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Existence of mild solution result for fractional neutral stochastic integro-differential equations with nonlocal conditions and infinite delay

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Abstract

We investigate in this paper the existence of mild solutions for the fractional differential equations of neutral type with nonlocal conditions and infinite delay in Hilbert spaces by employing fractional calculus and Krasnoselski-Schaefer fixed point theorem. Finally an example is provided to illustrate the application of the obtained results.

Keywords: Infinite delay, Stochastic fractional differential equations, mild solution, fixed point theorem.

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1 Introduction

The main purpose of this paper is to prove the Existence of the mild solution for fractional differential equations of neutral type with infinite delay in Hilbert spaces of the form.

$$\begin{cases} {}^cD_t^\alpha[x(t)-h(t,x_t)] = A[x(t)-h(t,x_t)] + f(t,x_t) + \int_{-\infty}^t \sigma(t,s,x_s)dW(s) & t \in J=[0,b] \\ x(0)+\mu(x) = x_0 = \phi(t) & t\in(-\infty,0], \end{cases} \tag{1.1}$$

Here, $x(.)$ takes value in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The fractional derivative ${}^cD^\alpha, \alpha\in(0,1)$, is understood in the Caputo sense. The operator A generates a strongly continuous semigroup of bounded linear operators $S(t), t\geq 0$, on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. W is a given $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a filtered complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. The histories $x_t:\Omega\to\mathcal{C}_v$ defined by $x_t=\{x(t+\theta),\theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_v$, which will be defined in section 2. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$- measurable, $\mathcal{C}_v$-valued random variable independent of W with finite second moments, and $h: J\times\mathcal{C}_\vartheta\to\mathbb{H}$, $h: J\times\mathcal{C}_v\to\mathbb{H}$, $\sigma: J\times J_1\times\mathcal{C}_v\to\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ are appropriate functions, where $J_1=(-\infty,b]$ and $\mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ denotes the space of all Q-Hilbert Schmidt operators from $\mathbb{K}$. into $\mathbb{H}$. $\mu: C(J,\mathbb{H})\to\mathbb{H}$ is bounded and the initial data x_0 is an $\mathcal{F}$ adapted $\mathbb{H}$-valued random variable independent of Wiener process W.

The fractional differential equations arise in many engineering and scientific disciplines as the mathematica modeling of systems and processes in the fields of physics,
chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology,etc.,
involves derivatives of fractional order. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations

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of various types (see the books by Hale and Verduyn Lunel [16] , Wu [31] , Liang et al [17], Liang and Xiao [18], and the references therein).

In particular the nonlocal condition problems for some fractional differential equations have been attractive to many researchers Mophou et al [23] studied existence of mild solution for some fractional differential equations with nonlocal condition. Chang et al [7] investigate the fractional order integro-differential equations with nonlocal conditions in the Riemann-Liouville fractional derivative sense.

In this paper, we prove the existence theorem of mild solution for neutral differential equation with nonlocal conditions and infinite delay by using the Krasnoselski-Schaefer fixed point theorem. An example is provided to illustrate the application of the obtained results.

2 Preliminaries

Next we mention a few results and notations needed to establish our results.
Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ be two real separable Hilbert spaces. We denote by $\mathcal{L}(\mathbb{K},\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|.\|$. In this article, we use the symbol $\|.\|$ to denote norms of operators regardless of the spaces involved when no confusion possibly arises.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(W(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t) \qquad e \in \mathbb{K} \qquad t \geq 0$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle\varphi, \phi\rangle_{\mathcal{L}_2^0} = tr[\varphi Q\phi^*]$.

The semigroup $S(\cdot)$ is uniformly bounded. That is to say, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$.

Assume that $\upsilon : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} \upsilon(t)dt < +\infty$ a continuous function.
Recall that the abstract phase space $\mathcal{C}_\upsilon$ is defined by

$$\mathcal{C}_\upsilon = \{\varphi : (-\infty, 0] \to \mathbb{H}, \text{ for any } a > 0, (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable}$$

$$\text{function on } [-a, 0] \text{ and } \int_{-\infty}^{0} \upsilon(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} ds < +\infty\}.$$

If $\mathcal{C}_\upsilon$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_\upsilon} = \int_{-\infty}^{0} \upsilon(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{\frac{1}{2}} ds, \qquad \varphi \in \mathcal{C}_\upsilon$$

then $(\mathcal{C}_\upsilon, \|.\|_{\mathcal{C}_\upsilon})$ is a Banach space (see [20]).

Let us now recall some basic definitions and results of fractional calculus.

Definition 2.1. [21] *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \qquad t > 0 \qquad \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. *The Caputo derivative of order α with the lower limit 0 for a function f can be written as*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \; t > 0, \; 0 \leq n-1 < \alpha < n$$

The Caputo derivative of a constant equal to zero. If f is an abstract function with values in $\mathbb{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense (see [21]).

Lemma 2.1. *5 Let H be a Hilbert space and Φ_1, Φ_2 two operators on H such that*

i) *Φ_1 is a contraction and*

ii) *Φ_2 is completely continuous.*

Then either

a) *the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution or*

b) *$G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.*

Lemma 2.2. [15] *Let $v(.), w(.) : [0, b] \to [0, \infty)$ be continuous function. If $w(.)$ is nondecreasing and there exist two constants $\theta \geq 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \quad t \in J$$

then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0, b]$ and every $n \in N$ such that $n\alpha > 1$.

3 Existence results

Definition 3.3. *An $\mathbb{H}$- valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is a mild solution of the system 1.1 if $x(0)+\mu(x) = x_0 = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{C}_v} < +\infty$, the process x satisfies the following integral equation*

$$\begin{aligned} x(t) = S_\alpha(t)[\phi(0) - \mu(x) - h(0, \phi)] + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau)\right] ds \end{aligned}$$

where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta) S(t^\alpha\theta) x d\theta, \quad T_\alpha(t)x = \alpha \int_0^\infty \theta\zeta_\alpha(\theta) S(t^\alpha\theta) x d\theta$$

and ζ_α is a probability density function defined on $(0, \infty)$

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ appeared in [34] are useful.

Lemma 3.3. *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*

i) *For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators such that for any $x \in \mathbb{H}$*

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M \|x\|_{\mathbb{H}} \text{ and } \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)} \|x\|_{\mathbb{H}}$$

ii) *$S_\alpha(t)$ and $T_\alpha(t)$ are strongly continuous and compact.*

To study existence of mild solutions of 1.1, we introduce the following hypotheses.

(H_1) : The function $h, f : J \times \mathcal{C}_v \to \mathbb{H}$ are continuous and there exist some constants M_h, M_f, such that

$$E\|h(t,x) - h(t,y)\|^2_{\mathbb{H}} \leq M_h \|x-y\|^2_{\mathcal{C}_v}, x, y \in \mathcal{C}_v, \quad t \in J$$

$$E\|h(t,x)\|^2_{\mathbb{H}} \leq M_h(1 + \|x\|^2_{\mathcal{C}_v})$$

$$E\|f(t,x) - f(t,y)\|^2_{\mathbb{H}} \leq M_f \|x-y\|^2_{\mathcal{C}_v}, x, y \in \mathcal{C}_v, \quad t \in J$$

$$E\|f(t,x)\|^2_{\mathbb{H}} \leq M_f(1 + \|x\|^2_{\mathcal{C}_v})$$

(H_2) : μ is continuous and there exists some positive constants M_μ such that

$$E\|\mu(x)-\mu(y)\|_{\mathbb{H}}^2 \leq M_\mu \|x-y\|_{\mathcal{C}_\upsilon}^2, x,y \in \mathcal{C}_\upsilon, \quad t \in J$$

$$E\|\mu(x)\|_{\mathbb{H}}^2 \leq M_\mu(1+\|x\|_{\mathcal{C}_\upsilon}^2)$$

(H_3) : For each $\varphi \in \mathcal{C}_\upsilon$,

$$k(t) = \lim_{a\to\infty} \int_{-a}^{0} \sigma(t,s,\varphi)dW(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$E\|k(t)\|_{\mathbb{H}}^2 \leq M_k$$

(H_4) The function $\sigma : J \times J_1 \times \mathcal{C}_\upsilon \to L(\mathbb{K},\mathbb{H})$ satisfies the following:

i) for each $(t,s) \in J \times J \times, \sigma(t,s,.) : \mathcal{C}_\upsilon \to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x \in \mathcal{C}_\upsilon$, $\sigma(.,.,x) : J \times J \to L(\mathbb{K},\mathbb{H})$ is strongly measurable;

ii) there is a positive integrable function $m \in L^1([0,b])$ and a continuous nondecreasing function M_σ : $[0,\infty) \to (0,\infty)$ such that for every $(t,s,x) \in J \times J \times \mathcal{C}_\upsilon$, we have

$$\int_0^t E\|\sigma(t,s,x)\|_{\mathcal{L}_2^0}^2 ds \leq m(t)M_\sigma(\|x\|_{\mathcal{C}_\upsilon}^2), \quad \liminf_{r\to\infty} \frac{M_\sigma(r)}{r} ds = \Delta < \infty$$

iii) For any $x,y \in \mathcal{C}_\upsilon, t \geq 0$, there exists a positive constant L_σ such that

$$\int_0^t E\|\sigma(t,s,x)-\sigma(t,s,y)\|_{\mathcal{L}_2^0}^2 ds \leq L_\sigma \|x-y\|_{\mathcal{C}_\upsilon}^2$$

(H_5) :

$$N_0 = 2l^2\{12M^2M_\mu + 4M_h\} \tag{3.1}$$

$$N_1 = 2\|\phi\|_{\mathcal{C}_\upsilon}^2 + 2l^2\bar{F} \tag{3.2}$$

$$N_2 = 8l^2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} M_f \tag{3.3}$$

$$N_3 = 16bl^2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} Tr(Q) \tag{3.4}$$

$$K_1 = \frac{N_1}{1-N_0}, \quad K_2 = \frac{N_2}{1-N_0}, \quad K_3 = \frac{N_3}{1-N_0} \tag{3.5}$$

$$\bar{F} = 12M^2(C_1+C_2) + 12M^2M_\mu + 4M_h + 4\left(\tfrac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \tfrac{b^{2\alpha}}{\alpha^2} M_f + 8b\left(\tfrac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \tfrac{b^{2\alpha}}{\alpha^2} M_k \tag{3.6}$$

Now, we consider the space,

$$\mathcal{C}_\upsilon^{'} = \{x : (-\infty,b] \to \mathbb{H}, x_0 = \phi \in \mathcal{C}_\upsilon\}$$

Set $\|.\|_b$ be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_\upsilon} + \sup_{s\in[0,b]} (E|x(s)|^2)^{\frac{1}{2}}, \quad x \in \mathcal{C}_\upsilon^{'}$$

We have the following useful lemma appeared in [20].

Lemma 3.4. [6] *Assume that* $x \in \mathcal{C}_v^{'}$*, then for all* $t \in J$*,* $x_t \in \mathcal{C}_v$*, Moreover,*

$$l(E|x(t)|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{C}_v} \leq l \sup_{s\in[0,t]} (E|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_v}$$

where $l = \int_{-\infty}^{0} v(s)ds < \infty$

The main object of this paper is to explain and prove the following theorem.

Theorem 3.1. *Assume that assumptions* $(H_0) - (H_5)$ *hold.Then there exists a mild solution*

Proof Consider the map $\Pi : \mathcal{C}_v^{'} \to \mathcal{C}_v^{'}$ defined by

$$(\Pi x)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ \\ S_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,x_\tau) dW(\tau)\right] ds \quad t \in J \end{cases} \tag{3.7}$$

In what follows, we shall show that the operator Π has a fixed point, which is then a mild solution for system 1.1.
For $\phi \in \mathcal{C}_v$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ \\ S_\alpha(t)\phi(0) & t \in J \end{cases} \tag{3.8}$$

Then $\tilde{\phi} \in \mathcal{C}_v^{'}$. Let $x(t) = \tilde{\phi}(t) + z(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies 1.1 if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = S_\alpha(t)\left[-\mu(\tilde{\phi} + z) - h(0,\phi)\right] + h(t, \tilde{\phi}_t + z_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dW(\tau)\right] ds \end{aligned}$$

Let

$$\mathcal{C}_v^{''} = \{z \in \mathcal{C}_v^{'}, z_0 = 0 \in \mathcal{C}_v\}$$

For any $z \in \mathcal{C}_v^{''}$, we have

$$\|z\|_b = \|z_0\|_{\mathcal{C}_v} + \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}}$$

Thus $(\mathcal{C}_v^{''}, \|.\|_b)$ is a Banach space, set

$$B_q = \{z \in \mathcal{C}_v^{''}, \|z\|_b^2 \leq q\}, \text{ for some } q \geq 0$$

then, $B_q \subset \mathcal{C}_v^{''}$ is uniformly bounded.

then, for each q, B_q is clearly a bounded closed convex set in $\mathcal{C}^{''}$. For $z \in B_q$, from Lemma 3.3, we have

$$\begin{aligned} \left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_v}^2 &\leq 2(\|z_t\|_{\mathcal{C}_v}^2 + \left\|\tilde{\phi}_t\right\|_{\mathcal{C}_v}^2) \\ &\leq 4(l^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_v}^2 + l^2 \sup_{s\in[0,t]} E\left\|\tilde{\phi}(s)\right\|^2 + \left\|\tilde{\phi}_0\right\|_{\mathcal{C}_v}^2) \\ &\leq 4l^2(q + M^2 E\|\phi(0)\|_{\mathbb{H}}^2) + 4\|\phi\|_{\mathcal{C}_v}^2 \\ &= \grave{q} \end{aligned}$$

Define the operator $\Phi : C_v^{''} \to C_v^{''}$ by

$$(\Phi z)(t)=\begin{cases}0 \quad t\in(-\infty,0]\\ \\ S_\alpha(t)[-\mu(\tilde{\phi}+z)-h(0,\phi)]+h(t,\tilde{\phi}_t+z_t)+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)ds\\ +\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\right]ds \ \ t\in J\end{cases}$$

Observe that Φ is well defined on B_q for each $q>0$.

Now we will show that the operator Φ has a fixed point on B_q, which implies that E.q 1.1 has a mild solution. To this end, we decompose Φ as $\Phi=\Phi_1+\Phi_2$, where the operators Φ_1 and Φ_2 are defined on B_q, respectively, by

$$(\Phi_1 z)(t)=S_\alpha(t)[-\mu(\tilde{\phi}+z)-h(0,\phi)]+h(t,\tilde{\phi}_t+z_t)$$

$$\begin{aligned}(\Phi_2 z)(t)=&\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)ds\\ &+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)\right]ds\end{aligned}$$

Thus, the theorem follows from the next theorem

Theorem 3.2. *If assumption $(H_1)-(H_5)$ hold, then Φ_1 is a contraction and Φ_2 is completely continuous.*

Proof To prove that Φ_1 is a contraction on $\mathcal{C}_\upsilon^{''}$, we take $u,v\in\mathcal{C}_\upsilon^{''}$. Then for each $t\in J$ we have

$$\begin{aligned}E\left\|\Phi_1 u(t)-\Phi_1 v(t)\right\|_{\mathbb{H}}^2\leq\ &2E\left\|S_\alpha(t)(\mu(\tilde{\phi}+u)-\mu(\tilde{\phi}+v))\right\|_{\mathbb{H}}^2\\ &+2E\left\|h(t,\tilde{\phi}_t+u_t)-h(t,\tilde{\phi}_t+v_t)\right\|_{\mathbb{H}}^2\\ &\leq 2M^2M_\mu\left\|u-v\right\|_{\mathcal{C}_\upsilon}^2+2M_h\left\|u_t-v_t\right\|_{\mathcal{C}_\upsilon}^2\\ &\leq 2(M^2M_\mu+M_h)\left\|u_t-v_t\right\|_{\mathcal{C}_\upsilon}^2\\ &\leq 2(M^2M_\mu+M_h)\\ &[2l^2\sup_{s\in[0,t]}E\left\|u(s)-v(s)\right\|^2+2\left\|u_0\right\|_{\mathcal{C}_\upsilon}^2+2\left\|v_0\right\|_{\mathcal{C}_\upsilon}^2]\\ &\leq 4l^2(M^2M_\mu+M_h)E\left\|u(s)-v(s)\right\|^2\\ &\leq \sup_{s\in[0,b]}L_0E\left\|u(s)-v(s)\right\|^2\end{aligned}$$

where we have used the fact that $\left\|u_0\right\|_{\mathcal{C}_\upsilon}^2=0,\left\|v_0\right\|_{\mathcal{C}_\upsilon}^2=0$.
Thus,

$$\left\|\Phi_1 u-\Phi_1 v\right\|\leq L_0\left\|u-v\right\|$$

and by assumption $0\leq L_0\leq 1$ it is clear that Φ_1 is contraction.

Now, we show that the operator Φ_2 is completely continuous, firstly we prove that $\Phi_2:\mathcal{C}_h^{''}\to\mathcal{C}_h^{''}$ is continuous.

Let $\{z^n(t)\}_{n=0}^\infty$,with $z^n\to z$ in $\mathcal{C}_h^{''}$. Then, there is a number $q\geq 0$ such that $|z^n(t)|\leq q$, for all n and a.e. $t\in J$. So $z^{(n)}\in B_q$ and $z\in B_q$.

$$f(t,z_t^{(n)}+\tilde{\phi}_t)\to f(t,z_t+\tilde{\phi}_t)$$
$$\sigma(s,\tau,z_\tau^{(n)}+\tilde{\phi}_\tau)\to\sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)$$

for $t \in J$, and since

$$E\left\|[f(t, z_t^{(n)} + \tilde{\phi}_t) - f(t, z_t + \tilde{\phi}_t)]\right\|^2 \leq 2M_{q'}(t)$$

$$E\left\|[\sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau) - \sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau)]\right\|^2 \leq 2m(t)M_\sigma(q^{'})$$

By the dominated convergence theorem we obtain continuity of Φ_2

$$\begin{aligned}
E\left\|\Phi z_t^{(n)} - \Phi_t z\right\|^2 \leq & 2\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(t, z_s^{(n)}) - f(t, z_s)]ds\right\|^2 \\
& + 2b\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s [\sigma(s,\tau,z_\tau^{(n)}) - \sigma(s,\tau,z_\tau)]dw(\tau)\right]ds\right\|^2 \\
& \leq 2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\int_0^t E\left\|[f(t, z_s^{(n)}) - f(t, z_s)]\right\|^2 ds \\
& + 2b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\int_0^t E\left\|\left[\int_{-\infty}^s [\sigma(s,\tau,z_\tau^{(n)}) - \sigma(s,\tau,z_\tau)]dw(\tau)\right]ds\right\|^2 \\
& \to 0 \text{ as } n \to \infty
\end{aligned}$$

Next, we prove that Φ_2 maps bounded sets into bounded sets in $\mathcal{C}_\upsilon^{''}$.
For each $z \in B_q$ from [3.4], we have

$$\left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_\upsilon}^2 \leq 4l^2(q + M^2E\|\phi(0)\|_{\mathbb{H}}^2) + 4\|\phi\|_{\mathcal{C}_\upsilon}^2 = q^{'}$$

$$\begin{aligned}
E\|\Phi_2 z(t)\|_{\mathbb{H}}^2 \leq & 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)f(s, \tilde{\phi}_s + z_s)\right\|_{\mathbb{H}}^2 \\
& + 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)]ds\right\|_{\mathbb{H}}^2 \\
& \leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}M_f(1 + \left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_\upsilon}^2)ds \\
& + \left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q)m(s)M_\sigma(\left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_\upsilon}^2)ds. \\
& \leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^{2\alpha}}{\alpha^2}M_f(1+q^{'}) \\
& + 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \frac{b^{2\alpha}}{\alpha^2}(M_k + Tr(Q)M_\sigma(q^{'})\sup_{t\in J} m(s)) \\
& \leq r
\end{aligned}$$

Which implies that for each $z \in B_q$, $|\Phi_2 z\|_b^2 \leq r$.

Next, we establish the compactness of Φ_2. We employ the Arzela-Ascoli theorem to show the set $V(t) = \{(\Phi_2 z)(t), z \in B_q\}$ is relatively compact in $\mathbb{H}$. Le $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon \leq t$. For $\delta > 0$, for $z \in B_q$, We define

$$
\begin{aligned}
(\Phi_2^{\epsilon,\delta} z)(t) &= \alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ \alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds \\
&= S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds
\end{aligned}
$$

Since $S(t), t>0$, is a compact operator, the set $V_{\epsilon,\delta}=\{\Phi_2^{\epsilon,\delta}(t), z\in B_q\}$ is relatively compact in $\mathbb{H}$ for every $\epsilon\in(0,t)$, $\delta>0$. Moreover, for each $z\in B_q$, we have

$$
\begin{aligned}
& E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta}z)(t)\right\|_{\mathbb{H}}^2 \\
&\leq 4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)dt\theta ds\right\|_{\mathbb{H}}^2 \\
&+ 4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)d\theta ds\right\|_{\mathbb{H}}^2 \\
&+ 4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2 \\
&+ 4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\infty^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2 \\
&\leq 4M^2b^{2\alpha}M_f(1+q^{'})\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2+\frac{4M^2\epsilon^{2\alpha}M_f(1+q^{'})}{\Gamma^2(1+\alpha)} \\
&+ 4\alpha M^2b^\alpha\int_0^t(t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2 \\
&+ \frac{4\alpha M^2\epsilon^\alpha}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t(t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds
\end{aligned}
$$

where we have used the equality (see [22, 29])

$$
\int_0^\infty \theta^\varsigma\eta_\alpha(\theta)=\frac{\Gamma(1+\varsigma)}{\Gamma(1+\alpha\varsigma)}
$$

We see that for each $z\in B_q$

$$
E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta})\right\|_{\mathbb{H}}^2 \to 0 \text{ as } \epsilon^+\to 0, \delta\longrightarrow 0.
$$

Since the right-hand side of the above inequality can be made arbitrarily small, there is relatively compact $V_{\epsilon,\delta}$ arbitrarily close to the set $V(t)$. Hence, the set $V(t)$ is relatively compact in B_q. It remains to showt hat Φ_2 maps is bounded set into equicontinuous sets of $\mathcal{C}_v^{''}$.
Let $0<\epsilon<t<b$ and $\delta>0$ such that $\|T_\alpha(s_1)-T_\alpha(s_2)\|\leq\epsilon$, for every $s_1,s_2\in J$.

with $|s_1 - s_2| < \delta$. For $z \in B_q$, we have

$$
\begin{aligned}
& E\left\|\Phi_2 z(t+h) - \Phi_2 z(t)\right\|_{\mathbb{H}}^2 \\
& \leq 6E\left\|\int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T_\alpha(t+h-s) f(s, \tilde{\phi}_s + z_s) ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} T_\alpha(t+h-s) f(s, \tilde{\phi}_s + z_s) ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_0^t (t-s)^{\alpha-1} [T_\alpha(t+h-s) - T_\alpha(t-s)] f(s, \tilde{\phi}_s + z_s) ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T_\alpha(t+h-s) [\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) ds]\right\|_{\mathbb{H}}^2
\end{aligned}
$$

$$
\begin{aligned}
& + 6E\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} T_\alpha(t+h-s) [\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) ds]\right\|_{\mathbb{H}}^2. \\
& + 6E\left\|\int_0^t (t-s)^{\alpha-1} [T_\alpha(t+h-s) - T_\alpha(t-s)] [\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) ds]\right\|_{\mathbb{H}}^2 \\
& \leq 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 M_f(1+q^{'}) ds \\
& + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_t^{t+h} \left|(t+h-s)^{\alpha-1}\right|^2 M_f(1+q^{'}) ds \\
& + 6\epsilon^2 \int_0^t \left|(t-s)^{\alpha-1}\right|^2 M_f(1+q^{'}) ds + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 \\
& \times (2M_k + 2Tr(Q) m(s) M_\sigma(q^{'})) ds \\
& + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1}\right|^2 (2M_k + 2Tr(Q) m(s) M_\sigma(q^{'})) ds \\
& + 6\epsilon^2 \int_0^t \left|(t-s)^{\alpha-1}\right| (2M_k + 2Tr(Q) m(s) M_\sigma(q^{'})) ds
\end{aligned}
$$

It is known that the compactness of $T_\alpha(t), t > 0$ implies the continuity in the uniform operator topology. Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $h \to 0$. Thus, the set $\{\Phi_2 z, z \in B_q\}$ is equicontinuous.
This completes the proof that Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$
G = \{x \in \mathbb{H} : \lambda \Phi_1(\frac{x}{\lambda}) + \lambda \Phi_2 x = x\} \text{ is bounded for } \lambda \in (0,1)
$$

.

We consider the following nonlinear operator equation,

$$
\begin{aligned}
x(t) &= \lambda \left(S_\alpha(t)[\phi(0) - \mu(x) - h(0, \phi)]\right) + \lambda h(t, x_t) \\
&+ \lambda \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\
&+ \lambda \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau)\right] ds
\end{aligned}
$$

$$
\begin{aligned}
E\left\|x(t)\right\|^2 &\leq E\left\|S_\alpha(t)(\phi(0)-\mu(x)-h(0,\phi))\right\|_{\mathbb{H}}^2 + 4\left\|h(t,x_t)\right\|_{\mathbb{H}}^2 \\
&+ 4E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds\right\|_{\mathbb{H}}^2 \\
&+ 4E\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds\right\|_{\mathbb{H}}^2 \\
&\leq 12M^2(C_1+C_2+M_\mu) + 12M^2(1+\|x\|_{\mathcal{C}_\upsilon}^2) \\
&+ 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}M_f(1+\|x\|_{\mathcal{C}_\upsilon}^2) \\
&+ 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k+2Tr(Q)m(s)M_\sigma(\|x_s\|_{\mathcal{C}_\upsilon}^2)ds
\end{aligned}
$$

Now, we consider the function ν defined by

$$\vartheta(t) = \sup\{E\left\|x(s)\right\|^2, 0 \leq s \leq t\}, 0 \leq t \leq b$$

From lemma [3.4] and the above inequality, we have

$$E\left\|x(t)\right\|^2 = 2\left\|\phi\right\|_{\mathcal{C}_\upsilon}^2 + 2l^2 \sup_{0\leq s\leq t}(E\left\|x(s)\right\|^2)$$

Therefore, we get

$$
\begin{aligned}
\vartheta(t) \leq 2\left\|\phi\right\|_{\mathcal{C}_\upsilon}^2 + 2l^2\{\bar{F} + 12M^2\vartheta(t) + 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}M_f\vartheta(t) \\
+ 8b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}Tr(Q)m(s)M_\sigma(\vartheta(s))ds\}
\end{aligned}
$$

where $\bar{F}$ is given in (3.6). Thus, we have

$$\vartheta(t) \leq K_1 + K_2\int_0^t \frac{\vartheta(s)}{(t-s)^{1-\alpha}}ds + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds$$

where K_1, K_2, K_3 are given in (3.5). By Lemma [2.2], we have

$$\vartheta \leq B_0(K_1 + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds)$$

Where

$$B_0 = e^{K_2^n(\Gamma(\alpha))^n b^{n\alpha}/\Gamma(n\alpha)}\sum_{j=0}^{n-1}\left(\frac{K_2b^\alpha}{\alpha}\right)^j$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = B_0K_1$

$$\grave{\nu}(t) \leq B_0K_3m(t)M_\sigma\vartheta(t)$$

$$\grave{\nu}(t) \leq B_0K_3m(t)M_\sigma(\vartheta(t))$$

This implies

$$\int_{\nu(0)}^{\nu(t)}\frac{ds}{M_\sigma(s)} \leq \int_0^b \pi(s)ds < \int_{B_0K_1}^\infty \frac{ds}{M_\sigma(s)}$$

This inequality implies that there is a constant ρ such that $\nu(t) \leq \rho$, $t \in J$ and hence $\vartheta(t) \leq \rho$, $t \in J$. Furthermore, we get $\|x_t\|_{\mathcal{C}_\upsilon}^2 \leq \vartheta(t) \leq \nu(t) \leq \rho$, $t \in J$, where ρ depends only on b and on the functions $\pi(s)$ and $M_\sigma(s)$.

Theorem 3.3. *Assume that the hypotheses $(H_1)-(H_5)$ hold. Then problem has at least one mild solution on J.*

Proof. Let us take the set

$$D(\Phi) = \{z \in \mathcal{C}_v^{''} : z = \lambda\Phi_1(\frac{z}{x}) + \lambda\Phi_2 z \text{ for some } \lambda \in [0,1]\} \tag{3.9}$$

Then, for any $z \in D(\Phi)$, we have by theorem ... that $\|x\|^2_{\mathcal{C}_v} \leq K, t \in J$, and hence

$$\begin{aligned}
\|z\|_b^2 &= \|z_0\|^2_{\mathcal{C}_v} + \sup\{E\|z(t)\|^2 ; 0 \leq t \leq b\} \\
&= \sup\{E\|z(t)\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{E\|x(t)\|^2 : 0 \leq t \leq b\} + \sup\{E\left\|\tilde{\phi}(t)\right\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{l^-\|x(t)\|^2_{\mathcal{C}_v} : 0 \leq t \leq b\} + \sup\{\|s_\alpha(t)\phi(0)\| : 0 \leq t \leq b\} \\
&\leq l^-\rho + M_1\|\phi(0)\|^2
\end{aligned}$$

□

This implies that D is bounded on J. Consequently by Lemma 2.1 , the operator Φ has a fixed point $z \in \mathcal{C}_h^{''}$. So Eq.(1.1) has a mild solution.Theorem is proved.

Example 3.1. *As an application of the above result,consider the following fractional order neutral stochastic partial differential system with non local conditions and infinite delay in Hilbert space.*

$$\begin{cases}
{}^cD_t^\alpha[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] = \frac{\partial^2}{\partial x^2}[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] + \eta(t,x) \\
+\int_{-\infty}^0 \hat{a}(s)\sin z(t+s,x)ds + \int_{-\infty}^t\int_{-\infty}^t \sigma(t,x,s-t)dsd\beta(s,x) \quad t \in J = [0,b] \\
z(t,0) = z(t,\pi) = 0 \quad t \in J \\
z(0,x) + \int_0^\pi k_1(x,y)z(t,y)dy = x_0 = \varphi(t,x) \quad t \in (-\infty,0],
\end{cases} \tag{3.10}$$

Where ${}^cD^\alpha$ is a Caputo fractional partial derivative of order $\alpha \in (0,1)$, and $K_1(x,y) \in \mathbb{H} = L^2([0,\pi]\times[0,\pi])$ and $\int_{-\infty}^0 |\hat{a}(s)|\,ds < +\infty$. $\beta(t)$ is a one-dimensional standard Wiener process on filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. To rewrite this system into the abstract form (1.1), let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|.\|$. Define $A : \mathbb{H} \to \mathbb{H}$ by $A(t)z = z^{''}$ with the domain

$$\mathcal{D}(A) = \left\{z \in \mathbb{H} : z, z^{'} \text{ are absolutely continuous }, z^{''} \in \mathbb{H}, z(0) = z(\pi) = 0\right\}$$

It is well known that A generates a strongly continuous semigroup T(.),which is compact,analytic and self adjoint. Then

$$Az = \sum_{n=1}^\infty n^2\langle z, z_n\rangle z_n, \quad z \in D(A)$$

where $z_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns), n = 1,2,....$ is the orthonormal set of eigenvector of A. It is well known that A is thein infinitesimal generator of ananalytic semigroup T(t)in H and is given by

$$T(t)z = \sum_{n=1}^\infty e^{-n^2t}\langle z, z_n\rangle z_n$$

Then the operator $A^{-\frac{1}{2}}$ is given by

$$A^{-\frac{1}{2}}z = \sum_{n=1}^\infty n\langle z, z_n\rangle z_n$$

on the space $\mathcal{D}(A^{-\frac{1}{2}}) = \{z(.) \in \mathbb{H} : \sum_{n=1}^\infty n\langle \zeta, z_n\rangle z_n \in \mathbb{H}\}$.

Now, we present a special $\mathcal{C}_\upsilon$ space. Let $\vartheta(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} \vartheta(s)ds = \frac{1}{2}$.
Let

$$\|\varphi\|_{\mathcal{C}_\upsilon} = \int_{-\infty}^{0} h(s) \sup_{s\le\theta\le 0} E\left(\|\varphi(\theta)\|^2\right)^{\frac{1}{2}} ds$$

Then $(\mathcal{C}_\upsilon, \|.\|_{\mathcal{C}_\upsilon})$ is a Banach space.

For $(t, \varphi) \in J \times \mathcal{C}_\upsilon$ where $\varphi(\theta)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$, and define the Lipschitz continuous functions $h, f : J \times \mathcal{C}_\upsilon \to H$, $\sigma : J \times \mathcal{C}_\upsilon \to L_Q(\mathbb{H})$, for the infinite delay as follows

$$h(t, \varphi)(x) = \int_{-\infty}^{0} e^{-4\theta}\varphi(\theta)(x)d\theta$$

$$f(t, \varphi)(x) = \int_{-\infty}^{0} \hat{a}(\theta)\sin(\varphi(\theta)(x))d\theta$$

$$\sigma(t, \varphi)(x) = \int_{-\infty}^{0} \varsigma(t, x, \theta)\sigma(\varphi(\theta)(x))d\theta$$

Then, the equation (3.10) can be rewritten as the abstract form as the system 1.1. Thus, under the appropriate condition so the functions h,f, and σ are satisfies the hypotheses $(H_1) - (H_5)$. All conditions of the Theorem 3.2 are satisfied, therefore the system (3.10) has a mild solution.

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