Malaya Journal of Matematik

MIM

an international journal of mathematical sciences with computer applications...



www.malayajournal.org

Some curvature tensors on a generalized Sasakian space form

B. Sumangala^a and Venkatesha^{b,*}

^{a,b}Department of Mathematics, Kuvempu University, Shankaraghatta–577451, Shimoga, Karnataka, India.

Abstract

In the present paper, we have studied the geometry of generalized Sasakian space form with the condition satisfying $W^*(\xi, X)W^* = 0$, $W^*(\xi, X)S = 0$, $W^*(\xi, X)P = 0$ and $P(\xi, X)P = 0$.

Keywords: Generalized Sasakian space form, M-projective curvature tensor, Projective curvature tensor.

2010 MSC: 53D10, 53D15, 53C25.

©2012 MJM. All rights reserved.

1 Introduction

A Sasakian manifold (M, ϕ, ξ, η, g) is said to be a Sasakian space form [3], if all the ϕ -sectional curvatures $K(X \land \phi X)$ are equal to a constant C, where $K(X \land \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX . In such a case, the Riemannian curvature tensor of M is given by,

$$R(X,Y)Z = \frac{C+3}{4} \{ g(Y,Z)X - g(X,Z)Y \}$$

$$+ \frac{C-1}{4} \{ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \}$$

$$+ \frac{C-1}{4} \{ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \}$$

$$+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \}. \tag{1.1}$$

As a natural generalization of these manifolds, Alegre P., Blair D. E. and Carriazo A. [1, 3] introduced the notion of generalized Sasakian space form.

Sasakian space form and generalized Sasakian space form have been studied by several authors, viz., [2], [3], [9], [14], [15].

De U. C. and Sarkar A. [9] studied properties of projective curvature tensor to generalized Sasakian space form. Mehmet Atceken [10] studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor.

The properties of the *M*-projective curvature tensor in Sasakian and Kaehler manifolds were studied by Ojha R. H. [11, 12]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. Chaubey S. K. and Ojha R. H. [7] studied the properties of the *M*-projective curvature tensor in Riemannian and Kenmotsu manifolds. Chaubey S. K. [8] also studied the properties of *M*-projective curvature tensor in LP-Sasakian manifold. Present authors [4] have studied some properties of *M*-projective curvature tensor in a generalized Sasakian space form. Motivated by these ideas, in the present paper we have extended the study of further properties of *M*-projective curvature tensor to generalized Sasakian space form. The present paper is organized as follows:

^{*}Corresponding author.

In section 2, we review some preliminary results. From section 3 onwards we have obtained necessary and sufficient condition for a generalized Sasakian space form satisfying the derivation conditions $W^*(\xi, X)W^* = 0$, $W^*(\xi, X)S = 0$, $W^*(\xi, X)P = 0$ and $P(\xi, X)P = 0$. We have proved that these conditions are satisfied if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

2 Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists on M, a (1,1) tensor field ϕ , a vector field ξ and a 1-form η [6] such that,

$$\phi^{2}(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$\eta(\phi X) = 0, (2.3)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\phi \xi = 0, \quad \eta(\xi) = 0, \quad g(X, \xi) = \eta(X),$$
 (2.5)

for any vector fields *X*, *Y* on *M*.

If in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well known that a Contact metric manifold is a K-contact manifold if and only if $(\nabla_X \xi) = -\phi(X)$ for any vector field X on M.

Given an almost contact metric manifold (M, ϕ, ξ, η, g) we say that M is an generalized Sasakian space form [1], if there exists three functions f_1 , f_2 and f_3 on M such that

$$R(X,Y)Z = f_{1}\{g(Y,Z)X - g(X,Z)Y\}$$

$$+ f_{2}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$

$$+ f_{3}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},$$
(2.6)

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian space form M(C), which can be obtained as particular cases of generalized Sasakian space form by taking $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$. Further in a (2n+1)-dimensional generalized Sasakian space form, we have [1]

$$(\nabla_X \phi)(Y) = (f_1 - f_3)(g(X, Y)\xi - \eta(Y)X), \tag{2.7}$$

$$(\nabla_X \xi) = -(f_1 - f_3)\phi(X), \tag{2.8}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.9}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.10}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, (2.11)$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{2.13}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.14}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X).$$
 (2.15)

In 1971, Pokhariyal G. P. and Mishra R. S. [13] defined M-projective curvature tensor W^* on a Riemannian manifold as

$$W^{*}(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],$$
(2.16)

and projective curvature tensor [16] is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y].$$
(2.17)

3 Generalized Sasakian space form satisfying $W^*(\xi, X)W^* = 0$

Let us consider a generalized Sasakian space form satisfying

$$W^*(\xi, X)W^* = 0. (3.18)$$

The above equation can be written as

$$W^{*}(\xi, X)W^{*}(Y, Z)U - W^{*}(W^{*}(\xi, X)Y, Z)U$$
$$-W^{*}(Y, W^{*}(\xi, X)Z)U - W^{*}(Y, Z)W^{*}(\xi, X)U = 0,$$
(3.19)

for any vector field X, Y, Z, U on M.

In view of (2.5), (2.9), (2.10) and (2.13), (2.16) becomes

$$W^*(\xi, X)Y = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2](g(X, Y)\xi - \eta(Y)X)$$
(3.20)

and

$$\eta(W^*(X,Y)Z) = \frac{1}{4n}[(1-2n)f_3 - 3f_2][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{3.21}$$

From (2.16) and (3.20), we find

$$W^{*}(\xi, X)W^{*}(Y, Z)U = \frac{1}{4n}[(1 - 2n)f_{3} - 3f_{2}][g(X, W^{*}(Y, Z)U)\xi - \frac{1}{4n}[(1 - 2n)f_{3} - 3f_{2}][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]]$$
(3.22)

and

$$W^{*}(W^{*}(\xi,X)Y,Z)U = \frac{1}{4n}[(1-2n)f_{3}-3f_{2}][\frac{(1-2n)f_{3}-3f_{2}}{4n}[g(X,Y)g(Z,U)\xi - g(X,Y)\eta(U)Z] - \eta(Y)W^{*}(X,Z)U].$$
(3.23)

Substituting $Z = \xi$ in (2.16), we get

$$W^*(X,Y)\xi = \frac{1}{4n}[(1-2n)f_3 - 3f_2](\eta(Y)X - \eta(X)Y), \tag{3.24}$$

Substituting (3.20), (3.22), (3.23) in (3.19), we get

$$\frac{(1-2n)f_3-3f_2}{4n}[g(W^*(Y,Z)U,X)\xi-\frac{(1-2n)f_3-3f_2}{4n}[g(Z,U)\eta(Y)X] - g(Y,U)\eta(Z)X + g(X,Y)g(Z,U)\xi - g(X,Y)\eta(U)Z + g(X,Z)\eta(U)Y - g(X,Z)g(U,Y)\xi + g(X,U)\eta(Z)Y - g(X,U)\eta(Y)Z] + \eta(Y)W^*(X,Z)U + \eta(Z)W^*(Y,X)U + \eta(U)W^*(Y,Z)X] = 0.$$
 (3.25)

Taking inner product of (3.25) with respect to ξ and using (2.16) and (3.21), we get

$$\frac{(1-2n)f_3 - 3f_2}{4n} [g(R(Y,Z)U,X) - \frac{4nf_1 + 3f_2 - (1+2n)f_3}{4n} [g(X,Y)g(Z,U) - g(X,Z)g(Y,U)] - \frac{3f_2 + (2n-1)f_3}{4n} [g(X,Z)\eta(U)\eta(Y) + g(Y,U)\eta(Z)\eta(X) - g(X,Y)\eta(Z)\eta(U) - g(Z,U)\eta(X)\eta(Y)]] = 0.$$
(3.26)

This implies either

$$f_3 = \frac{3f_2}{(1-2n)} \tag{3.27}$$

or

$$g(R(Y,Z)U,X) = \frac{4nf_1 + 3f_2 - (1+2n)f_3}{4n} [g(X,Y)g(Z,U) - g(X,Z)g(Y,U)] + \frac{3f_2 + (2n-1)f_3}{4n} [g(X,Z)\eta(U)\eta(Y) + g(Y,U)\eta(Z)\eta(X) - g(X,Y)\eta(Z)\eta(U) - g(Z,U)\eta(X)\eta(Y)].$$
(3.28)

Let $\{e_i\}$, i=1,2,...,2n+1 be an orthonormal basis of the tangent space at any point of the space form. Then putting $X=Y=e_i$, in (3.28) and taking summation over $i,1 \le i \le 2n+1$, we get

$$S(Z,U) = \frac{1}{4n} [[2n(4nf_1 + 3f_2 - (1+2n)f_3) - (3f_2 + (2n-1)f_3)]g(Z,U) - (2n-1)(3f_2 + (2n-1)f_3)\eta(U)\eta(Z)].$$
(3.29)

Contracting the above equation we get,

$$r = \frac{1}{2}[(2n+1)(4nf_1+3f_2-(1+2n)f_3)-2(3f_2+(2n-1)f_3)], \tag{3.30}$$

using (2.11) we get

$$f_3 = \frac{3f_2}{(1-2n)}. (3.31)$$

This leads us to state the following:

Theorem 3.1. A (2n+1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition $W^*(\xi, X)W^* = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

4 Generalized Sasakian space form satisfying $W^*(\xi, X)S = 0$

The condition $W^*(\xi, X)S = 0$ implies that

$$S(W^*(\xi, X)Y, Z) + S(Y, W^*(\xi, X)Z) = 0.$$
(4.32)

Substituting (3.20) in (4.32), we obtain

$$\frac{(1-2n)f_3 - 3f_2}{4n} [g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) + S(Y,\xi)g(X,Z) - \eta(Z)S(X,Y)] = 0.$$
(4.33)

Again substituting $Z = \xi$ in (4.33), we get

$$\frac{(1-2n)f_3-3f_2}{4n}[S(X,Y)-2n(f_1-f_3)g(X,Y)]=0. (4.34)$$

This implies either

$$f_3 = \frac{3f_2}{(1-2n)} \tag{4.35}$$

or

$$S(X,Y) = 2n(f_1 - f_3)g(X,Y). \tag{4.36}$$

On contracting (4.36), we find

$$r = 2n(2n+1)(f_1 - f_3)$$
 and so $f_3 = \frac{3f_2}{(1-2n)}$. (4.37)

Thus, we state

Theorem 4.2. A (2n+1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition $W^*(\xi, X)S = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

5 Generalized Sasakian space form satisfying $W^*(\xi, X)P = 0$

We know that,

$$(W^*(\xi, X)P)(Y, Z)U = W^*(\xi, X)P(Y, Z)U - P(W^*(\xi, X)Y, Z)U - P(Y, W^*(\xi, X)Z)U - P(Y, Z)W^*(\xi, X)U.$$
(5.38)

But as we assume $W^*(\xi, X)P = 0$, (5.38) takes the form

$$W^{*}(\xi, X)P(Y, Z)U - P(W^{*}(\xi, X)Y, Z)U$$

$$- P(Y, W^{*}(\xi, X)Z)U - P(Y, Z)W^{*}(\xi, X)U = 0.$$
(5.39)

In view of (2.14), we obtain from (2.17) that

$$\eta(P(X,Y)Z) = \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{5.40}$$

From (2.17) and (3.20), we find

$$W^{*}(\xi, X)P(Y, Z)U = \frac{1}{4n}[(1 - 2n)f_{3} - 3f_{2}][g(X, R(Y, Z)U)\xi - \frac{1}{2n}[S(U, Z)g(X, Y) - S(Y, U)g(X, Z)]\xi - \frac{1}{2n}[(1 - 2n)f_{3} - 3f_{2}][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]]$$
(5.41)

and

$$P(W^{*}(\xi, X)Y, Z)U = \frac{1}{4n}[(1-2n)f_{3} - 3f_{2}][(f_{1} - f_{3})g(X, Y)g(Z, U)\xi - \frac{1}{2n}S(U, Z)g(X, Y)\xi - \eta(Y)P(X, Z)U].$$
(5.42)

Also

$$P(Y,Z)W^*(\xi,X)U = -\frac{1}{4n}[(1-2n)f_3 - 3f_2]\eta(U)P(Y,Z)X].$$
 (5.43)

Substituting (5.41), (5.42) and (5.43) in (5.39), we get

$$\frac{(1-2n)f_3 - 3f_2}{4n} [g(R(Y,Z)U,X)\xi]$$

$$- \frac{1}{2n} [S(U,Z)g(X,Y) - S(Y,U)g(X,Z)]\xi$$

$$- \frac{1}{2n} [(1-2n)f_3 - 3f_2] [g(Z,U)\eta(Y)X - g(Y,U)\eta(Z)X]$$

$$- (f_1 - f_3)g(X,Y)g(Z,U)\xi + \frac{1}{2n}S(U,Z)g(X,Y)\xi$$

$$+ (f_1 - f_3)g(X,Z)g(Y,U)\xi - \frac{1}{2n}S(Y,U)g(X,Z)\xi$$

$$+ \eta(Y)P(X,Z)U + \eta(Z)P(Y,X)U + \eta(U)P(Y,Z)X] = 0$$
(5.44)

Taking inner product of (5.44) with respect to the Riemannian metric g and then using (2.5) and (5.40), we have

$$\frac{1}{4n}[(1-2n)f_3 - 3f_2][g(R(Y,Z)U,X)
- (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}
+ \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)]] = 0.$$
(5.45)

This implies either

$$f_3 = \frac{3f_2}{(1-2n)} \tag{5.46}$$

or

$$g(R(Y,Z)U,X) = (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}$$

$$- \frac{1}{2n}[(1-2n)f_3 - 3f_2][g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)].$$
(5.47)

Let $\{e_i\}$, i=1,2,...,2n+1 be an orthonormal basis of the tangent space at any point of the space form. Then putting $X=Y=e_i$, in (5.47) and taking summation over $i,1 \le i \le 2n+1$, we get

$$S(Z,U) = 2n(f_1 - f_3)g(Z,U) + [(1-2n)f_3 - 3f_2]\eta(Z)\eta(U).$$
(5.48)

Contracting (5.48), we find

$$r = 2n(2n+1)(f_1 - f_3) + (1-2n)f_3 - 3f_2.$$
(5.49)

Using (2.11), the above equation gives

$$f_3 = \frac{3f_2}{(1-2n)} \tag{5.50}$$

Thus, we state

Theorem 5.3. A (2n+1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition $W^*(\xi, X)P = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

6 An generalized Sasakian space form satisfying $P(\xi, X)P = 0$

The condition $P(\xi, X)P = 0$ implies that

$$(P(\xi, X)P)(Y, Z)U = P(\xi, X)P(Y, Z)U - P(P(\xi, X)Y, Z)U - P(Y, P(\xi, X)Z)U - P(Y, Z)P(\xi, X)U = 0.$$
(6.51)

In view of (2.5), (2.10) and (2.13), (2.17) becomes

$$P(\xi, X)Y = (f_1 - f_3)g(X, Y)\xi - \frac{1}{2n}S(X, Y)\xi$$
(6.52)

Using (6.52)in (6.51), we get

$$(f_{1} - f_{3})g(P(Y,Z)U,X)\xi - \frac{1}{2n}S(P(Y,Z)U,X)\xi$$

$$- [(f_{1} - f_{3})g(X,Y) - \frac{1}{2n}S(X,Y)][(f_{1} - f_{3})g(Z,U) - \frac{1}{2n}S(Z,U)]\xi$$

$$- [(f_{1} - f_{3})g(X,Z) - \frac{1}{2n}S(X,Z)][\frac{1}{2n}S(Y,U) - (f_{1} - f_{3})g(Y,U)]\xi$$

$$- [(f_{1} - f_{3})g(X,U) - \frac{1}{2n}S(X,U)]P(Y,Z)\xi = 0,$$
(6.53)

Taking inner product of (6.53) with respect to the Riemannian metric g and then using (2.10), (2.17) and (5.40), we have

$$\frac{1}{2n}[(1-2n)f_3 - 3f_2][g(R(Y,Z)U,X) - (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}] = 0.$$
(6.54)

$$\Rightarrow \qquad f_3 = \frac{3f_2}{(1-2n)} \qquad \text{or} \qquad$$

$$g(R(Y,Z)U,X) = (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}.$$
(6.55)

(6.55) implies

$$R(Y,Z)U = (f_1 - f_3)\{g(Z,U)Y - g(Y,U)Z\}.$$
(6.56)

Contracting (6.56) with respect to the vector field Y, we find

$$S(Z, U) = 2n(f_1 - f_3)g(Z, U). (6.57)$$

On contracting the above equation, we get

$$r = 2n(2n+1)(f_1 - f_3)$$
 and so $f_3 = \frac{3f_2}{(1-2n)}$. (6.58)

Thus, we state

Theorem 6.4. A (2n+1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition $P(\xi, X)P = 0$ if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

References

- [1] Alegre P., Blair D. E. and Carriazo A., Generalized Sasakian-space-forms, Israel J. Math. 14 (2004), 157-183.
- [2] Alegre P. and Carriazo A., *Structures on generalized Sasakian-space-form*, Differential Geom. and its application 26 (2008), 656-666.
- [3] Alfonso Carriazo, David. E. Blair and Pablo Alegre, *Proceedings of the Ninth International Workshop on Differential Geometry*, 9 (2005), 31-39.
- [4] Venkatesha and Sumangala B, On M-projective curvature tensor of a generalized Sasakian space form, Acta Math. Univ. Comenianae, LXXXII (2) (2013), 209 217.
- [5] Belkhelfa M., Deszcz R., Verstraelen L., *Symmetric properties of Sasakian-space-forms*, Soochow J.math., 31 (2005), 611-616.
- [6] Blair D. E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, 509 Springer-Verlag, Berlin, 1976.
- [7] Chaubey S. K. and Ojha R. H., On the M-projective curvature tensor of a Kenmotsu manifold, Differential Geomerty-Dynamical Systems, 12 (2010), 52-60.
- [8] Chaubey S. K., Some properties of LP-Sasakian manifolds equipped with M-Projective curvature tensor, Bulletin of Mathematical Analysis and Applications, 3 (4) (2011), 50-58.
- [9] De U. C. and Sarkar A., On the Projective Curvature tensor of Generalized Sasakian-space-forms, Quaestiones Mathematicae, 33 (2010), 245-252.
- [10] Mehmet Atceken, On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor, Bulletin of Mathematical Analysis and applications, 6 (1) (2014), 1-8.
- [11] Ojha R. H., A note on the m-projective curvature tensor, Indian J. pure Applied Math., 8 (12) (1975), 1531-1534.
- [12] Ojha R. H., On Sasakian manifold, Kyungpook Math. J., 13 (1973), 211-215.
- [13] Pokhariyal G. P. and Mishra R. S., *Curvature tensor and their relativistic significance II*, Yokohama Mathematical Journal, 19 (1971), 97-103.
- [14] Prakasha D. G., On Generalized Sasakian-Space-Forms with Weyl-Conformal Curvature Tensor, Lobachevskii Journal of Mathematics, 33 (3) (2012), 223228.
- [15] Prakasha D. G. and Nagaraja H. G., On quasi-Conformally flat and quasi-Conformally semisymmetric generalized Sasakian-space-forms, CUBO A Mathematical Journal, 15 (3) (2013), 59-70.

[16] Yano K. and Kon M., *Structures on manifolds*, Series in pure mathematics, Vol.3, World Scientific Publishing Co., Singapore, 1984.

Received: July 09, 2014; Accepted: August 19, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/