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Drazin invertibility of sum and product of closed linear operators

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Abstract

The paper present a survey of results concerning the fundamental properties of the Drazin inverse for bounded operators and an interesting study of the Drazin inverse for a closed operator in a Banach space. Some necessary and sufficient conditions for A closed linear operator to possess a Drazin inverse A^D are given, we obtain also a useful caracterization and explicit formula for the Drazin inverse $(A+B)^D$ and $(AB)^D$ if A and B are closed operators.

Keywords: Drazin inverse, Closed linear operators, Gap metric.

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1 Introduction

In recent years, the representation and characterization of the Drazin inverses of matrices or operators on a Banach space have been considered by many authors (see [2], [6], [7], [14], [15], [31],...). It is shown that the Drazin inverse has proved helpful in analyzing Markov chains, difference equation, differential equations, Cauchy problems and iterative procedures.

For bounded linear operators and elements of a Banach algebra the Drazin inverse was introduced and studied by Ben-Israel in [2], Caradus [5], Koliha in [15] and other authors.

In this paper, we give a survey of results concerning the fundamental properties of the Drazin inverse for bounded operators on a Banach space. We continue our investigation with a representation and characterization of Drazin inverse for a closed linear operator in a Banach space. We give some necessary and sufficient conditions for A closed to possess a Drazin inverse A^D, we obtain also a useful characterization and explicit formula for the Drazin inverse $(A+B)^D$ and $(AB)^D$ if A and B are closed linear operators satisfying certain topological conditions via the gap metric between their respective graphs.

Precisely, if A and B are two closed linear operators Drazin invertible, it is a question when $A+B$ and AB are closed and Drazin invertible. This question was partially studied by Messirdi and Mortad [21], Azzouz, Messirdi and Djellouli [1] and Koliha and Tran [18], it finds its applications in a number of areas such that differential and difference equations, linear and non linear analysis. Koliha and Tran consider in [18], the Drazin inverse of $A+B$ and AB where A is closed and B bounded, what assures the closedness. Their method can not be applied to arbitrary closed operators.

As main result, we give sufficient conditions for a sum and product of two Drazin invertible closed operators to be closed and Drazin invertible in a Hilbert space. Thus, using a different method via the gap metric, we generalize the result from [18].

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2 A concise introduction to Drazin inverse for bounded operators

Let E be a complex Banach space (resp. H be a separable Hilbert space). Let us denote by $B(E)$ the algebra of bounded linear operators on E and $C(E)$ the set of all densely defined closed linear operators on E. If $A \in C(E)$, the domain of A is denoted by $D(A)$ and $G(A) = \{(x, Ax) ; x \in D(A)\}$ is its graph, in particular $G(A)$ is a closed subspace of $E \times E$. The null space and the range of A will be denoted by $N(A)$ and $R(A)$ respectively. A^* is the adjoint of A and I is the operator identity on E. The orthogonal complement of a subset M of H is denoted by $M^{\perp}$.

We write $\sigma(A), \rho(A)$ and $r(A)$ for the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$ we denote the resolvent $(\lambda I - A)^{-1}$ by $R(\lambda, A)$. If 0 is an isolated point of $\sigma(A)$, then the spectral projection of A associated with $\{0\}$ is defined by (see [13]) :

$$P_A = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, A) d\lambda$$

where γ is a small circle surrounding 0 and separating 0 from $\sigma(A) \backslash \{0\}$. An element $A \in B(E)$ whose spectrum $\sigma(A)$ consists of the set $\{0\}$ is said to be quasinilpotent. It is clear that A is quasinilpotent if and only if the spectral radius $r(A) = 0$.

For bounded linear operators and elements of a Banach algebra the Drazin inverse was introduced and studied by Ben-Israel [2], Koliha in [15] and others.

Let $A \in B(E)$ if there exists an operator $X \in B(E)$ satisfied the following three operator equations

$$\begin{cases} AX = XA \\ XAX = X \\ A^{k+1}X = A^k \end{cases}$$

then X is called a Drazin inverse of A and denoted by A^D. The smallest non-negative integer k in the third equation is called the index of A, denoted by $i(A)$.

The above conditions are equivalent to $AA^D = A^D A$; $A^D A A^D = A^D$ and $A(I - AA^D)$ is nilpotent.

If A is Drazin invertible and $i(A) = r \geq 1$, then $R(\lambda, A)$ has a pole of order r at $\lambda = 0$ and it can be expressed in the region $0 < |\lambda| < (r(A^D))^{-1}$, by (see [5]) :

$$R(\lambda, A) = \sum_{n=1}^{r} \frac{A^{n-1} P_A}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n (A^D)^{n+1}$$

An operator $A \in B(E)$ has its Drazin inverse A^D if and only if it has finite ascent and descent, which is equivalent with that 0 is a finite order pole of the resolvent operator $R(\lambda, A)$, say of order p. In such case $i(A) = asc(A) = des(A) = p$. Recall that $asc(A)$ (resp. $des(A)$), the ascent (resp. descent) of $A \in B(E)$, is the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$ (resp. $R(A^n) = R(A^{n+1})$). If no such n exists, then $asc(A) = \infty$ (resp. $des(A) = \infty$). It is well known, $des(A) = asc(A)$ if $asc(A)$ and $des(A)$ are finite [12, 14]. Otherwise, we say $i(A) = +\infty$. When $i(A) = 0$, then the Drazin inverse is reduced into the regular inverse, i.e., $A^D = A^{-1}$. Moreover, we know that for $A \in B(E)$, A^D exists if and only if $0 \notin acc[\sigma(A)]$ ($acc[\sigma(A)]$ is the set of all accumulation points of the spectrum $\sigma(A)$ of A) and in that case A^D is unique [15]. We note that if A is nilpotent, then it is Drazin invertible, $A^D = 0$, and $i(A) = r$, where r is the power of nilpotency of A. Chen F. King showed in [14] that if $A \in B(E)$ has a Drazin inverse with index $i(A) = k$ then A can be written as $A = S + T$ where S has index 0 or 1, T is nilpotent of order k and $ST = TS = 0$.

The problems of the invertibility of sums and products of idempotent operators on a Hilbert space were studied by several researchers, Groß and Trenkler in [12]; Buckholtz in [3], Rakocevic in [25], Vidav in [29] and Wimmer in [30]. Rakocevic and Wei in [26] investigated this question in the case of C^*-algebra. The formulae for the Drazin inverse of sums, differences and products of idempotents are established by Deng and Wei in [10].

Generally, commutation relations between operators in a Banach space E play an important role in the representations of the Drazin inverse. Some properties of the Drazin inverse according to such relations have been extensively studied in the mathematical literature. Djordjevic and Wei in 2002 showed in [11] the following results, see also [18]:

Theorem 2.1. *Let $A, B \in B(E)$ be Drazin invertible.*
If $AB = BA = 0$ then $(A+B)^D = A^D + B^D$.
If $AB = 0$, then $A + B$ is Drazin invertible and

$$(A+B)^D = (I - BB^D)[\sum_{n=0}^{\infty} B^n (A^D)^n]A^D + B^D[\sum_{n=0}^{\infty}(B^D)^n A^n](I - AA^D)$$

This result was later refined by Castro-González, Dopazo and Martínez-Serrano in [6] :

Theorem 2.2. *Let $A, B \in B(E)$ be Drazin invertible. If $B^2A = BA^2 = 0$, and let BA de Drazin invertible, then $A + B$ is Drazin invertible and*

$$\begin{aligned}(A+B)^D &= UP_B + P_A V + X(I+YB)P_B + P_A(I+AX)Y + AUV + UVB \\ &+ \sum_{k=0}^{2r+t-2}(A^D)^{k+1}\Gamma_{k+2}B + \sum_{k=0}^{2r+s-2} A\Lambda_{k+2}(B^D)^{k+1}\end{aligned}$$

where $s = i(A)$, $t = i(B)$ and $r = i(BA)$. Γ_{k+2} is the coefficient at $(\frac{1}{\lambda})^{k+2}$ of $R(\lambda^2, BA)R(\lambda, B)$, Λ_{k+2} is the coefficient at $(\frac{1}{\lambda})^{k+2}$ of $R(\lambda, A)R(\lambda^2, BA)$.

$$\begin{aligned} X &= \sum_{j=1}^{[s/2]} A^{2j-1}P_A((BA)^D)^j \, , \, Y = \sum_{j=1}^{[t/2]}((BA)^D)^j B^{2j-1}P_B \\ U &= \sum_{j=0}^{r-1}(A^D)^{2j+1}(BA)^j P_{BA} \, , \, V = \sum_{j=0}^{r-1} P_{BA}(BA)^j(B^D)^{2j+1}\end{aligned}$$

Moreover, $i(A+B) \leq 2r + s + t - 1$.
If B is nilpotent, then

$$(A+B)^D = U + X(I+YB) + P_A(I+AX)Y + \sum_{k=0}^{2r+t-2}(A^D)^{k+1}(T_k - YB^k)B$$

where $T_k = \sum_{j=0}^{k'} P_{BA}(BA)^j B^{k-1-2j}P_B$, $k \geq 1$ with $T_0 = 0$.
In the case $B^2 = 0$, then

$$(A+B)^D = U + X + P_A(I+AX)(BA)^D B + A^D UB$$

If A and B are nilpotent, then

$$(A+B)^D = X(I+YB) + (I+AX)Y$$

Patricio and Hartwig in 2009 investigated in [24] the existence of the Drazin inverse $(A+B)^D$ under the condition $A^2B + AB^2 = 0$.

Theorem 2.3. *Let $A, B \in B(E)$. Supose that $A^2 + AB$ and $AB + B^2$ are Drazin invertible, and that $A^2B + AB^2 = 0$. Then $A + B$ is Drazin invertible with :*

$$\begin{aligned}(A+B)^D &= (A^2+AB)^D A + B(AB+B^2)^D + BCA \\ C &= -(AB+B^2)^D(A+B)(A^2+AB)^D + \\ & [I - (AB+B^2)(AB+B^2)^D]C_k[(A^2+AB)^D]^{k+1} \\ & +[(AB+B^2)^D]^{k+1}C_k[I - (A^2+AB)(A^2+AB)^D]\end{aligned}$$

$$C_k = \sum_{r=0}^{k-1}(AB+B^2)^{k-r-1}(A+B)(AB+A^2)^r$$

and $\max(i(A^2+AB), i(B^2+AB)) \leq k \leq i(A^2+AB) + i(B^2+AB)$.

Xiaoji Liu, Liang Xu and Yaoming Yu in 2010 gave the explicit representations of $(A \pm B)$ when $n \times n$ matrices A, B satisfied $AB = B^3A$, $BA = A^3B$ (see [31]). Precisely, if a pair of bounded invertible operators A and B have dual power commutativity $AB = B^mA$ and $BA = A^nB$, $m, n \geq 1$, then $A + B$ is always Drazin invertible.

On the other hand, for a given pair of bounded operators (A, B) and an arbitrary operator X, expressions for the inverse and the Drazin inverse of the operator $A - XB$ are established :

Theorem 2.4. *(i) Let A and B be matrices of the same size. If A is invertible, then $A - XB$ is invertible iff the Schur complement $I - BA^{-1}X$ is invertible and*

$$(A - XB)^{-1} = A^{-1} + A^{-1}X(I - BA^{-1}X)^{-1}BA^{-1}$$

(ii) Let $A, B \in B(E)$ and A be Drazin invertible
- If $(I - AA^D)X = 0$, $B(I - AA^D) = 0$ and $X(I - AA^D)B = 0$, then

$$(A - XB)^D = A^D + A^DX(I - BA^DX)^DBA^D$$

- If there exists an idempotent operator P (ie $P^2 = P$) such that $AP = PA$ and $PX = 0$, then

$$\begin{aligned}(A - XB)^D \quad = \quad & R^D + PA^D + R^DXBPA^D \\ & - \sum_{n=0}^{\infty}(R^D)^{n+2}XBPA^n(I - AA^D) \\ & -(I - RR^D)\sum_{n=0}^{\infty}(A - XB)^nXBP(A^D)^{n+2}\end{aligned}$$

where $R = (A - XB)(I - P)$.

Let us recall that the original Sherman-Morrison-Woodbury formula has been used to consider the inverse of matrices. We will consider here directly the more generalized case for bounded linear operators.

Theorem 2.5. *Let $A, B, Y, Z \in B(E)$.*

(i) Suppose that A and B are both invertible.Then, $A + YBZ^*$ is invertible iff $B^{-1} + Z^*A^{-1}Y$ is invertible. In which case

$$(A + YBZ^*)^{-1} = A^{-1} - A^{-1}Y(B^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}$$

(ii) Suppose that A and B are both Drazin invertible. Let $C = A + YBZ^*$ and $T = B^D + Z^*A^DY$. If

$$\begin{aligned} R(A^D) \quad & \subset \quad R(C^D)\,;\, N(A^D) \subset N(C^D) \\ N(B^D) \quad & \subset \quad N(Y)\,;\, N(T^D) \subset N(B) \end{aligned}$$

then,

$$(A + YBZ^*)^D = A^D - A^DY(B^D + Z^*A^DY)^DZ^*A^D$$

The question of the invertibility of $P - Q$ where P and Q are idempotent operators on a Hilbert space H, is of great interest in operator theory as it is connected with the question of when the space H is the direct sum $H = R(P) \oplus R(Q)$ of the ranges, and with the existence of an idempotent operator X satisfying the equations $PX = X$, $XP = P$, $Q(I - X) = I - X$ and $(I - X)Q = Q$.

These problems were studied by several researchers, Groß and Trenkler in [12] considered the case of general matrix projectors; Buckholtz [3], [4], Rakocevic [25], Vidav [29], Wimmer [30] discussed the invertibility in the setting of Hilbert spaces. Koliha, Rakocevic and Straskraba [17], Rakocevic and Wei [26] investigated this question in the setting of C* -algebra. Consequently, if P and Q are idempotents some equivalent conditions for the Drazin invertibility of $P + Q$, $P - Q$, PQ and $PQ \pm QP$ are listed as following.

Theorem 2.6. *Let P and Q be idempotents.*

(i) The following statements are equivalent :
$P - Q$ is Drazin invertible,
$P + Q$ is Drazin invertible,
$I - PQ$ is Drazin invertible.

(ii) PQ is Drazin invertible iff $I - P - Q$ is Drazin invertible.
(iii) $PQ - QP$ is Drazin invertible,
$PQ + QP$ is Drazin invertible,
PQ and $P - Q$ are Drazin invertible.

Let P and Q be idempotents and $P - Q$ is invertible, Koliha and Rakocevic in [16] first use the denotations $F = P(P-Q)^{-1}$ and $G = (P-Q)^{-1}P$ to give the representation of $(P-Q)^{-1}$. To give explicit formulae for $(P-Q)^D$ and $(P+Q)^D$ we define $F = P(P-Q)^D$ and $G = (P-Q)^D P$ and $K = (P-Q)^D(P-Q)$. The following results are showed by Deng and Wei in [10].

Theorem 2.7. *Let $P, Q \in B(E)$ be idempotents, then*
(i) $(P-Q)^D = F + G - K$.
(ii) $(P+Q)^D = (2G-K)(F+G-K)$.
(iii) $(P+Q)^D = (P-Q)^D(P+Q)(P-Q)^D$.
(iv) $(P-Q)^D = (P+Q)^D(P-Q)(P+Q)^D$.
(v) $(P-Q)^D = (I-PQ)^D(P-PQ) + (P+Q-PQ)^D(PQ-Q)$.
(vi) $(PQ-QP)^D = (PQP)^D(P-Q)^D - (P-Q)^D(PQP)^D$.
(vii) $(PQ+QP)^D = (P+Q)^D(P+Q-I)^D$.
(viii) $(I-PQP)^D = I - P + P[(P-Q)^D]^2$.
(ix) if PQ is Drazin invertible,

$$\begin{aligned} (PQP)^D &= [(I-P-Q)^D]^2 P \\ (PQ)^D &= [(PQP)^D]^2 Q = [(I-P-Q)^D]^4 Q \end{aligned}$$

As is well known, that AB is invertible does not imply that BA is invertible for A and $B \in B(E)$ (let S be the unilateral shift operator on H, then $S^*S = I$ is invertible, but SS^* is not invertible). The following result insure the equivalence between Drazin invertibility of AB and BA, it was successively shown by Dajic and Koliha [8], Schmoeger [27], Deng [9] and Lu Jian Ming, Du Hong Ke and Wei Xiao Mei [20].

Theorem 2.8. *Let A,$B \in B(E)$ be Drazin invertible. AB is Drazin invertible if and only if BA is Drazin invertible, $i(AB) \leq i(BA) + 1$ and $(AB)^D = A[(BA)^D]^2 B$.*
If A is idempotent, then $A^D = A$.
If $AB = BA$, then $(AB)^D = B^D A^D = A^D B^D$, $A^D B = BA^D$ and $AB^D = B^D A$.

In the following section we introduce the notion of Drazin inverse into the class of the closed operators on a Banach space E and we investigate some basic properties of A^D.

3 Drazin inverse of closed operators

The conventional Drazin inverse was extended to closed linear operators by Nashed and Zhao in [23]; it exists if and only if 0 is at most a pole of the resolvent $R(\lambda, A)$ of the operator A.

The purpose of this section is to introduce the Drazin inverse A^D of a closed linear operator A on a Banach space E which is defined if 0 is merely an isolated spectral point of A, and to investigate basic properties of A^D. We also study, the Drazin invertibility of sums and products of closed linear operators in the case where the respective domains are not trivial and where the sum and the product remain closed operators. The result of Theorem 4.11 is a generalization to densely defined closed linear operators of results obtained by several authors on bounded operators (see e.g. [7, 15, 18, 24, 27]).

We start with a definition of the Drazin inverse of a closed operator and we recall afterward the results on the stability of the closedness of sum and product of closed operators established by Azzouz, Messirdi and Djellouli in [1], and Messirdi, Mortad, Azzouz and Djellouli in [22].

Definition 3.1. *([18]) Let $A \in C(E)$. A is called Drazin invertible (or generalized Drazin invertible) if it can be expressed in the form $A = A_1 \oplus A_2$ where A_1 is bounded and quasinilpotent and A_2 is closed and invertible on E. Thus, $A_2^{-1} \in B(E)$, the operators $A^D = 0 \oplus A_2^{-1}$ is the Drazin inverse of A.*

The Drazin index $i(A)$ is defined to be $i(A) = 0$ if A is invertible, $i(A) = q$ if A is not invertible and A_1 is nilpotent of index q, and $i(A) = \infty$ otherwise.

Remark 3.1. *Drazin invertible operators include closed invertible and quasinilpotent operators when* $A_1 = 0$ *and* $A_2 = 0$ *respectively and projections.*

We deduct directly from the definition the following properties for a Drazin invertible densely defined closed linear operator.

Lemma 3.1. *Let* $A \in C(E)$ *be Drazin invertible with Drazin inverse* $A^D \in B(E)$*. Then,*
(i) A^D *is unique and* $R(A^D) \subset D(A)$*.*
(ii) $R(I - AA^D) \subset D(A)$*.*
(iii) $A^D AA^D = A^D$*.*
(iv) $AA^D = A^D A$*.*
(v) $(AA^D)^2 = AA^D$*.*
(vi) $P_A = I - AA^D$ *is the spectral projection of* A *corresponding to* 0*.*
(vii) $\sigma(A(I - AA^D)) = \{0\}$*.*

Proof. The properties (i) to (v) follow from Definition 3.1. (v) implies $P_A^2 = P_A$. We can deduce from [28] that 0 is not an accumulation point of the spectrum of A and P_A is the spectral projection of A at 0. □

Remark 3.2. *Koliha and Tran showed in* [18] *that the properties (i) to (iv) and (vi) in the Lemma* 3.1 *are necessary and sufficient conditions for* $A \in C(X)$ *to possess a Drazin inverse.*

Example 3.1. *Let* $L^2([0,1]) = \left\{ f \,;\, f \;:\; [0,1] \longrightarrow \mathbb{C} \text{ such that } \int_0^1 |f(x)|^2\, dx < +\infty \right\}$
be a Hilbert space with the natural inner product. Set $M : [0,1] \longrightarrow \mathbb{C}$ *by*

$$M(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \end{cases}$$

and define the maximal operator of multiplication A *by* M *on* $L^2([0,1])$*, that is,*

$$Af = Mf, \text{for } f \in D(A) = \{f \in L^2([0,1]) : Mf \in L^2([0,1])\}$$

Then A *is a densely defined closed linear operator on* $L^2([0,1])$*. Since* $|M(x)| \geq 1$ *for all* $x \in [0,1]$*,* $R(A) = L^2([0,1])$ *and* A *has a bounded inverse* $A^{-1} : L^2([0,1]) \longrightarrow L^2([0,1])$ *defined by* $A^{-1}g = \frac{g}{M}$*, for all* $g \in L^2([0,1])$*.*
Therefore, A *is a closed operator on* $L^2([0,1])$ *with a bounded inverse* A^{-1}*which is the Drazin inverse of* A*.*

The following results, easily verifiable by a manipulation of direct operator sums, are often useful (see e.g. [18]).

Theorem 3.9. *Let* $A \in C(E)$ *be Drazin invertible. Then,*
(i) P_A *and* A *commute on* $D(A)$*.*
(ii) $A + P_A$ *is invertible and* $A^D = (A + P_A)^{-1}(I - P_A) \in B(E)$*.*
(iii) If $B \in C(E)$ *be such that* $R(B) \subset D(A)$*,* $R(A) \subset D(B)$ *and* $AB = BA$*, then,* $A^D B = BA^D$*.*
(iv) For each $n \geq 1$*,* A^n *is Drazin invertible and* $(A^n)^D = (A^D)^n$*.*
(v) A^* *is Drazin invertible and* $(A^D)^* = (A^*)^D$*.*
(vi) If $B \in B(E)$ *is quasinilpotent such that* $R(B) \subset D(A)$ *and* $AB = BA$*, then* $A + B \in C(E)$ *is Drazin invertible and*

$$(A + B)^D = (A + B + P_A)^{-1}(I - P_A)$$

(vii) If $B \in B(E)$ *is Drazin invertible such that* $R(B) \subset D(A)$*,* $R(A) \subset D(B)$ *and* $AB = BA = 0$*, then* $(A + B)^D = A^D + B^D$*.*
(viii) There exist $B \in C(E)$*,* $D(A) = D(B)$ *and* B *is Drazin invertible with index* $i(B) \leq 1$*, and* $C \in B(E)$ *quasinilpotent with* $R(C) \subset D(A)$ *such that* $A = B + C$ *and* $BC = CB = 0$*.* $B^D = A^D$ *and a such decomposition is unique.*

Example 3.2. *(*[18]*) Let the operator* A_1 *on* l^1 *the space of all complex sequences* $(x_k)_k$ *such that* $\sum_{k=0}^{\infty} |x_k| < +\infty$*, defined by*

$$A_1 x = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$$

Then $A_1 \in B(l^1)$ and is quasinilpotent but not nilpotent. The right shift $Rx = (0, x_1, x_2, ...)$ is an injective bounded operator on l^1 with spectrum equal to the closed unit ball in $\mathbb{C}$. Its inverse A_2 is a closed linear operator with the domain $D(A_2) = \{x \in l^1 ; x_1 = 0\}$ and $\sigma(A_2) = \{\lambda \in \mathbb{C} ; |\lambda| \geq 1\}$.

Define $A = A_1 \oplus A_2$ on $E = l^1 \oplus l^1$. Then, $\sigma(A) = \{0\} \cup \{\lambda \in \mathbb{C} ; |\lambda| \geq 1\}$, 0 is isolated in $\sigma(A)$,A is Drazin invertible with $A^D = 0 \oplus A_2^{-1} = 0 \oplus R$ and $i(A) = \infty$. A^D is not Drazin invertible since $\sigma(A^D) = \{\lambda \in \mathbb{C} ; |\lambda| \leq 1\}$ and 0 is an accumulation point of $\sigma(A)$.

4 Drazin inverse of sum and product of closed operators

Let us remind at first a perturbation result of Castro Gonzalez, Koliha and Yimin Wei for the Drazin inverse of closed linear operators (see [7]).

Theorem 4.10. *Let $A \in C(E)$ be a Drazin invertible operator. Let $B \in C(E)$ with $D(A) = D(B)$ such that $(A - B)A^D \in B(E)$.*

If $\|(A-B)A^D\| < 1$, $P_A B A^D = A^D B P_A = 0$ and $\sigma(BP_A) = \{0\}$, then B is Drazin invertible operator and

$$\begin{aligned} B^D &= A^D(I-(A-B)A^D)^{-1} \\ \frac{\|B^D - A^D\|}{\|A^D\|} &\leq \frac{\|(A-B)A^D\|}{1-\|(A-B)A^D\|} \\ \frac{\|A^D\|}{1+\|(A-B)A^D\|} &\leq \left\|B^D\right\| \leq \frac{\|A^D\|}{1-\|(A-B)A^D\|} \end{aligned}$$

A general result of stability of the Drazin inverse is obtained via the gap metrics in the case of Hilbert spaces.

Let $\mathcal{P}_F$ be the orthogonal projection on a closed linear subspace F of H. If $M,\ N$ are two closed linear subspaces of H let us put :

$$g(M,N) = \|\mathcal{P}_M - \mathcal{P}_N\|_{B(H)}$$

We notice that g is a distance on the set of all closed linear subspaces of H and we can easily verify that g have the following properties (see [21], [19]).

Proposition 4.1. *Let M, N be closed linear subspaces of H, we have*

$g(M,N) < 1 \Rightarrow M \cap N^{\perp} = M^{\perp} \cap N = \{0\}.$

$g(M,N) < 1 \Longleftrightarrow M \oplus N^{\perp} = H.$

Now $C(H)$ equipped with the metrics g called "gap" metric becomes a metric space :

$$A, B \in C(H), \qquad g(A,B) = g(G(A),G(B)) = \left\|\mathcal{P}_{G(A)} - \mathcal{P}_{G(B)}\right\|_{B(H\times H)}$$

where $\mathcal{P}_{G(A)}$ and $\mathcal{P}_{G(B)}$ denote respectively the orthogonal projection in $H \times H$ on the graph $G(A)$ of the operator A and the graph $G(B)$ of the operator B. Nevertheless, $C(H)$ is not complete for the metric g, and the natural laws of addition and multiplication are partially defined without being stable in $C(H)$. In fact, the sum and product of two operators A, B of $C(H)$ can be trivial ($D(A+B) = \{0\}$ and $D(AB) = \{0\}$) or else an operator not closed on H.

Azzouz, Messirdi and Djellouli in [1] have established, by means of the metrics g and by using Proposition 4.1, sufficient conditions under different perturbations to ensure that the closed and selfadjoint character is preserved and that the adjoint of the sum is the sum of adjoints. They showed that $(A+B) \in C(H)$ and $(A+B)^* = \overline{A^*+B^*}$, if $A, B \in C(H)$ such that $D(A) \cap D(B)$, $D(A^*) \cap D(B^*)$ and $D((A^*+B^*)^*)$ are dense in H, $0 \notin \sigma(A+B)$ and $g\left(G(A), G(-B)^{\perp}\right) < 1$.

Messirdi, Mortad, Azzouz and Djellouli in [22] have already found a topological condition such that the product of two operators of $C(H)$ remains in $C(H)$. Indeed, they show that if $A,\ B \in C(H)$ are such that $g(A, B^*) < 1$ then $D(AB)$ and $D(BA)$ are dense in H and $AB,\ BA \in C(H)$.

We show here the main results of this paper. We establish a topological characterization on the Drazin inverse of the sum and product of two operators of $C(H)$.

Theorem 4.11. *Let $A, B \in C(H)$ be Drazin invertible operators such that $R(B) \subset D(A)$, $R(A) \subset D(B)$ and $AB = BA$ on $D(A)$.*

(i) If $AB = BA = 0$ on $D(A)$, $0 \notin \sigma(A+B) \cup \sigma(A^* + B^*)$ and $g\left(G(A), G(-B)^{\perp}\right) < 1$, then $(A+B), (A^*+B^*) \in C(H)$, $(A+B)^* = A^* + B^*$ and

$$\begin{aligned} (A+B)^{-1} &= A^D + B^D \\ (A^*+B^*)^{-1} &= (A^D)^* + (B^D)^* = (A^*)^D + (B^*)^D \end{aligned}$$

(ii) If $P_A = P_B$ on $D(A)$ and $g(A, B^*) < 1$, then $AB \in C(H)$ is Drazin invertible and

$$(AB)^D = A^D B^D$$

Proof. (i) Under the assumptions $0 \notin \sigma(A+B) \cup \sigma(A^* + B^*)$ and $g\left(G(A), G(-B)^{\perp}\right) < 1$, Theorem 2.8 in [1] implies that $(A+B), (A^*+B^*) \in C(H)$ and $(A+B)^* = A^* + B^*$, furthermore $(A+B)^{-1}, (A^*+B^*)^{-1} \in B(H)$.

Remark that A, B, A^D, B^D all commute. Then from Lemma 3.1 (iii) and (iv), we obtain

$$AB^D = ABB^DB^D = 0\,;\, A^DB = A^DA^DAB = 0$$

Hence

$$\begin{aligned} (A^D+B^D)(A+B)(A^D+B^D) &= (A+B)(A^D+B^D)^2 \\ &= A^D + B^D \end{aligned}$$

Furthermore, $(A+B)[I - (A+B)(A^D+B^D)]$ is well defined on H since

$$(A+B)[I - (A+B)(A^D+B^D)] = AP_A + BP_B$$

As A and B are Drazin invertible $\sigma(AP_A) = \sigma(BP_B) = \{0\}$. Thus, $(A+B)[I - (A+B)(A^D+B^D)]$ is quasinilpotent operator on H, which shows that $A+B$ is Drazin invertible in H and $(A+B)^{-1} = (A+B)^D = A^D + B^D$ by uniqueness of the inverse.

By the same process we obtain the Drazin inverse of $A^* + B^*$.

(ii) Under the assumption $g(A, B^*) < 1$, then $AB \in C(H)$ by Theorem 2 in [22]. The operators A, B, A^D, B^D all commute. Then from Lemma 3.1 (iii) and (iv), we have

$$\begin{aligned} A^DB^DABA^DB^D &= A(A^D)^2B(B^D)^2 \\ &= A^DB^D \end{aligned}$$

Furthermore, $AB[I - ABA^DB^D] \in B(H)$ since

$$AB[I - ABA^DB^D] = AP_ABP_B + AP_A(I-P_A)BP_B + BP_B(I-P_B)AP_A$$

and $AP_ABP_B, AP_A(I-P_A)BP_B, BP_B(I-P_B)AP_A \in B(H)$. Consequently, $AB[I - ABA^DB^D]$ is quasinilpotent since the operators AP_A and BP_B are too. ☐

Remark 4.3. *(1) The assuption $P_A = P_B$ become useless if the operators A and B are bounded.*

(2) Drazin invertible operators having the same spectral projection were studied by Castro Gonzalez, Koliha and Yimin Wei in the paper [7].

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