sum_{s=n_2}^{n-1} \Delta v(s)\delta^\alpha(s) p_0^\alpha + \sum_{s=n_2}^{n-1} \delta^\alpha(s) \left[\frac{1}{2^{\alpha-1}} Q(s) + \frac{\alpha[-u(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(s)]} - \frac{\alpha p_0^\alpha [-v(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(s))]}\right] \leq 0. \tag{2.64}$$

By using summation by parts formula, we obtain

$$[u(s)\delta^\alpha(s)]_{n_2}^n - \sum_{s=n_2}^{n-1} u(s+1)\Delta\delta^\alpha(s) + [p_0^\alpha v(s)\delta^\alpha(s)]_{n_2}^n - p_0^\alpha \sum_{s=n_2}^{n-1} v(s+1)\Delta\delta^\alpha(s) \sum_{s=n_2}^{n-1} \delta^\alpha(s) \left[\frac{1}{2^{\alpha-1}} Q(s) + \frac{\alpha[-u(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(s)]} - \frac{\alpha p_0^\alpha [-v(s+1)]^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}[\eta(\tau(s))]}\right] \leq 0.$$

Now

$$u(n)\delta^\alpha(n) - u(n_2)\delta^\alpha(n_2) + p_0^\alpha v(n)\delta^\alpha(n) - p_0^\alpha v(n_2)\delta^\alpha(n_2) + \alpha \sum_{s=n_2}^{n-1} \left[\frac{\delta^{\alpha-1}(s)u(s+1)}{a^{\frac{1}{\alpha}}\eta(s)} - \frac{\delta^\alpha(s)u(s+1)^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}\eta(s)}\right] + \alpha p_0^\alpha \sum_{s=n_2}^{n-1} \left[\frac{\delta^{\alpha-1}(s)v(s+1)}{a^{\frac{1}{\alpha}}\eta(s)} - \frac{\delta^\alpha(s)v(s+1)^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}\eta(s)}\right] + \sum_{s=n_2}^{n-1} \frac{\delta^\alpha(s)}{2^{\alpha-1}} Q(s) \leq 0. \tag{2.65}$$

By using the inequality

$$Bu - Au^{\alpha+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{(\alpha+1)}} \frac{B^{\alpha+1}}{A^\alpha} \tag{2.66}$$

in fifth and sixth parts of the left hand side of the last inequality, we have

$$\sum_{s=n_2}^{n-1} \left[\frac{Q(s)\delta^\alpha(s)}{2^{\alpha-1}} + (1+p_0^\alpha)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(\frac{1}{\delta(s)a^{1/\alpha}[\eta(s)]}\right)\right] \leq u(n_2)\delta^\alpha(n_2) + p_0^\alpha v(n_2)\delta^\alpha(n_2) + 1 + p_0^\alpha. \tag{2.67}$$

Letting $n \to \infty$, we get a contradiction with (2.45). This completes the proof. □

3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta(n(\Delta(x(n) + \frac{1}{2}x(n-2)))^3) + n^2 x^3(n-3) = 0,\ n \geq 3. \tag{3.68}$$

Here $a(n) = n$, $p(n) = \frac{1}{2}$, $q(n) = n^2$, $\alpha = 3$, $\tau(n) = n-2$ and $\sigma(n) = n-3$. By taking $\rho(n) = n$, it is easy to see that all conditions of Theorem 2.1 are satisfied and hence all solutions of equation (3.68) *are oscillatory.*

Example 3.2. *Consider the neutral difference equation*

$$\Delta(n^4(\Delta(x(n) + \frac{1}{3}x(n-2)))^3) + n^6 x(n-4) = 0,\ n \geq 4. \tag{3.69}$$

Here $a(n) = n^4$, $p(n) = \frac{1}{3}$, $q(n) = n^6$, $\alpha = 3$, $\tau(n) = n-2$ *and* $\sigma(n) = n-4$. *By taking* $\rho(n) = 1$ *and* $\eta(n) = n$, *it is easy to see that all conditions of Theorem 2.5 are satisfied and hence all solutions of equation* (3.69) *are oscillatory.*

We conclude this paper with the following remark.

Remark 3.1. *The method used in this paper can be applied to the following difference equation*

$$\Delta(a(n)\Delta(x(n) + p(n)x(\tau(n)))) + q(n)|x(\delta(n))|^{\alpha-1}x(\delta(n)) = 0$$

where $\alpha \geq 1$, *to obtain oscillation results. Also it would be interesting to find oscillation criteria for the equation* (1.1) *when* $\tau \circ \sigma \neq \sigma \circ \tau$.

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