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Comparison of four different obstacle models of fluid flow with a slip-like boundary condition

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Abstract

In this paper, we investigate a time-discretized 2-dimensional Navier-Stokes equation with a slip-like boundary condition, which arises in the melting ice problem with obstacle. We study the existence and uniqueness of a approximate solution. We also study the numerical solution of melting ice problem using Continuous Galerkin method.

Keywords: Navier-Stokes equation; obstacle modeling; slip-like boundary; Continuous Galerkin finite element method.

2010 MSC: 65L60; 34K28, 37D35.

1 Introduction

The incompressible Navier-Stokes system is one of the main equations studied in mathematical physics and fluid mechanics fields and there is a huge literature written on the subject. For example, we quote [1] for finite difference methods, [2, 3, 4] for finite element methods, and [5] for finite volume methods. Computational fluid dynamics models are in general based on the solution of the Navier-Stokes equations and its discretization scheme, for instance, finite element methods and finite volume methods. To accurately capture the physical properties of the fluid flow being simulated, we usually need highly refined meshes on the entire flow domain which can cause a large scale computation possibly beyond the capability of a single computer. Therefore, to utilize the computational power of modern high-performance computers, much effort is thrown into the development of efficient computing methods for the Navier-Stokes equations.

Let Ω be an open and bounded domain in R^2 with Lipschitz continuous boundary Γ. Throughout the paper we will use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ with norm $||.||_{m,p,\Omega}$ (see[6]). Specially $H^m(\Omega) = W^{m,2}(\Omega)$, where m is an integer greater than zero, will denotes the Sobolev space of real-valued functions with square integrable derivatives of order up to m equipped with the usual norm which we denote $||.||_m$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$ and the standard L^2 inner product by $(.,.)$. Also $H^m(\Omega)$ will denote the space of vector-valued functions each of whose n components belong to $H^m(\Omega)$, and the dual space of $H^m(\Omega)$ will be denoted by $H^{-m}(\Omega)$ of particular interest to us will be the constrained space

$$L_0^2(\Omega) = \{\xi \in L^2(\Omega), \quad \int_\Omega \xi d\Omega = 0\}$$

and

$$H_0^1(\Omega) = \{v \in H^1(\Omega), \quad v|_\Gamma = 0\}.$$

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In [8], the authors studied only the existence and uniqueness of weak solution of Navier-Stokes equation with slip-like boundary condition. In this paper we study the existence and uniqueness of approximate solution and numerical solution of melting ice problem using Continuous Galerkin finite element method.

2 Problem Formulation

Consider an ice plate, placed upright, whose vertical face is exposed to the air and melting. So this face is covered by the layer of flowing water, and the shapes of the ice and the water-layer vary as time t goes on. Therefore, in the water region, this system can be described by Navier-Stokes equations with two free boundaries of the ice-water interface Γ_1 and the water-air interface Γ_2, whose movements would depend on the unknown functions. However, as a first step of analysis, we here consider the discretized Navier-Stokes equation in the time variable t with the discretization parameters $\tau > 0$ in the fixed domain Ω with given interfaces Γ_1 and Γ_2. Experiments for this kind of problems can be found in [14] and mathematical treatments for problems similar to ours are discussed by several authors see [11, 12].

Fix the x-axis vertically and downward, the y-axis in the direction of the thickness and outward, and the z-axis orthogonally to the x and y axes. The ice-water interface and the water-air interface are represented by $y = l(x, z)$ and $y = d(x, z)$ respectively. Further suppose that the size of ice plate in z-direction is so large that we can regard l and d as constant in z. So our problem can be formulated in the following 2-dimensional setting.

Define the domain Ω which is occupied by water by

$$\Omega = \{(x, y) : 0 < x < 1,\ \ l(x) < y < d(x)\},$$

where $l, d \in C^{0,1}([0, 1])$; that is, l and d are Lipschitsz continuous on $[0, 1]$ and

$$0 \le l(x) < d(x) \le 1 \quad \text{for all } \ 0 \le x \le 1.$$

Hence Ω is of class $C^{0,1}([0, 1])$. Define the ice-water interface Γ_1, the water-air interface Γ_2, the lower boundary Γ_3, and the upper boundary Γ_4 by

$$\begin{aligned}
\Gamma_1 &= \{(x, y) : 0 \le x \le 1,\ \ y = l(x)\},\\
\Gamma_2 &= \{(x, y) : 0 \le x \le 1,\ \ y = d(x)\},\\
\Gamma_3 &= \{(x, y) : x = 1,\ \ l(1) \le y \le d(1)\},\\
\Gamma_4 &= \{(x, y) : x = 0,\ \ l(0) \le y \le d(0)\},
\end{aligned}$$

by respectively, We consider the following two-dimensional Navier-Stokes equations with Slip-like boundary condition

$$\begin{gathered}
\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = \mathbf{g} \quad \text{in } \Omega,\\
\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega,
\end{gathered} \tag{1}$$

for the fixed discretization parameter $\tau > 0$ with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{2}$$

$$U_Y = V_Y = 0 \quad \text{on } \Gamma_2, \tag{3}$$

$$v = 0 \quad \text{on } \Gamma_3, \tag{4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_4. \tag{5}$$

Here, the velocity vector $\mathbf{u} = (u, v)$ and the pressure p are unknown functions of (x, y). The initial velocity $\mathbf{u}_0$, the gravity force $\mathbf{g}$, the density ρ, and the kinematic viscosity ν are given data. The unit time τ is to be determined later. Put $U = \mathbf{u} \cdot \mathbf{t}$, $V = \mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ designates the outer unit normal vector of Γ_2 and $\mathbf{t}$ designates the downward unit tangential vector of Γ_2. Denote by (X, Y) the local coordinate with directions $\mathbf{t}$ and $\mathbf{n}$. The original slip boundary condition is stated as

$$U_Y + V_X = 0 \quad \text{on } \Gamma_2,$$

(see [13]) and condition (3) is its linearized version. In the original problem, both Γ_1 and Γ_2 move after the unit time τ. But in our setting, the interfaces stay invariant.

To approximate the solutions of the governing equations derived in section 2 we use the Continuous Galerkin finite element method, which is also know as Ritz-Galerkin method. In this method we formulate a weak formulation of the 2-dimensional Navier-Stokes equation with a slip-like boundary condition that we observe. Discretizing the equations offers the possibility to obtain the approximated solution numerically.

First we discretize the Navier-Stokes equation with a slip-like boundary condition by using the arbitrariness of the variational derivatives with respect to each variable. Note that they are elements of the test function-space $C_0^\infty(D_h)$ on the domain, which can be restricted to the test functions on each element K_i with the test function space $C_0^\infty(K+)$. Hereby then we formulate the finite element weak formulations.

3 The Variational Formulation

The variational formulation for problem (1) is written as

$$\begin{aligned} (\frac{1}{\tau}(\mathbf{u}-\mathbf{u_0})-\mathbf{g},\delta\mathbf{u})+\nu a(\mathbf{u},\delta\mathbf{u})+a_1(\mathbf{u},\mathbf{u},\delta\mathbf{u})-b(p,\delta\mathbf{u}) &= 0 \quad \forall\delta\mathbf{u}\in\mathbf{V} \\ b(q,\delta\mathbf{u}) &= 0 \qquad \forall q\in\mathbf{H} \end{aligned} \tag{6}$$

where

$$\mathbf{V}=\{\mathbf{u}\in\left(H^1(\Omega)\right)^2 : \operatorname{div}\mathbf{u}=0,\ \mathbf{u}=\mathbf{0} \text{ on } \Gamma_1 \text{ and } \Gamma_4, v=0 \text{ on } \Gamma_3\},$$
$$\mathbf{H}=\{\mathbf{u}\in(L_2(\Omega))^2 : \operatorname{div}\mathbf{u}=0\},$$
$$P_\sigma \text{is the orthogonal projection from } (L_2(\Omega))^2 \text{ onto } \mathbf{H},$$
$$\mathbf{L}_4=\{\mathbf{u}\in(L_4(\Omega))^2 : \operatorname{div}\mathbf{u}=0\}$$

and let $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and the norm of the space $\mathbf{H}$.

Define a bounded positive bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$ by

$$a(\mathbf{u},\delta\mathbf{u}) := \int_\Omega (\nabla u\cdot\nabla\,\delta u+\nabla v\cdot\nabla\,\delta v)\,d\mathbf{x}$$

for $\mathbf{u}=(u,v)$, $\delta\mathbf{u}=(\delta u,\delta v)\in\mathbf{V}$. Also define a trilinear form $b(\cdot,\cdot,\cdot)$ on $(\mathbf{L}_4)^2\times\mathbf{V}$ by

$$a_1(\mathbf{w},\mathbf{u},\delta\mathbf{u}) := \int_\Omega (w_1u_x\delta u+w_1v_x\delta v+w_2u_y\delta u+w_2v_y\delta v)\,d\mathbf{x}$$

for $\mathbf{w}=(w_1,w_2)\in\mathbf{L}_4$, $\mathbf{u}=(u,v)\in\mathbf{V}$, $\delta\mathbf{u}=(\delta u,\delta v)\in\mathbf{L}_4$, where $\mathbf{x}=(x,y)$.

The above trilinear form $a_1(.;.,.)$ satisfies the following properties [2, 3]

$$a_1(\mathbf{u};\delta\mathbf{u},\delta\mathbf{u})=0, \qquad a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})=-a_1(\mathbf{u};\mathbf{w},\delta\mathbf{u}), \forall\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}$$

$$|a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})|\le N||\nabla\mathbf{u}||_0||\nabla\delta\mathbf{u}||_0||\nabla\mathbf{w}||_0, \qquad \forall\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}$$

where

$$N=\sup_{\mathbf{u},\delta\mathbf{u},\mathbf{w}\in\mathbf{V}}\frac{|a_1(\mathbf{u};\delta\mathbf{u},\mathbf{w})|}{||\nabla\mathbf{u}||_0||\nabla\delta\mathbf{u}||_0||\nabla\mathbf{w}||_0}$$

is a positive constant depending only on the domain Ω.

We note that Hölder's inequality gives

$$|a_1(\mathbf{w},\mathbf{u},\delta\mathbf{u})|\le|\mathbf{w}|_4|\nabla\mathbf{u}||\delta\mathbf{u}|_4 \quad \text{for } \tilde{\mathbf{u}}\in\mathbf{L}_4,\ \mathbf{u}\in\mathbf{V},\ \delta\mathbf{u}\in\mathbf{L}_4. \tag{7}$$

Here $|\cdot|_4$ denotes the norm of $\mathbf{L}_4$ and

$$|\nabla\mathbf{u}|=|(|\nabla u|,|\nabla v|)|, \quad |\nabla\mathbf{u}|=\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial u}{\partial y}\Big|^2\Big)^{1/2}.$$

Let $\mathbf{Z}=\{\delta\mathbf{u}\in\mathbf{V}, b(q,\delta\mathbf{u})=0 \forall q\in\mathbf{H}\}=\{\delta\mathbf{u}\in\mathbf{H}, div\delta\mathbf{u}=0\}$ denote the divergence-free subspace of $\mathbf{V}$. Then $\mathbf{u}\in\mathbf{V}$ is said to be a week solution of (1) with boundary conditions (2)-(5) if the following relations holds

$$\Big(\frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0)-\mathbf{g},\delta\mathbf{u}\Big)+\nu a(\mathbf{u},\delta\mathbf{u})+a_1(\mathbf{u},\mathbf{u},\delta\mathbf{u})=0 \quad \text{for all } \delta\mathbf{u}\in\mathbf{Z}. \tag{8}$$

We remark that if a sufficiently smooth function $\mathbf{u}$, say in $(C^2(\bar{\Omega}))^2\cap\mathbf{V}$, satisfies (8), then $\mathbf{u}$ should satisfy equation (1) and boundary condition (3) on Γ_2. In fact, let $\mathbf{u}\in(C^2(\bar{\Omega}))^2\cap\mathbf{V}$ and let
$\mathbf{f}=-\frac{1}{\nu}\left(\frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0)-\mathbf{g}+P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}\right)\in\mathbf{H}$, then (8) gives

$$a(\mathbf{u},\delta\mathbf{u})=(\mathbf{f},\delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u}\in\mathbf{V}.$$

Here we note that since $v\equiv 0$, $\operatorname{div}\mathbf{u}=u_x+v_y\equiv 0$ on Γ_3, $v_y\equiv 0$ and hence $u_x\equiv 0$ on Γ_3. Consequently, integration by parts yields

$$\begin{aligned} a(\mathbf{u},\delta\mathbf{u}) &= (\mathbf{f},\delta\mathbf{u}) \\ &= \int_\Omega(-\Delta u\delta u-\Delta v\delta v)dx+\int_{\Gamma_2}(u_Y\delta u+v_Y\delta v)dS+\int_{\Gamma_3}u_x\delta u dS \\ &= \int_\Omega(-\Delta u\delta u-\Delta v\delta v)dx+\int_{\Gamma_2}(U_Y\delta U+V_Y\delta V)dS, \end{aligned} \tag{9}$$

where δU and δV are $\mathbf{t}$ and $\mathbf{n}$ components of $\delta\mathbf{u}$ on Γ_2.

If we take $\delta\mathbf{u}\in(C_0^\infty(\Omega))^2\cap\mathbf{V}$, then the term of the integration on Γ_2 in (9) vanishes, whence follows

$$(\mathbf{f},\delta\mathbf{u})=(-\Delta\mathbf{u},\delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u}\in(C_0^\infty(\Omega))^2\cap\mathbf{V}.$$

This says that $\mathbf{f}=-P_\sigma\Delta\mathbf{u}$ in the sense of distribution. Hence $\mathbf{f}=-P_\sigma\Delta\mathbf{u}$ holds a.e. in Ω, which implies that $\mathbf{u}$ gives a solution of (1). Furthermore, plugging this relation into (9), we get

$$\int_{\Gamma_2}(U_Y\delta U+V_Y\delta V)dS=0 \quad \text{for any } \delta\mathbf{u}\in\mathbf{V},$$

whence easily follows that $\mathbf{u}$ should satisfy (3).

From [8], we have the following result.

Theorem 3.1. *Let* $\mathbf{u}_0\in\mathbf{V}$ *and* $\mathbf{g}\in\mathbf{H}$. *There exists a positive number* $\tau_0=\tau_0(|\mathbf{g}|,|\nabla\mathbf{u}_0|)$ *such that for all* $\tau\in(0,\tau_0]$, (8) *admits a unique weak solution* $\mathbf{u}\in\mathbf{V}$.

4 Existence and Uniqueness of the Approximated Solution

We introduce the bilinear forms $a^h(.,.)$, $b^h(.,.)$ and trilinear form $a_1^h(.;.,.)$ as follows

$$a^h(\mathbf{u_h},\delta\mathbf{u_h}):=\sum_{K\in T_h}\int_K(\nabla u_h\cdot\nabla\delta u_h+\nabla v_h\cdot\nabla\delta v_h)\,d\mathbf{x}$$

for $\mathbf{u_h}=(u_h,v_h)$, $\delta\mathbf{u_h}=(\delta u_h,\delta v_h)\in\mathbf{V_h}$.

$$a_1^h(\mathbf{w_h},\mathbf{u_h},\delta\mathbf{u_h}):=\sum_{K\in T_h}\int_K(w_{1h}\frac{\partial u_h}{\partial x}\delta u_h+w_{1h}\frac{\partial v_h}{\partial x}\delta v_h+w_{2h}\frac{\partial u_h}{\partial y}\delta u_h+w_{2h}\frac{\partial v_h}{\partial y}\delta v_h)\,d\mathbf{x}$$

for $\mathbf{w_h}=(w_{1h},w_{2h})\in\mathbf{L}_4$, $\mathbf{u_h}=(u_h,v_h)\in\mathbf{V_h}$, $\delta\mathbf{u_h}=(\delta u_h,\delta v_h)\in\mathbf{L}_4$, where $\mathbf{x}=(x,y)$, respectively. Then the approximation of problem (1) reads as follows.

Find $(u_h,p_h)\in V_h\times H_h$, such that

$$\begin{aligned} (\frac{1}{\tau}(\mathbf{u_h}-\mathbf{u_{0h}})-\mathbf{g},\delta\mathbf{u_h})+\nu a^h(\mathbf{u_h},\delta\mathbf{u_h}) & & \\ +a_1^h(\mathbf{u_h},\mathbf{u_h},\delta\mathbf{u_h})-b^h(p_h,\delta\mathbf{u_h}) &= 0 & \forall\delta\mathbf{u_h}\in\mathbf{V_h} \\ b^h(q_h,\delta\mathbf{u_h}) &= 0 & \forall q_h\in\mathbf{H_h} \end{aligned} \tag{10}$$

The above forms $a^h(.,.)$, $b^h(.,.)$ and $a_1^h(.;.,.)$ have the following properties [2, 3]

$$\begin{aligned} a^h(\delta\mathbf{u}_h,\delta\mathbf{u}_h) &= \nu||\delta\mathbf{u}_h||_h^2, \forall\delta\mathbf{u}_h \in V_h, \\ |a^h(\mathbf{u}_h,\delta\mathbf{u}_h)| &\leq C||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h, \quad |b^h(p_h,\delta\mathbf{u}_h)| \leq C||p_h||_0||\delta\mathbf{u}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h \in V_h, \;\; p_h \in H_h, \\ a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\delta\mathbf{u}_h) &= 0, \;\; a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h) = -a_1^h(\mathbf{u}_h;\mathbf{w}_h,\delta\mathbf{u}_h), \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h \in \delta\mathbf{u}_h, \\ |a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h)| &\leq N_h||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h||\mathbf{w}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h \in H_h^1(\Omega)^2 \cup V_h, \end{aligned} \tag{11}$$

where

$$N_h = \sup_{\mathbf{u_h},\delta\mathbf{u_h},\mathbf{w_h}\in\mathbf{V_h}} \frac{|a_1^h(\mathbf{u_h};\delta\mathbf{u_h},\mathbf{w_h})|}{||\nabla\mathbf{u_h}||_0||\nabla\delta\mathbf{u_h}||_0||\nabla\mathbf{w_h}||_0}$$

Now, we state the discrete embedding inequality over V_h, the same proof as one constructed by [17] ahows that

$$||\delta\mathbf{u}_h||_{0,2k,\Omega} \leq C(k)||\delta\mathbf{u}_h||_h, \qquad \forall\delta\mathbf{u}_h \in X_h, \quad k = 1,2,....$$

Using the discrete embedding inequality (12), we have

$$N_h \leq N_0, \quad N_0 > 1, \;\; \forall \;\; 0 < h \leq 1.$$

Therefore,

$$|a_1^h(\mathbf{u}_h;\delta\mathbf{u}_h,\mathbf{w}_h)| \leq N_0||\mathbf{u}_h||_h||\delta\mathbf{u}_h||_h||\mathbf{w}_h||_h, \quad \forall\mathbf{u}_h,\delta\mathbf{u}_h,\mathbf{w}_h \in H^1(\Omega)^2 \cup V_h,$$

Let $Z_h = \{\delta\mathbf{u} \in V_h, \;\; b^h(q,\delta\mathbf{u}) = 0, \;\; \forall q \in H_h\}$ denote the divergence-free subspace of V_h. Then, the solution u_h of (10) lies in Z_h and satisfies

$$\Big(\frac{1}{\tau}(\mathbf{u_h}-\mathbf{u}_0)_h - \mathbf{g}, \delta\mathbf{u_h}\Big) + \nu a(\mathbf{u_h},\delta\mathbf{u_h}) + a_1^h(\mathbf{u_h},\mathbf{u_h},\delta\mathbf{u_h}) = 0 \quad \text{for all } \delta\mathbf{u_h} \in \mathbf{Z_h}. \tag{12}$$

Next, we discuss the existence and uniqueness of the solution to problem (10). To do this, from [9], the following assumption is necessary.

$$\frac{N_h||f||_h^*}{\nu^2} \leq 1-\delta_1, \quad 0 < \delta_1 < 1, \tag{13}$$

where

$$||f||_h^* = \sup_{\delta\mathbf{u}_h \in V_h} \frac{(f,\delta\mathbf{u}_h)}{||\delta\mathbf{u}_h||_h}.$$

Lemma 4.1. *The space V_h and H_h satisfy the discrete inf-sup condition, that is,*

$$\sup_{\delta\mathbf{u}_h \in V_h} \frac{b^h(q_h,\delta\mathbf{u}_h)}{||\delta\mathbf{u}_h||_h} \geq \beta||q_h||_0, \;\; \forall \;\; q_h \in V_h,$$

where β is a positive constant independent of h.
Under condition (13), by (11) and lemma 4.1, the existence and uniqueness of the approximated solution is obvious (see [2] for details).

5 Numerical Examples

In this section, we present numerical example to conform our theoretical analysis with the algorithms described below.

We consider the following Navier-Stokes equations with slip-like boundary conditions,

$$\begin{gathered} \frac{1}{\tau}(\mathbf{u}-\mathbf{u}_0) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = \mathbf{g} \quad \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{ in } \Omega, \end{gathered} \tag{14}$$

with $\Omega = ([0, 2.2] \times [0, .41]$ - $\Omega_s)$ where Ω_s is semi circular and rectangular obstacle with diameter=0.1 and hight = 0.1 respectively for the fixed discretizing parameter $\tau > 0$ with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{15}$$

$$U_Y = V_Y = 0 \quad \text{on } \Gamma_2, \tag{16}$$

$$v = 0 \quad \text{on } \Gamma_3, \tag{17}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_4. \tag{18}$$

Navier-Stokes equations are difficult to be solved directly due to its nonlinearity. So many iteration algorithms, such as the Uzawa type algorithm, the Arrow Hurwicz algorithm, Schur complement pre-conditioners, and so on, are proposed in the literature [2, 3]. In this paper, we adopt the simpler and often more efficient method such as the Chorin-Teman Projection method with the following scheme.

5.1 Numerical Algorithm

This section describes the essential steps of the classical Chorin-Teman projection method
Step 1
Computing tentative velocity u^* by

$$(\frac{u^* - u^n}{\Delta t}, v) + ((u^*.\nabla)u^{**}, v) + (\nabla u^*, \nabla v) - (g, v) = 0$$

including boundary conditions for the velocity.
Step 2
Computing new pressure p^{n+1} by

$$(\nabla p^{n+1}, \nabla q) + \frac{1}{\Delta t}(\nabla.u^*, q) = 0$$

including boundary conditions for pressure,
Step 3
Compute corrected velocity by

$$(u^{n+1} - u^*, v) + \Delta t(\nabla p^{n+1}, v) = 0$$

including boundary conditions for the velocity.

Set the kinematic viscosity $\nu = 0.001 \ \ m^2/s$ and $\rho = 1.0 \ \ kg/m^2$. A do-nothing boundary condition is assumed at the outlet. Defining the inflow condition is given by

$$U = 4y(H - y)sin(\pi t/8)/H^2, \qquad V = 0 \tag{19}$$

and computing the flow on the time interval $[0, 8]$ with time-step $dt = 0.001$.

The figures of numerical solutions of problem (14) are shown in Figs $[1 - 8]$. The discrete velocity u_h, while the discrete pressure p_h are shown in Figs $[1 - 8]$, for different time interval for the Navier-Stokes equation with Slip-Like boundary condition.

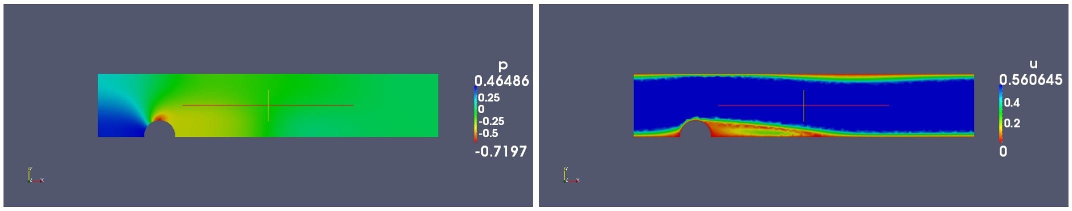

Figure 1: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

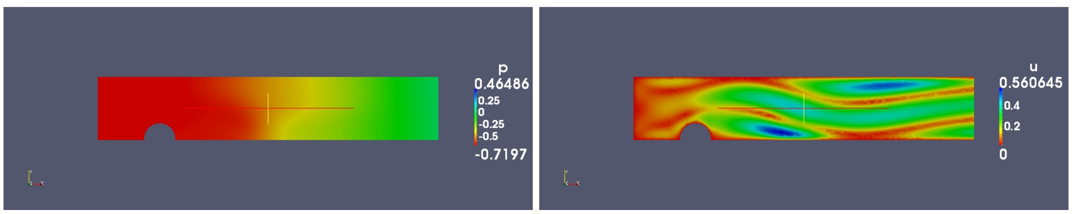

Figure 2: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

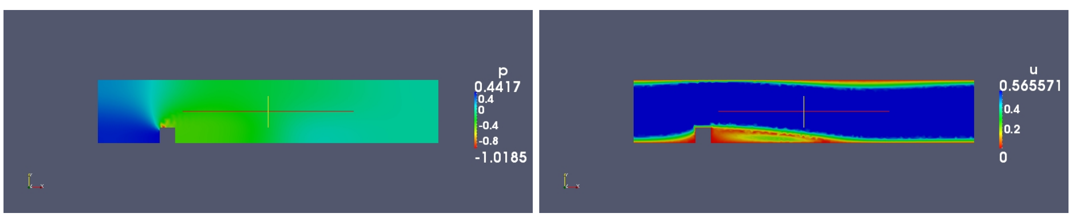

Figure 3: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

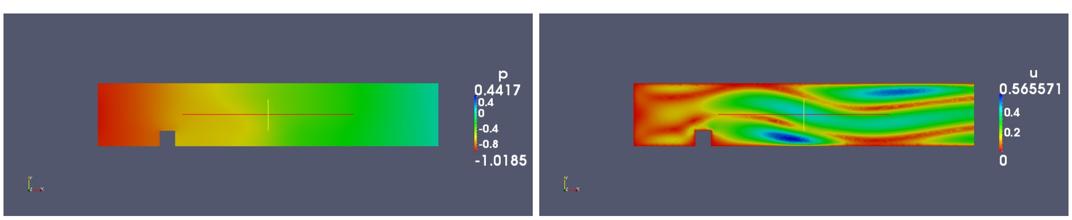

Figure 4: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

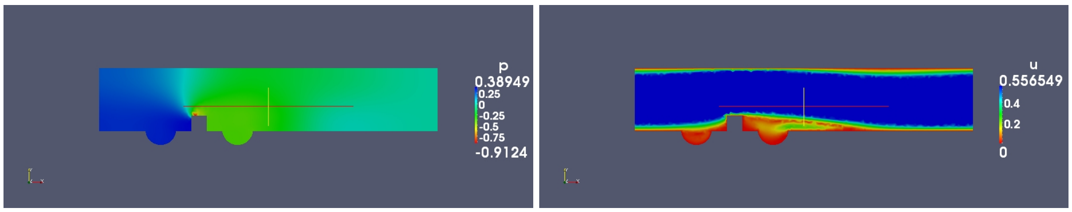

Figure 5: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

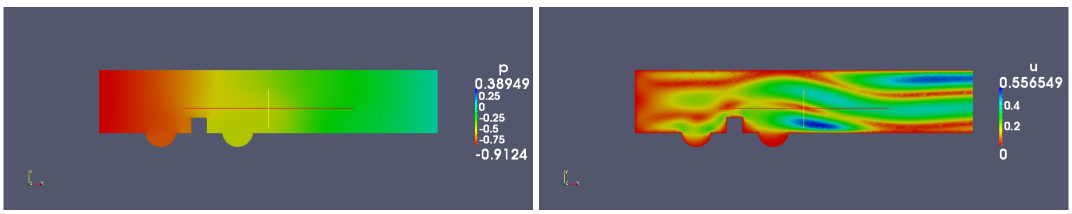

Figure 6: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

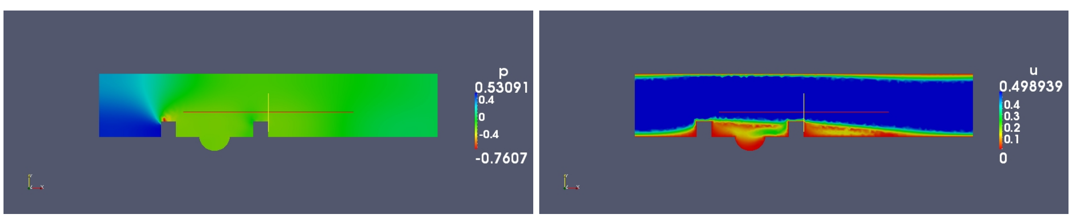

Figure 7: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=4

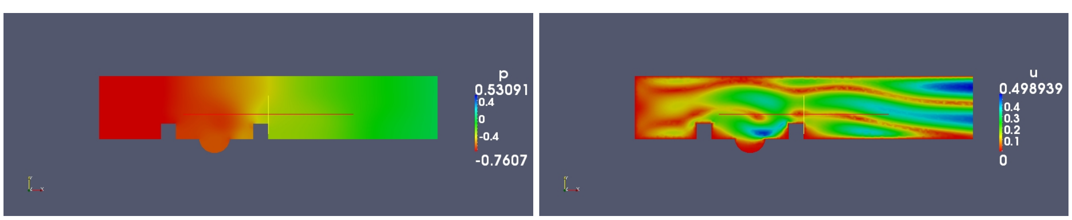

Figure 8: The Numerical pressure p_h and Numerical velocity u_h for Navier-Stokes equations at time t=8

5.2 Conclusion

The flow around obstacle in the ice plate flow was simulated. The effects of four different sets of boundary conditions on a two-dimensional fluid flow across a fixed, rectangular, Semi Circular solid obstruction have been studied numerically using Continuous Galerkin method. Both wave structure far away the obstacle and boundary layer are well resolved. The number, position and wave length are practically identical and are in the good agreement with the theoretical prediction.

The Navier-Stokes Equation and the continuity equation, have been used under the approximations of incompressibility of the fluid, time independence and absence of external, potential dependent forces. The differentiations required have been carried out by the finite element method. The final flow field, the stream function, the vorticity and the velocity fields along at grid points upstream from the leading face of a fixed obstacle have been examined. The variation in the final solutions during the change in the Reynolds number and the size of the obstacle was investigated. The starting field was tested for higher values of the relaxation parameter to generate the reliable solutions.

The small differences are in the predicted maxima and minima of the computed quantities, which are higher in the multidomain approach. For the deeper understanding of the behavior of these models (e.g. dependency on the mesh density) further research is necessary.

For the kinematic viscosity $\nu = 0.001 \ \ m^2/s$ and density $\rho = 1.0 \ \ kg/m^2$, the region of almost dead flow behind the obstacle of size 1×1 was vanished before ending the domain length for the forced conditions at the downstream edge of free flow. The weak flow region remained up to the downstream edge for the conditions of no x derivatives at the downstream edge. The dead region was wider for the flow with higher Reynolds number. The region behind the bigger obstacle of size was also found to have a bigger dead region. The stream function drawn just ahead of the leading edge of the obstacle was such that it was bent to cross the height and went linearly. Vorticity was non zero only for a small region, near the leading corner of the obstacle. It was found to be more disturbed for higher Reynolds numbers for certain number of iterations. The curling of the fluid was more while using a taller obstacle. The velocity profiles along the x and y directions were found to obey the same nature for all the studied values of the kinamatic viscosity and density. They had higher higher peaks while using a taller obstacle.

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