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On the oscillation of third order quasilinear delay differential equations with Maxima

R. Arul[a,*] and M. Mani[b]

[a,b] *Department of Mathematics, Kandaswami Kandar's College, Velur–638 182, Tamil Nadu, India.*

Abstract

In this paper, we study the oscillation and asymptotic properties of third order quasilinear neutral delay differential equation

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t)\max_{[\sigma(t),t]} x^{\alpha}(s) = 0, \quad t \geq t_0 \geq 0 \tag{0.1}$$

where α is a ratio of odd positive integers and $\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)}dt = \infty$. We establish a new condition which guarantees that every solution of (0.1) is either oscillatory or converges to zero. There results extend some known results in the literature without "maxima". Examples are given to illustrate the main results.

Keywords: Oscillation, quasilinear, neutral, delay, third order, differential equations with maxima.

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1 Introduction

We are concerned with the oscillation problem of third order quasilinear neutral delay differential equation with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t)\max_{[\sigma(t),t]} x^{\alpha}(s) = 0, \quad t \geq t_0 \geq 0 \tag{1.1}$$

where $\alpha > 0$ is the quotient of odd positive integers. Throughout this paper, we will assume that the following conditions hold:

(C_1) $\tau(t) \leq t$ and $\sigma(t) < t$ are continuous functions in $[t_0, \infty)$l;

(C_2) $p(t) \in C^3([t_0,\infty),\ R)$ with $0 \leq p(t) \leq p < 1$, and $q(t) \in C([t_0,\infty),\ R_+)$ with $q(t)$ is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \geq t_0$;

(C_3) $a(t) \in C^1([t_0,\infty),\ a(t) > 0$ and nondecreasing for all $t \geq t_0$ and $\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(1)}dt = \infty$.

By a solution of equation (1.1) we mean a continuous function $x(t) \in C^2([T_x,\infty)),\ T_x \geq t_)$, which has the property $((x(t)+p(t)x(\tau(t)))'')^{\alpha}$ are continuously differentiable and $x(t)$ satisfies the equation (1.1) on $[T_x, \infty)$. We consider only those solution $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $t \geq T_x$. We assume that the equation (1.1) is called oscillatory if it has arbitrary large zeros on $[T_x, \infty)$, otherwise it is called nonoscillatory. A solution $x(t)$ of equation (1.1) is said to be almost oscillatory if $x(t)$ is either oscillatory or $|x(t)| \to 0$ monotonically as $t \to \infty$.

*Corresponding author.
E-mail address: rarulkkc@gmail.com (R. Arul), mani.varmans03@gmail.com (M. Mani).

In the last few years, the qualitative theory of differential equations with "maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real system, see for example [1, 10, 12]. The oscillatory behavior of solutions of differential equations with "maxima" are discussed in [1-6, 11, 13, 14], and the references cited therein.

The great attention has been devoted to the oscillation of third order differential equation without "maxima" see for example [15-24, 26, 27] and the references cited therein. Compared to second order differential equations with "maxima" less attention has received the third order differential equation with "maxiam". Motivated by these observations, in this paper, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). The result extend that of in [25] for equation (1.1) with $p(t) \equiv 0$ and without "maxima".

In Section 2, we obtain criteria for the oscillation of all solution of equation (1.1) and is Section 3 we present some examples to illustrate the main results.

Remark 1.1. *All functional inequalities consider in this paper assumed to hold eventually, that is they are satisfied for all t large enough.*

Remark 1.2. *Without loss of generality we can deal only with the positive solution of equation* (1.1).

2 Oscillation Results

In this section, we obtain a oscillatory criterion for equation (1.1). For a solution $x(t)$ of (1.1) we define the corresponding function $z(t)$ by

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.2}$$

To obtain sufficient condition for the oscillation of solutions of equation (1.1), we need the following lemmas.

Lemma 2.1. *Let $x(t)$ be a positive solution of equation* (1.1)*, then there are only the following two cases for $z(t)$ defined in* (2.2) *hold:*

(I) $z(t) > 0,\ z'(t) > 0$ *and* $z''(t) > 0$;

(II) $z(t) > 0,\ z'(t) < 0$ *and* $z''(t) > 0$ *for* $t \ge t_1 \ge t_0$;

where t_1 is sufficiently large.

Proof. Assume that $x(t)$ is a positive solution of (1.1) on $[t_0, \infty)$. We see that $z(t) > x(t) > 0$ and

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' = -q(t) \max_{[\sigma(t),t]} x^{\alpha}(s) < 0. \tag{2.3}$$

Thus, $a(t)(z''(t))^{\alpha}$ is nonincreasing and of one sign. Therefore $z''(t)$ is also of one sign and so we have two possibilities

$$z''(t) < 0 \ \text{ or } \ z''(t) > 0 \ \text{ for } \ t \ge t_1.$$

If we admit that $z''(t) < 0$, then there exists a constant $M > 0$ such that

$$aa(t)(z''(t))^{\alpha} \le -M < 0.$$

Integrating the last inequality from t_1 to t we obtain

$$z'(t) \le z'(t_1) - M^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s) ds.$$

Letting $t \to \infty$ and using (C_2) we get $z'(t) \to \infty$. Thus $z'(t) < 0$ eventually. But $z''(t) < 0$ and $z'(t) < 0$ eventually imply $z(t) < 0$ for $t \ge t_1$ a contradiction. This contradiction proves that $z''(t) > 0$ and we have only tow cases (I) and (II) for $z(t)$. The proof is now complete. □

Lemma 2.2. *Assume that $u(t) > 0,\ u'(t) \ge 0,\ u''(t) \le 0$, on $[t_0, \infty)$. Then for each $\ell \in (0, 1)$ there exists a $T_\ell \ge t_0$ such that*

$$\frac{u(\tau(t))}{u(t)} \ge \ell \frac{u(t)}{t} \ \textit{ for } \ t \ge T_\ell.$$

Lemma 2.3. *Assume that* $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \le 0$, *on* $[T_\ell, \infty)$. *Then*

$$\frac{z(t)}{z'(t)} \ge \frac{t - T_\ell}{2} \ \text{for} \ t \ge T_\ell.$$

The proofs of Lemma 2.2 and Lemma 2.3 are found in [25].

Lemma 2.4. *The function* $x(t)$ *is a negative solutions of equation* (1.1) *if and only if* $-x(t)$ *is a positive solution of the equation*

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \min_{[\sigma(t),t]} x^{\beta}(s) = 0. \tag{2.4}$$

Proof. The assertion of Lemma 2.4 can be verified easily. □

Lemma 2.5. *Let* $x(t)$ *be a positive solution of equation* (1.1) *and let the corresponding* $z(t)$ *satisfy Lemma 2.1 (II). If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{a(u)} \int_{u}^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} dudv = \infty \tag{2.5}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. The proof is similar to that of in [25] and hence the details are omitted. □

Lemma 2.6. *Assume that* $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \le 0$ *on* $[T_\ell, \infty)$. *Then*

$$(t - T_\ell)\frac{z''(t)}{z'(t)} \le 1 \ \text{for} \ t \ge T_\ell.$$

Proof. The proof is similar to that of in [25] and hence the details are omitted. □

Now, we present the main results. For simplicity we introduce the following notations:

$$p_* = \lim_{t\to\infty} \frac{t^{\alpha}}{a(t)} \int_{t}^{\infty} P_\ell(s)ds,$$
$$q_* = \lim_{t\to\infty} \sup \frac{1}{t} \int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds$$

where

$$P_\ell(s) = \ell^{\alpha} \max_{[\sigma(t),t]} (1 - p(s))^{\alpha} q(s) \left(\frac{\tau(s)}{s}\right)^{\alpha} \left(\frac{\tau(s) - T_\ell}{2}\right)^{\alpha} \tag{2.6}$$

with $\ell \in (0, 1)$ arbitrarily chosen and T_ℓ large enough. Moreover for $z(t)$ satisfying case (I), we define

$$w(t) = a(t)\left(\frac{z''(t)}{z(t)}\right)^{\alpha} \tag{2.7}$$

$$r = \lim_{t\to\infty} \inf \frac{t^{\alpha}}{a(t)},$$

and

$$r = \lim_{t\to\infty} \sup \frac{t^{\alpha}}{a(t)}. \tag{2.8}$$

Theorem 2.1. *Assume that condition* (2.5) *holds and* $a'(t) \ge 0$ *for all* $t \ge t_0$. *If*

$$p_* = \lim_{t\to\infty} \inf \frac{t^{\alpha}}{a(t)} \int_{t}^{\infty} P_\ell(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{2.9}$$

Then the solution $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a positive solution of equation (1.1) and the corresponding function $z(t)$ satisfies case(I) of Lemma 2.1. First note that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1 - p(t))z(t) \tag{2.10}$$

or

$$\max_{[\tau(t),t]} x^{\alpha}(s) \geq z^{\alpha} \max_{[\sigma(t),t]} (1 - p(s))^{\alpha}.$$

Using the above inequality in (1.1) we obatin

$$(a(t)(z''(t))^{\alpha})' \leq 0 \tag{2.11}$$

The last inequality together with $a'(t) \geq 0$ gives that $z(t)$ satisfies $z(\tau(t)) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ for $t \in [T, \infty]$. From the definition of $w(t)$ we see that $w(t) > 0$ and from (1.1) we have

$$\begin{aligned} w'(t) &= \frac{(z'(t))^{\alpha}(a(t)(z''(t))^{\alpha})' - (a(t)(z''(t))^{\alpha})\alpha(z'(t))^{\alpha-1}z''(t)}{(z'(t))^{2\alpha}} \\ &= \frac{-q(t)z^{\alpha}(t)\max_{[\sigma(t),t]}(1-p(s))^{\alpha}}{(z'(t))^{\alpha}} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \end{aligned} \tag{2.12}$$

From Lemma 2.2 with $u(t) = z'(t)$, we have for ℓ the same $P_\ell(t)$,

$$\frac{1}{z'(t)} \geq \ell \frac{\tau(t)}{t} \frac{1}{z'(tau(t))}, \quad t \geq T_\ell$$

which with (2.12) gives

$$w'(t) \leq -q(t)\ell^{\alpha} \left(\frac{\tau(t)}{t}\right)^{\alpha} \frac{z^{\alpha}(t)}{(z'(\tau(t)))^{\alpha}} \max_{[\sigma(s),s]} (1 - p(s))^{\alpha} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t).$$

Using the fact from Lemma 2.3 that $z(t) \geq \frac{(t-T_\ell)}{2} z'(t)$, we have

$$w'(t) + P_\ell(t) + \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \leq 0. \tag{2.13}$$

Since $P_\ell(t) > 0$ and $w(t) > 0$ for $t \geq T_\ell$, we have from (2.13) that $w'(t) \leq 0$ and

$$-\left(\frac{w'(t)}{\alpha w^{\frac{\alpha+1}{\alpha}}(t)}\right) > \frac{1}{a^{1/\alpha}(t)}, \quad \text{for } t \geq T_\ell. \tag{2.14}$$

This implies that

$$\left(\frac{1}{w^{1/\alpha}(t)}\right)' > \frac{1}{a^{1/\alpha}(t)} \tag{2.15}$$

Integrating the last inequality from T_ℓ to t, we obtain

$$w(t) = \frac{1}{\left(\int_{T_\ell}^{t} \frac{ds}{a^{1/\alpha}(t)}\right)^{\alpha}} \tag{2.16}$$

which inview of (C_3) implies that $\lim_{t\to\infty} w(t) = 0$. On the otherhand, from the definition of $w(t)$, and Lemma 2.3, we see that

$$0 \leq r \leq R \leq 1. \tag{2.17}$$

Now, let $\varepsilon > 0$, then from the definitions of p_* and r we can pick $t_2 \in [T_\ell, \infty)$ sufficiently large that

$$\frac{t^{\alpha}}{a(t)} \int_t^{\infty} P_\ell(s)ds \geq p_* - \varepsilon,$$

and

$$\frac{t^{\alpha}w(t)}{a(t)} \geq t - \varepsilon, \quad \text{for } t \in [t_0, \infty).$$

Integrating (2.13) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^\infty P_\ell(s)ds + \alpha \int_t^\infty \frac{w^{1+1/\alpha}(s)}{a^{1/\alpha}(s)}ds, \text{ for } t \in [t_2, \infty). \tag{2.18}$$

Assume $p_* = \infty$, then from (2.18), we have

$$\frac{t^\alpha w(t)}{a(t)} \geq \frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds.$$

Taking the limit infimum on both sides as $t \to \infty$, we get in view of (2.17) that $1 \geq r \geq \infty$. This is a contradiction. Next assume that $p_* < \infty$. Now from (2.18) and the fact $a'(t) \geq 0$, we have

$$\begin{aligned} \frac{t^\alpha}{a(t)} w(t) &\geq \frac{t^\alpha}{a(t)} \int_t^\alpha P_\ell(s)ds + \alpha \frac{t^\alpha}{a(t)} \int_t^\infty \frac{s^{\alpha+1} a(s) w^{\frac{1}{\alpha}+1}(s)}{s^{\alpha+1} a^{\frac{1}{\alpha}+1}(s)} ds \\ &\geq (p_* - \varepsilon) + \frac{t^\alpha (r-\varepsilon)^{1+\frac{1}{\alpha}}}{a(t)} \int_t^\infty \frac{\alpha a(s)}{s^{\alpha+1}} ds \\ &\geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}} t^\alpha \int_t^\infty \frac{\alpha}{s^{\alpha+1}} ds \end{aligned} \tag{2.19}$$

or

$$\frac{t^\alpha w(t)}{a(t)} \geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Taking the limit infimum on both sides as $t \to \infty$, we get

$$r \geq p_* - \varepsilon + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired result

$$p_* \leq r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$. With $A = B = 1$, we get $p_* \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$, which contradicts (2.9). This completes the proof. □

Corollary 2.1. *Assume that* (2.5) *holds and* $a'(t) \geq 0$. *Let* $x(t)$ *be a solution of equation* (1.1). *If*

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} P_\ell(s)ds > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}} \tag{2.20}$$

then $x(t)$ *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. We shall now show that (2.20) implies (2.19). First note that for any $\ell \in (0,1)$ there exists a t_1 such that $\tau(t) - T_\ell \geq \ell\tau(t),\ t \geq t_1$. Therefore

$$P_\ell \geq \frac{\ell^{2\alpha} \max_{[\sigma(t),t]} (1-p(t))^\alpha}{2^\alpha} \frac{q(t)\tau^{2\alpha}(t)}{t^\alpha}, \ t \geq t_1. \tag{2.21}$$

On the otherhand, (2.20) implies that for some $\ell \in (0,1)$

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} > \frac{1}{\ell^{2\alpha}} \frac{(2\alpha)^\alpha}{(\alpha+1)^(\alpha+1)} \tag{2.22}$$

Combining (2.21) with (2.22) we get (2.9). □

Theorem 2.2. *Assume that the condition* (2.5) *holds and* $a'(t) \geq 0$ *for all* $t \geq t_0$. *Let* $x(t)$ *be a solution of equation* (1.1). *If* $p_* + q_* > 1$, *then* $x(t)$ *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ be a positive solution of equation (1.1) and the corresponding function $z(t)$ satisfies case(I) of Lemma 2.1. Now multiply (2.13) by $\frac{t^{\alpha+1}}{a(t)}$, and integrating from t_2 to t $(t \geq t_2)$, we get

$$\int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} w'(s)ds \leq \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds - \alpha \int_{t_2}^{t} \left(\frac{s^\alpha w(t)}{a(s)} \right)^{\frac{s+1}{s}} ds \tag{2.23}$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\alpha+1}}{a(t)} w(t) \quad \leq \quad & \frac{t_2^{\alpha+1} w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds \\ & - \alpha \int_{t_2}^{t} \left(\frac{s^\alpha w(t)}{a(s)} \right)^{\frac{s+1}{s}} ds + \int_{t_2}^{t} \left(\frac{s^{\alpha+1}}{a(s)} \right)' w(s)ds. \end{aligned}$$

Since $a'(t) \geq 0$, we have

$$\left(\frac{s^{\alpha+1}}{a(s)} \right)' = \frac{a(s)(\alpha+1)s^\alpha - a'(s)s^\alpha}{(a(s))^2} \leq \frac{(\alpha+1)s^\alpha}{a(s)}.$$

Hence,

$$\begin{aligned} \frac{t^{\alpha+1}}{a(t)} w(t) \quad \leq \quad & \frac{t_2^{\alpha+1} w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds \\ & + \int_{t_2}^{t} \left[\frac{(\alpha+1)s^\alpha w(s)}{a(s)} - \alpha \left(\frac{s^\alpha w(s)}{a(s)} \right)^{\alpha+1} \right] ds. \end{aligned}$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$, with $u(s) = \frac{s^\alpha w(s)}{a(s)} > 0$, and positive constants. $A = \alpha$, $B = \alpha + 1$, we get

$$\frac{t^{\alpha+1}}{a(t)} w(t) \leq \frac{t_2^{\alpha+1}}{a(t_2)} w(t_2) - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds + \frac{t - t_2}{t}. \tag{2.24}$$

Taking limit supreme on both sides as $t \to \infty$ we obtain $R \leq q_* + 1$. Combining this with the inequality (2.20) we get

$$p_* + q_* \leq 1. \tag{2.25}$$

This is a contradiction. If $z(t)$ satisfies condition (2.5) then by Lemma 2.1 of case(II) with $\lim_{t\to\infty} z(t) = 0$. This completes the proof. □

Corollary 2.2. *Assume that* (2.5) *holds and* $a'(t) \geq 0$. *Let* $x(t)$ *be a solution of equation* (1.1). *If*

$$q_* = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds > 1 \tag{2.26}$$

then $x(t)$ *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

As a matter of fact we can slightly simplify the function $P_\ell(t)$ in (2.26).

Corollary 2.3. *Assume that* (2.5) *holds and* $a'(t) \geq 0$. *Let* $x(t)$ *be a solution of equation* (1.1). *If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s\tau^{2\alpha}(s)q(s)\max_{[\sigma(t),t]}(1-p(s))^\alpha}{a(s)} ds > 2^\alpha$$

then $x(t)$ *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

3 Examples

In this section we present some examples to illustrate the main results.

Example 3.1. *Consider the differential equation*

$$\left(t^3\left(\left(x(t)+\frac{1}{3}x(t/2)\right)''\right)^3\right)' + \frac{750}{27t^4}\max_{[t/2,t]} x^3(s) = 0, \quad t \geq 0. \tag{3.1}$$

One can easily verify that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (1.1) *is almost oscillatory. Infact* $x(t) = \frac{1}{t}$ *is one such solution of equation* (3.1).

Example 3.2. *Consider the differential equation*

$$\left(t^{1/3}\left(\left(x(t)+\frac{1}{2}x(t/2)\right)''\right)^{1/3}\right)' + \frac{1}{3}\left(\frac{2}{t}\right)^{4/3}\max_{[t/2,t]} x^{1/3}(s) = 0, \quad t \geq 1. \tag{3.2}$$

One can easily verify that all conditions of Theorem 2.2 are satisfied and hence every solution of equation (1.1) *is almost oscillatory. In fact* $x(t) = \frac{1}{t}$ *is one such solution of equation* (3.2).

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