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b-Chromatic number of some wheel related graphs

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Abstract

A proper coloring f is a b-coloring of the vertices of graph G such that in each color class there exists a vertex that has neighbours in every other color classes. The b-chromatic number $\varphi(G)$ of a graph G is the largest integer k for which G admits a b-coloring with k colors. If $\chi(G)$ is the chromatic number of G and b-coloring exists for every integer k satisfying the inequality $\chi(G) \leq k \leq \varphi(G)$ then G is called b-continuous. The b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. We investigate b-chromatic number for the graphs obtained from wheel W_n by means of duplication of vertices. We also discuss b-continuity and b-spectrum for such graphs.

Keywords: b-Coloring, b-Continuity, b-Spectrum.

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1 Introduction

A proper k-coloring of a graph $G = (V(G), E(G))$ is a mapping $f : V(G) \rightarrow \{1, 2, ..., k\}$ such that every two adjacent vertices receive different colors. The chromatic number of a graph G is denoted by $\chi(G)$, is the minimum number for which G has a proper k-coloring. The set of vertices with a specific color is called a color class. A b-coloring of a graph G is a variant of proper k-coloring such that every color class has a vertex which is adjacent to at least one vertex in every other color classes and such a vertex is called a color dominating vertex. If v is a color dominating vertex of color class c then we denote it as $cdv(c) = v$. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. The concept of b-coloring was originated by Irving and Manlove [6] and they also observed that every coloring of a graph G with $\chi(G)$ colors is obviously a b-coloring. In the same paper they have introduced the concepts of b-continuity and b-spectrum. If the b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ then G is called b-continuous and the b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. The concept of b-coloring has been extensively studied by Faik [4], Kratochvil *et al.*[7], Alkhateeb [1], Balakrishnan [2], Chandrakumar and Nicholas [3]. The b- chromatic number of some cycle realated graphs have investigated by Vaidya and Shukla [8] while b-chromatic number of some degree splitting graphs is studied by Vaidya and Rakhimol [9].
Throughout this work wheel W_n we mean $W_n = C_n + K_1$.

Proposition 1.1. *[2] For any graph G, $\varphi(G) \leq \Delta(G) + 1$.*

Definition 1.1. *[5] The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$.*

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Proposition 1.2. *[6] If graph G admits a b-coloring with m-colors, then G must have at least m vertices with degree at least $m-1$.*

Proposition 1.3. $\chi(W_n) = \begin{cases} 3, & n \text{ is even} \\ 4, & n \text{ is odd.} \end{cases}$

2 Some general Results

Definition 2.2. *Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$.*

Theorem 2.1. *Let G_1 be the graph obtained from graph G by duplication of vertices(vertex) then $\chi(G) = \chi(G_1)$.*

Proof. Let $v \in V(G)$ be an arbitrary vertex of G and $v' \in V(G_1)$ be its duplicated vertex. As $N(v) = N(v')$ in G_1 and v and v' are independent vertices we can assign the same color to v and v'. Thus no extra color is required for proper coloring of G_1.

As all the duplicated vertices are independent in G_1 this argument can be extended in the case when arbitrary number of vertices are duplicated . Hence $\chi(G) = \chi(G_1)$. □

Theorem 2.2. *Let G be the graph obtained by duplicating all the rim vertices in W_n then*

$$\varphi(G) = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 5, & n = 5, 6, 8 \\ 6, & n = 7 \\ 6, & n \geq 9. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be the rim vertices and u be the apex vertex of W_n and G be the graph obtained by duplication of all the rim vertices of W_n. Let $v'_1, v'_2, ..., v'_n$ be the duplicated vertices corresponding to $v_1, v_2, ..., v_n$. Then $|V(G)| = 2n + 1$ and $|E(G)| = 5n$. To define proper coloring we consider the following cases.
Case-1: $n = 3$.
In this case we have $V(G) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3, u\}$ and $|V(G)| = 7$. More precisely G has three vertices of degree three, three vertices of degree five and one vertex of degree six. Then by Proposition 1.1, $\varphi(G) \leq 7$ as $\Delta(G) = 6$.
If $\varphi(G) = 7$, then according to Proposition 1.3, the graph G must have seven vertices of degree six which is not possible as there is only one vertex of degree six. Hence $\varphi(G) \neq 7$.
If $\varphi(G) = 6$ then according to Proposition 1.3, the graph G must have six vertices of degree five which is not possible as there are only three vertices of degree five and the remaining vertices are of degree three. Hence $\varphi(G) \neq 6$.
We claim that $\varphi(G) \neq 5$ because to achieve $\varphi(G) = 5$ we need minimum five vertices of degree four, which is not possible by Proposition 1.3 as there are only three vertices of degree five and the remaining one vertex is of degree six while three vertices are of degree three. Hence $\varphi(G) \neq 5$.
If $\varphi(G) = 4$ then according to Proposition 1.3, the graph G must have four vertices of degree three, which is possible for G. For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = f(v'_1) = 1, f(v_2) = f(v'_2) = 2, f(v_3) = f(v'_3) = 3, f(u) = 4$. This proper coloring gives $cdv(1) = v'_1$, $cdv(2) = v'_2$, $cdv(3) = v'_3$, $cdv(4) = u$. Hence $\varphi(G) = 4$.
Case-2: $n = 4$.
For graph G we have $V(G) = \{v_1, v_2, v_3, v_4, v'_1, v'_2 v'_3, v'_4, u\}$ and $|V(G)| = 9$. More precisely graph G has four vertices of degree three, four vertices of degree five and one vertex of degree eight. Then by Proposition 1.3 we have $\varphi(G) \leq 9$ as $\Delta(G) = 8$. If $\varphi(G) = 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then according to Proposition 1.3, the graph G must have six vertices of degree five, which is not possible as there are only four vertices of degree five, four vertices of degree three and one vertex of degree

eight. Hence $\varphi(G) \neq 6$
If $\varphi(G) = 5$ then according to Proposition 1.3, the graph G must have five vertices of degree four that is not possible as there is no vertex of degree four. Hence $\varphi(G) \neq 5$
If $\varphi(G) = 4$ then by Proposition 1.3, the graph G must have four vertices of degree three which is possible. But due to nature of the graph G any proper coloring with four colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Hence $\varphi(G) \neq 4$. Thus we can color the graph by three colors.
For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3\}$ as $f(v_1) = f(v'_1) = 1$, $f(v_2) = f(v'_2) = 2$, $f(v_3) = f(v'_3) = 1, f(v_4) = f(v'_4) = 2, f(u) = 3$. This proper coloring gives $cdv(1) = v'_1, cdv(2) = v'_2, cdv(3) = u$. Hence $\varphi(G) = 3$.
Case-3: When $n = 5, 6, 8$.
Subcase-1: For $n = 5$.
In this case we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5, u\}$ and $|V(G)| = 11$. More precisely G has five vertices of degree three, five vertices of degree five and one vertex of degree ten. Then by Proposition 1.1, $\varphi(G) \leq 11$ as $\Delta(G) = 10$.
If $\varphi(G) = 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then the graph G must have six vertices of degree at least five which is not possible as there are only five vertices of degree five and the remaining vertices are of degree three while one vertex is of degree ten. Hence $\varphi(G) \neq 6$
If $\varphi(G) = 5$ then the graph G must have five vertices of degree at least four which is possible for the graph G. Thus we can color the graph by five colors.
Now for b-coloring consider the set of colors $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 2, f(v'_1) = 3, f(v'_2) = 3, f(v'_3) = 4, f(v'_4) = 4, f(v'_5) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_1, cdv(5) = u$. Hence $\varphi(G) = 5$.
Subcase-2: For $n = 6$.
For graph G we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, u\}$ and $|V(G)| = 13$. More precisely G has six vertices of degree three, six vertices of degree five and the remaining one vertex is of degree twelve. Then by Proposition 1.1, $\varphi(G) \leq 13$ as $\Delta(G) = 12$.
If $\varphi(G) = 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then the graph G must have six vertices of degree at least five which is possible. But due to the nature of the graph G any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Thus $\varphi(G) \neq 6$.
For b-coloring with five colors consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = f(v'_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = u$. Hence $\varphi(G) = 5$.
Subcase-3: For $n = 8$.
In this case we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, u\}$ and $|V(G)| = 17$. More precisely G has eight vertices of degree three, eight vertices of degree five and one vertex of degree sixteen. Then by Proposition 1.1, $\varphi(G) \leq 17$ as $\Delta(G) = 16$.
If $\varphi(G) = 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$, then graph G must have six vertices of degree at least five which is possible. But due to nature of the graph G any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph G. Thus $\varphi(G) \neq 6$.
For b-coloring with five colors consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = f(v'_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 3, f(v_8) = f(v'_8) = 1, f(u) = 5$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = u$. Hence $\varphi(G) = 5$.
Case-4: $n = 7$.

For graph G we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, u\}$ and $|V(G)| = 15$. More precisely G has seven vertices of degree three, seven vertices of degree five and one vertex is of degree fourteen. Then by Proposition 1.1, $\varphi(G) \leq 15$ as $\Delta(G) = 14$.
If $\varphi(G) = 15, 14, 13, 12, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
If $\varphi(G) = 6$ then according to Proposition 1.3, we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5, 6\}$ as $f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 4, f(v_7) = 2, f(v'_1) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 5, f(v'_5) = 5, f(v'_6) = 4, f(v'_7) = 3, f(u) = 6$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = u$. Which conforms that $\varphi(G) = 6$.
Case-5: $n \geq 9$.
For $n = 9$, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and $|V(G)| = 19$. More precisely G has nine vertices of degree five, nine vertices of degree three and one vertex of degree eighteen. Then by Proposition 1.1, $\varphi(G) \leq 19$ as $\Delta(G) = 18$.
If $\varphi(G) = 19, 18, 17, 16, 15, 14, 13, 11, 10, 9, 8, 7$ then the respective graphs do not have the required number of m-degree vertices so it is not possible to obtain b-coloring with said number of colors.
According to Proposition 1.3 if $\varphi(G) = 6$ then we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5, 6\}$ as $f(v_1) = 4, f(v_2) = 2, f(v_3) = 5, f(v_4) = 1, f(v_5) = 2, f(v_6) = 3, f(v_7) = 1, f(v_8) = 4, f(v_9) = 2, f(v'_1) = 4, f(v'_2) = 3, f(v'_3) = 3, f(v'_4) = 4, f(v'_5) = 4, f(v'_6) = 5, f(v'_7) = 5, f(v'_8) = 4, f(v'_9) = 3, f(u) = 6$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = u$. Hence $\varphi(G) = 6$.
When $n > 9$ we repeat the colors as in the above graph G for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and for the remaining vertices assign the colors as follows $f(v_i) = 1,\ f(v'_i) = 2;$ *when i is even* and $f(v_i) = 2,\ f(v'_i) = 1;$ *when i is odd* . Hence $\varphi(G) = 6,\ n \geq 9$. □

Theorem 2.3. *G is b-continuous.*

Proof. To prove this result we continue with the terminology and notations used in Theorem 2.3 and consider the following cases.
Case-1: $n = 3$.
In this case the graph G is b-continuous as $\chi(G) = \varphi(G) = 4$.
Case-2: $n = 4$.
In this case the graph G is b-continuous as $\chi(G) = \varphi(G) = 3$.
Case-3: $n = 5$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_5) = 4$. Also by Theorem 2.3, $\varphi(G) = 5$. Thus b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$(Here $k = 4, 5$). Hence G is b-continuous.
Case-4: $n = 6$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_6) = 3$. Also by Theorem 2.3, $\varphi(G) = 5$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 3, 5$.
Now for $k = 4$ the b-coloring for the graph G is as follows. Consider the color class $c = 1, 2, 3, 4$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(u) = 4$.
This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Thus G is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 3, 4, 5$).Hence G is b-continuous.
Case-5: $n = 7$.
In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_7) = 4$. Also by Theorem 2.3,$\varphi(G) = 6$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 4, f(v_6) = 2, f(v_7) = 4, f(v'_1) = 1, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 4, f(v'_6) = 3, f(v'_7) = 3, f(u) = 5$. This proper coloring gives dominating vertices $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_5, cdv(5) = u$. So the graph G is five colorable. Hence b-coloring exists for every integer k satis-

fying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 4, 5, 6$). Thus G is b-continuous.

Case-6: $n = 8$.

In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_8) = 3$. Also by Theorem 2.3, $\varphi(G) = 5$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(v_7) = 1 = f(v'_7), f(v_8) = 3 = f(v'_8), f(u) = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Thus G is four colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 3, 4, 5$). Consequently G is b-continuous.

Case-7: $n = 9$.

In this case by Theorem 2.2 and Proposition 1.4 we have $\chi(G) = \chi(W_9) = 4$. Also by Theorem 2.3, $\varphi(G) = 6$. It is obvious that b-coloring for the graph G is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph G is as follows. Consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v_1) = 2, f(v_2) = 4, f(v_3) = 1, f(v_4) = 2, f(v_5) = 3, f(v_6) = 1, f(v_7) = 2, f(v_8) = 1, f(v_9) = 3, f(v'_1) = 2, f(v'_2) = 3, f(v'_3) = 3, f(v'_4) = 4, f(v'_5) = 4, f(v'_6) = 1, f(v'_7) = 2, f(v'_8) = 1, f(v'_9) = 3, f(u) = 5$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_9, cdv(5) = u$. Thus G is five colorable. Hence b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ (Here $k = 4, 5, 6$).

Case-8: $n > 9$.

When $n > 9$ we repeat the color assignment as in the case $n = 9$ discussed above for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, u\}$ and for the remaining vertices give the colors as follows.

When $k = 5$

$$f(v_i) = f(v'_i) = \begin{cases} 1, & i \text{ even} \\ 3, & i \text{ odd} \end{cases}$$

Hence G is b-continuous. □

As any coloring with $\chi(G)$ colors is a b-coloring, we state the following obvious result.

Corollary 2.1.

$$S_b(G) = \begin{cases} \{4\} & n = 3 \\ \{3\} & n = 4 \\ \{4, 5\} & n = 5 \\ \{3, 4, 5\} & n = 6, 8 \\ \{4, 5, 6\} & n = 7 \\ \{4, 5, 6\} & n \geq 9 \end{cases}$$

Theorem 2.4. *Let G_1 be the graph obtained by duplicating the apex vertex in W_n then*

$$\varphi(G_1) = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 4, & n \geq 5 \end{cases}$$

Proof. For W_n, $v_1, v_2, ..., v_n$ be the vertices and u be the apex vertex of W_n. Let G_1 be the graph obtained by duplication of the vertex u of W_n. Let u' be the duplicated vertices corresponding to u. Then $|V(G_1)| = n + 2$ and $|E(G_1)| = 3n$. To define the proper coloring we consider the following two cases.

Case-1: $n = 3$.

In this case $V(G_1) = \{v_1, v_2, v_3, u, u'\}$ and $|V(G_1)| = 5$. More precisely G_1 has two vertices of degree three, three vertices of degree four. Then by Proposition 1.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 4$ then according to Proposition 1.3, the graph G_1 must have four vertices of degree at least three which is possible.

For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u) = 4 = f(u')$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = u$. Hence $\varphi(G_1) = 4$.

Case-2: $n = 4$.

In this case $V(G_1) = \{v_1, v_2, v_3, v_4, u, u'\}$ and $|V(G_1)| = 6$. More precisely G_1 has six vertices of degree four. Then by Proposition 1.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 5$ then according to Proposition 1.3 the graph G_1 must have five vertices of degree at least four which is possible. But due to the nature of graph G_1 any proper coloring with five colors have at least one color class which does not have color dominating vertices. Hence the graph G_1 is not b-colorable using five colors. Hence $\varphi(G_1) \neq 5$.
If possible let $\varphi(G_1) = 4$ and $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u) = 4$, which in turn forces us to assign $f(v_4) = 2, f(u') = 4$. This proper coloring gives the color dominating vertices for color classes 2 and 4 but not for 1 and 3 which is contradiction. Thus $\varphi(G_1) \neq 4$. Hence we can color the graph by three colors.
For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 1, f(v_4) = 2, f(u) = 3, f(u') = 3$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = u$. Hence $\varphi(G_1) = 3$.
Case-3: $n = 5$.
For graph G_1 we have $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, u, u'\}$ and $|V(G_1)| = 7$. More precisely G_1 has five vertices of degree four and two vertices of degree two. Then by Proposition 1.1, $\varphi(G_1) \leq 6$ as $\Delta(G_1) = 5$. According to Proposition 1.3, if $\varphi(G_1) = 6$ then we need six vertices of degree at least five, which is not possible as there are only two vertices of degree five and the remaining vertices are of degree four. Hence $\varphi(G_1) \neq 6$.
If $\varphi(G_1) = 5$ then according to Proposition 1.3 the graph G_1 must have five vertices of degree at least four which is possible. But due to the nature of graph G_1 any proper coloring with five colors have at least one color class which does not have any color dominating vertex. Hence G_1 is not b-colorable with five colors. Hence $\varphi(G_1) \neq 5$. Thus we can color the graph by four colors.
For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(u) = 4, f(u') = 4$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = u$. Hence $\varphi(G_1) = 4$.
Case-3: $n > 5$.
When $n > 5$ we repeat the color assignment as in the case when $n = 5$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, u, u'\}$ and for the remaining vertices assign the colors as follows.

$$f(v_i) = \begin{cases} 2, & i \text{ is even} \\ 1, & i \text{ is odd} \end{cases}$$

Hence $\varphi(G_1) = 4\,;\ n \geq 5$. □

3 Concluding Remarks

We explore the concept of b-coloring in the context of duplication of vertex in graph and prove that the chromatic number of graph G is same as the chromatic number of the graph obtained by duplication of vertices in G. We investigate the b-chromatic number for the larger graphs obtained from wheel W_n by means of duplication of a vertex. The graph obtained by duplication of the apex of W_n is obviously b-continuous while we have shown that the graph obtained by duplication of rim vertices altogether is a b-continuous. We also determine the b-spectrum for the same.

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