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Adapted linear approximation for singular integral equations

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Abstract

The aim of this work is to solve singular integral equations (S.I.E), of Cauchy type on a closed smooth curve. This method presented by the author is based on the adapted linear approximation of the singular integral of the dominant part, where we reduce a (S.I.E) to an algebraic linear system and we realize numerically this approach by examples.

Keywords: singular integral, interpolation, linear approximation, Holder space and Holder condition.

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1 Introduction

Many studies devoted to the numerical procedure are developed for solving singular integral equations (S.I.E) over a contour. Cauchy type singular integral equations are often encountered in electrostatics, fluid dynamics and in simulation of cracks. Computational efficient quadrature methods for the solution of (S.I.E) have recently been introduced and analyzed in the case of a closed curve [5]. A different quadrature method for a closed curve, involving subtraction of the singularity, was analyzed in [8]. Noting that, the solution of a large class of boundary-value problems in mathematical physics can be reduced to singular integral equations (S.I.E) of the form

$$a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma k(t,t_0)\,\varphi(t)\,dt = f(t_0). \tag{1.1}$$

It will often be useful to write equation (1) in the form

$$\left(a_0(t_0)\,I + b_0(t_0)\,S_\Gamma + K_\Gamma\right)\varphi(t_0) = f(t_0), \tag{1.2}$$

where I is the identity operator and the operators S_Γ and K_Γ are defined by

$$S_\Gamma\varphi(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt; \qquad K_\Gamma\varphi(t_0) = \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt. \tag{1.3}$$

In this work we present a direct method for an approximative solution of a singular integral equation (S.I.E) on a piecewise smooth integration path Γ, where Γ is any piecewise smooth closed contour [2], t_0 and t are points on Γ, the known functions $a_0(t)$, $b_0(t)$ and $k(t,t_0)$ are defined on Γ and satisfying the Holder condition $H(\alpha)$, $0 < \alpha \leq 1$ [2]. Further, anywhere on Γ we have

$$a_0^2(t) - b_0^2(t) \neq 0. \tag{1.4}$$

As it is known, the integral of the dominant part of the above equation (1) exists in the sense of a Cauchy principal value integral for all density φ satisfies the Holder condition $H(\alpha)$ and also exists for all function $\varphi \in L^2(\Gamma)$.

The present note is divided into two parts. In the first one, we present a formulation of the quadrature formula for the evaluation of Cauchy type integrals proposed by the author [5], this quadrature formula is based on the adapted quadratic approximation of the density $\varphi(x)$.

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In the second part, we present the numerical realization of this approximation; also the estimate of the error of the approximation integral was established. Besides, pointwise convergence of the approximate solutions to an exact solution is obtained [5,6].

A method to proceed is to solve the (S.I.E) by numerical means, like the reduction to a system of linear algebraic equations after the use of an appropriate quadrature rule.

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) \ = x(s) + iy(s), \ \ a \leq s \leq b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a,b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null.

2 The Quadrature

Theorem 2.1. *Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a,b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points*

$$s_\sigma = a + \sigma \frac{l}{N} \ , \ l = b - a \ , \ \sigma = 0, 1, 2, ..., N.$$

We introduce the notation

$$t_\sigma = t(s_\sigma); \ \ \sigma = 0, 1, 2, ..., N.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, ..., N-1$, the points t and t_0 belong respectively to the arcs $t_\sigma \overset{\frown}{t_{\sigma+1}}$ and $t_\nu \overset{\frown}{t_{\nu+1}}$ where $t_\alpha \overset{\frown}{t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ [3,5,6,7].

Following [6], we define the approximation $\psi_{\sigma\nu}(\varphi; t, t_0)$ for the density $\varphi(t)$ by the following expression

$$\begin{aligned} \psi_{\sigma\nu}(\varphi; t, t_0) &= \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0) \\ &= \varphi(t_0) + U(\varphi; t, t_0, \sigma) - V(\varphi; t, t_0, \sigma, \nu), \end{aligned} \tag{2.1}$$

where the expression $\psi_{\sigma\nu}(\varphi; t_0, t)$, designates the approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ [5], destined for the first integral of the left hand side of the equation (1).

Indeed, for $t_\sigma \leq t \leq t_{\sigma+1}$ we put

$$U(\varphi; t, t_0, \sigma) = \frac{(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_\sigma) \frac{t - t_0}{t_\sigma - t_0} + \frac{(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)} \varphi(t_{\sigma+1}) \frac{t - t_0}{t_{\sigma+1} - t_0},$$

and the function $V(\varphi; t_0, \sigma, \nu)$ is given by

$$V(\varphi; t, t_0, \sigma, \nu) = \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma+1} - t)}{(t_{\sigma+1} - t_\sigma)} + \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_\sigma)}{(t_{\sigma+1} - t_\sigma)},$$

with the function $S_1(\varphi; t_0, \nu)$ represents the piecewise linear interpolating polynomial of the function density $\varphi(t_0)$ given by

$$S_1(\varphi; t, \nu) = \frac{(t_{\nu+1} - t)}{(t_{\nu+1} - t_\nu)} \varphi(t_\nu) + \frac{(t - t_\nu)}{(t_{\nu+1} - t_\nu)} \varphi(t_{\nu+1}).$$

Let $A\varphi(t_0)$ denote the left side of the equation (1)

$$\begin{aligned} A\varphi(t_0) &= (a_0(t_0)I + b_0(t_0)S_\Gamma + K_\Gamma)\varphi(t_0) \\ &= a_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0} dt + \frac{1}{\pi i} \int_\Gamma k(t, t_0)\varphi(t) dt \\ &= (a_0(t_0) + b_0(t_0))\varphi(t_0) + \frac{b_0(t_0)}{\pi i} \int_\Gamma \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt + \frac{1}{\pi i} \int_\Gamma k(t, t_0)\varphi(t) dt \\ &= ((a_0(t_0) + b_0(t_0))I + b_0(t_0)S_\Gamma^1 + K_\Gamma)\varphi(t_0), \end{aligned}$$

where the operator S_Γ^1 is defined by

$$S_\Gamma^1 \varphi(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt, \tag{2.2}$$

and $\widetilde{A}\widetilde{\varphi}(t_0)$ be the adapted linear interpolation formula for the operator $A\varphi(t)$ given by

$$\begin{aligned}
\widetilde{A}\widetilde{\varphi}(t_0) &= (a_0(t_0)I + b_0(t_0)\widetilde{S}_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0) \\
&= a_0(t_0)\widetilde{\varphi}(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\psi_{\sigma v}(\varphi;t,t_0)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \widetilde{k}(t,t_0)\widetilde{\varphi}(t)dt \\
&= (a_0(t_0)+b_0(t_0))\widetilde{\varphi}(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\beta_{\sigma v}(\varphi;t,t_0)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \widetilde{k}(t,t_0)\widetilde{\varphi}(t)dt \\
&= ((a_0(t_0)+b_0(t_0))I + b_0(t_0)\widetilde{S}^1_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0),
\end{aligned}$$

where the function $\widetilde{\varphi}(t)$ represents the approximate solution of equation (1), obtained by the equality of the functions $\widetilde{A}\widetilde{\varphi}(t_0)$ and $f(t_0)$ at the points t_σ, $\sigma = 0, 1, .., N-1$.

3 Main results

Theorem 3.1. *he singular integral equation of the form* (1) *with the condition* (2) *has a unique solution* $\varphi(t)$ *and an approximate solution* $\widetilde{\varphi}(t)$ *converges to the solution* $\varphi(t)$ *with the following estimation*

$$\|\varphi(t) - \widetilde{\varphi}(t)\| \leq \frac{C_1 \ln(N)}{N^\mu} + \frac{C_2}{N^2}; \quad N > 1,$$

where the constant C_1 and C_2 *depend only on the curve* Γ *and the Holder constant of the function* φ.

We can written the integral equation (1) as

$$A\varphi(t_0) = (I + S_\Gamma + K_\Gamma)\varphi(t_0) = f(t_0),$$

while as an approximating equation n the space $H(\alpha)$ we consider

$$\widetilde{A}\widetilde{\varphi}(t_0) = (I + \widetilde{S}_\Gamma + \widetilde{K}_\Gamma)\widetilde{\varphi}(t_0) = f.$$

It follows from [5] that, for all $\varphi(t)$ in $H^\alpha(\Gamma)$ we have

$$\|S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)\| \leq \frac{C\ln(N)}{N^\mu}, \quad C > 0,$$

and also it is known that

$$\|K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0)\| \leq \frac{C'}{N^2}, \quad C' > 0,$$

for all K compact and $\varphi \in H(\alpha)$.

It is easily to see that

$$\begin{aligned}
\|\varphi(t_0) - \widetilde{\varphi}(t_0)\| &= \|\frac{1}{a_0(t_0)+b_0(t_0)}\|\|b_0(t_0)(S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)) + (K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0))\| \\
&\leq \|\frac{b_0(t)}{a_0(t_0)+b_0(t_0)}\|\|S_\Gamma\varphi(t_0) - \widetilde{S}_\Gamma\widetilde{\varphi}(t_0)\| + \|\frac{1}{a_0(t_0)+b_0(t_0)}\|\|K_\Gamma\varphi(t_0) - \widetilde{K}_\Gamma\widetilde{\varphi}(t_0)\| \\
&\leq \frac{C_1\ln(N)}{N^\mu} + \frac{C_2}{N^2}; \quad N > 1.
\end{aligned}$$

4 Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the singular integral equations (1). In all cases, the curve Γ designate the unit circle and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct.

We apply the algorithms described in [1,3,5] to solve S.I.E and we present results concerning the accuracy of the calculations; in this numerical experiments it is easily to see that the matrix of the system of algebraic equation given by our approximation is invertible, confirmed in [3,6,7].

In each table, φ represents the exact solution given in the sense of the principal value of Cauchy and $\widetilde{\varphi}$ corresponds to the approximate solution produced by the approximation at points values interpolation [3,4,5].

Consider the singular integral equation

$$\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = 2t_0,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t$$

Table 1. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	8.0902e-001 +5.8779e-001i	8.0902e-001 +5.8779e-001i	1.2311e-014
-3.0902e-001 +9.5106e-001i	-3.0902e-001 +9.5106e-001i	-3.0902e-001 +9.5106e-001i	1.2152e-014
-1.0000e+000 +1.2246e-016i	-1.0000e+000 +1.2246e-016i	-1.0000e+000 +2.2760e-015i	8.7082e-015
-3.0902e-001 -9.5106e-001i	-3.0902e-001 -9.5106e-001i	-3.0902e-001 -9.5106e-001i	2.2892e-014
8.0902e-001 -5.8779e-001i	8.0902e-001 -5.8779e-001i	8.0902e-001 -5.8779e-001i	2.5337e-014

Consider the singular integral equation

$$\frac{2}{t_0}\varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = \frac{2-t_0}{t_0^2},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t}$$

Table 2. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	8.0902e-001 -5.8779e-001i	8.0676e-001 -5.8909e-001i	2.6098e-003
-3.0902e-001 +9.5106e-001i	-3.0902e-001 -9.5106e-001i	-3.1278e-001 -9.5190e-001i	3.8573e-003
-1.0000e+000 +1.2246e-016i	-1.0000e+000 -1.2246e-016i	-1.0055e+000 -5.0190e-003i	7.4789e-003
-3.0902e-001 -9.5106e-001i	-3.0902e-001 +9.5106e-001i	-3.1026e-001 +9.4740e-001i	3.8618e-003
8.0902e-001 -5.8779e-001i	8.0902e-001 +5.8779e-001i	8.0747e-001 +5.8568e-001i	2.6087e-003

Consider the singular integral equation

$$t_0(t_0+3)\varphi(t_0) + \frac{t_0^2+2}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = \frac{t_0(t_0+3)-(t_0^2+2)}{2t_0+1},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{2t+1}$$

Table 3. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	3.1787e-001 -1.4273e-001i	3.1957e-001 -1.4146e-001i	2.1209e-003
-3.0902e-001 +9.5106e-001i	1.0148e-001 -5.0535e-001i	9.9874e-002 -5.0612e-001i	1.7823e-003
-1.0000e+000 +1.2246e-016i	-1.0000e+000 -2.4493e-016i	-1.0023e+000 -1.3759e-002i	1.3942e-002
-3.0902e-001 -9.5106e-001i	1.0148e-001 +5.0535e-001i	1.0204e-001 +5.0625e-001i	1.0605e-003
8.0902e-001 -5.8779e-001i	3.1787e-001 +1.4273e-001i	3.1644e-001 +1.4133e-001i	2.0082e-003

Consider the singular integral equation

$$-t_0(t_0-2)\varphi(t_0) - \frac{t_0(t_0+5)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \frac{1}{\pi i}\int_\Gamma \frac{t_0(t+2)}{t}\varphi(t)dt = \frac{t_0}{t_0+2},$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t+2}$$

Table 4. The values of points, the exact solution of the singular integral equation, the approximate solution and the error for N=50.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.0902e-001 +5.8779e-001i	3.4106e-001 -7.1367e-002i	3.4502e-001 -7.0920e-002i	3.9853e-003
-3.0902e-001 +9.5106e-001i	4.4926e-001 -2.5268e-001i	4.5626e-001 -2.5387e-001i	7.1025e-003
-1.0000e+000 +1.2246e-016i	1.0000e+000 -1.2246e-016i	1.0139e+000 -1.1282e-002i	1.7870e-002
-3.0902e-001 -9.5106e-001i	4.4926e-001 +2.5268e-001i	4.5100e-001 +2.5844e-001i	6.0213e-003
8.0902e-001 -5.8779e-001i	3.4106e-001 +7.1367e-002i	3.4398e-001 +7.3563e-002i	3.6542e-003

5 Conclusion

We have considered the numerical solution of singular integral equations and have presented an efficient scheme to compute this singular integrals. The essential idea is to find a combination of functions of approximation for the function density where we can be used it to remove integrable singularities. The regular part where it is the remaining integrands are well behaved and pose no serious numerical problem. Typical examples taken from the literature with known closed form solutions, were used to illustrate the stability and convergence of the approach. The stability of the numerical solution was verified by comparing the analytical and numerical solutions which agree well.

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