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Radial symmetry of positive solutions for nonlinear elliptic boundary value problems

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Abstract

The aim of this paper is to study the symmetry properties of positive solutions of nonlinear elliptic boundary value problems of type

$$\Delta u + f(|x|, u, \nabla u) = 0 \text{ in } R^n.$$

$$u(x) \to 0 \text{ as } |x| \to \infty$$

We employ the moving plane method based on maximum principle on unbounded domains to obtain the result on symmetry.

Keywords: Maximum principle; Moving plane method; Semilinear elliptic boundary value problems.

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1 Introduction

The moving plane method is a clever way of using the maximum principle to obtain the qualitative properties of positive solutions of some elliptic equations, notably the symmetry of solutions. It was introduced by Alexandroff [6] in his study of surfaces of constant mean curvature. In 1971, Serrin [13] first proved the symmetry properties of some overdetermined elliptic problems. It has become welknown through the work of Gidas, Ni and Nirenberg [3],[4] where it was used to obtain the symmetry results for positive solutions of nonlinear elliptic equations. Since then, this method has been further developed and used in variety of problems by many researchers. Pucci, Sciunzi and Serrin [12] studied symmetry of solutions of degenerate quasilinear elliptic problems by applying comparison principle. Farina, Montoro and Sciunzi [2] obtained symmetry results for semilinear p-Laplacian equation. In this paper we present an approach based on the maximum principle in unbounded domains together with the method of moving planes. Recently Dhaigude and Patil [1] proved the symmetry result for same equation in unit ball. Naito [10] obtained symmetry result for semilinear elliptic equations in R^2. Further Naito [9] studied the semilinear elliptic problem $\Delta u + f(|x|, u) = 0$ in R^n where $u(x) \to 0$ as $|x| \to \infty$.

In this paper we study the radial symmetry of positive solutions for nonlinear elliptic boundary value problems for second order elliptic equations in R^n. We consider the problem of the form

$$\Delta u + f(|x|, u, \nabla u) = 0 \text{ in } R^n \tag{1.1}$$

$$u(x) \to 0 \text{ as } |x| \to \infty$$

where $n \geq 3$. We organise the paper as follows: In section 2 the preliminary results and some useful lemmas are proved. The symmetry result and corollaries are proved in the last section.

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2 Preliminaries

In this section, first we state some lemmas and theorem which are useful to prove our main result.

Lemma 2.1. *Hopf Boundary lemma[5] : Let Ω be closed subset of R^n. Suppose that Ω satisfies the interior sphere condition at $x_0 \in \partial\Omega$. Let L be strictly elliptic with $c \leq 0$ where*

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} + b_i(x)\sum_{j=1}^{n}\frac{\partial}{\partial x_j} + c(x)$$

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $L(u) \geq 0$ and $\max_{\bar{\Omega}} u(x) = u(x_0)$ then either $u = u(x_0)$ on Ω or

$$\liminf_{t\to 0}\frac{u(x) - u(x_0 + t\nu)}{t} > 0$$

for every direction ν, pointing into an interior sphere. If $u \in C^1 \subset \Omega \cup \{0\}$ then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.

Lemma 2.2. *[8] Let Ω be unbounded domain in R^n . Suppose that $u \neq 0$ satisfies*

$$L(u) \leq 0 \quad in \quad \Omega \quad and \quad u \geq 0 \quad on \quad \partial\Omega.$$

Suppose furthermore that there exist a function w such that $w > 0$ on $\Omega \cup \partial\Omega$ and $\quad L(w) \leq 0 \quad in \quad \Omega$. If

$$\frac{u(x)}{w(x)} \to 0 \quad as \quad |x| \to \infty, \quad x \in \Omega$$

then $u > 0 \quad in \quad \Omega$.

Theorem 2.1. *[11] Let $u(x)$ satisfies differential inequality*

$$(L)(u) \geq 0,$$

in a domain Ω where L is uniformly elliptic. If there exist a function w(x) such that, $w(x) > 0 \quad on \quad \Omega \cup \partial\Omega$

$$(L)(w) \leq 0 \quad in \quad \Omega$$

then $\frac{u(x)}{w(x)}$ can not attain a non negative maximum at a point p on $\partial\Omega$, which lies on the boundary of a ball in Ω and if $\frac{u}{w}$ is not constant then,

$$\frac{\partial}{\partial \nu}\left(\frac{u}{v}\right) > 0 \ \ at\ P$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.

3 Main Results

We define following,

Let $\lambda > 0$ a real number. Define the plane $T_\lambda = \{x : x = (x_1, x_2, x_3, ..., x_n), x_1 = \lambda\}$, which is the plane perpendicular to x_1-axis. We will move this plane continuously normal to itself to new position till it begins to intersect Ω. After that point the plane advances in Ω along x_1- axis and cut of cap Σ_λ; which is the portion of Ω and lies in the same side of the plane T_λ as the original plane T.

$\Sigma_\lambda = \{x : x_1 < \lambda, x \in \Omega\}$.

Let $x^\lambda = (2\lambda - x_1, x_2, x_3, ...x_n)$ be the reflection of the point $x = (x_1, x_2, x_3, ..., x_n)$, about the plane T_λ.

Define $V_\lambda(x) = u(x) - u(x^\lambda)$. We have $|x^\lambda| \geq |x|$ and $u(x^\lambda) = u(2\lambda - x_1, x_2, x_3, ..., x_n)$.

Define $\Lambda = \{\lambda \in (0, \infty) : V_\lambda(x) > 0\}$ for $x \in \Sigma_\lambda$.

In (1.1), we assume that $f(|x|, u(x), \nabla u(x))$ is continuous and C^1 in $u \geq 0$. Also assume that $f(|x|, u(x), \nabla u(x))$ is nonincreasing in $|x| = r > 0$, for each fixed $u \geq 0$.

Our main result is the following

Theorem 3.2. *Let $u \in C^2(R^n)$ be a positive solution of* (1.1) *with following conditions*

1. *f is continuous in all of its variables and Lipschitz in u*
2. $f(|x|, u, (p_1, p_2, p_3, ..., p_{i-1}, -p_i, p_{i+1}, ..., p_n)) = f(|x|, u, (p_1, p_2, p_3, ..., p_n))$ *for all* $1 \leq i \leq n$
3. *f is nonincreasing in* $|x| = r > 0$*, for each fixed* $u \geq 0$.

Define U and Φ *as*

$$U(r) = Sup\{u(x) : |x| \geq r\} \tag{3.1}$$

$$\Phi(r) = Sup\Big\{\frac{\partial f}{\partial u}(|x|, u(x), \nabla u(x)) : 0 \leq u(x) \leq U(r)\Big\} \tag{3.2}$$

respectively. Assume that there exist a positive function w on $|x| \geq R_0$ *for some* $R_0 > 0$ *satisfying*

$$\Delta w + \phi(|x|)w \leq 0 \ in \ |x| > R_0 \tag{3.3}$$

$$\lim_{|x|\to\infty} \frac{u(|x|)}{w(x)} = 0, \tag{3.4}$$

then u must be radially symmetric about some point $x_0 \in R^n$ *and* $u_r < 0$ *for* $r > 0$.

Before proceeding to the proof of main result we shall state and prove some lemmas.

Lemma 3.1. *Let* $\lambda \geq 0$ *then*

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \leq 0 \ in \ \Sigma_\lambda, \tag{3.5}$$

where

$$C_\lambda(x) = \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt.$$

Proof. Let u be the positive solution of (1.1), $u(x^\lambda)$ satisfies the same equation that u does.

$$\Delta u(x^\lambda) + f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda)) = 0 \ \text{ in } \ R^n. \tag{3.6}$$

Since $u(x) = u(x_1, x_2, x_3, ..., x_n)$

$$\begin{aligned}\nabla u(x) &= \hat{i_1}\frac{\partial u}{\partial x_1} + \hat{i_2}\frac{\partial u}{\partial x_2} + \hat{i_3}\frac{\partial u}{\partial x_3} + ... + \hat{i_n}\frac{\partial u}{\partial x_n} \\ &= (p_1, p_2, p_3, ..., p_n).\end{aligned}$$

Since $u(x^\lambda) = u(2\lambda - x_1, x_2, x_3, ..., x_n)$

$$\begin{aligned}\nabla u(x^\lambda) &= \hat{i_1}\frac{\partial u}{\partial x_1}(-1) + \hat{i_2}\frac{\partial u}{\partial x_2} + \hat{i_3}\frac{\partial u}{\partial x_3} + ... + \hat{i_n}\frac{\partial u}{\partial x_n} \\ &= (-p_1, p_2, p_3, ..., p_n).\end{aligned}$$

Subtracting equation (3.6) from equation (1.1) we get

$$\begin{aligned}0 &= [\Delta u(x) + f(|x|, u(x), \nabla u(x))] - [\Delta u(x^\lambda) + f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda))] \\ &= \Delta u(x) - \Delta u(x^\lambda) + f(|x|, u(x), \nabla u(x)) - f(|x^\lambda|, u(x^\lambda), \nabla u(x^\lambda)) \\ &= \Delta V_\lambda(x) + f(|x|, u(x), (p_1, p_2, p_3, ..., p_n)) - f(|x^\lambda|, u(x^\lambda), (-p_1, p_2, p_3, ..., p_n)) \\ &= \Delta V_\lambda(x) + f(|x|, u(x), (p_1, p_2, p_3, ..., p_n)) - f(|x^\lambda|, u(x^\lambda), (p_1, p_2, p_3, ..., p_n)) \\ &\geq \Delta V_\lambda(x) + f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x)) \\ &\geq \Delta V_\lambda(x) + \frac{f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x))}{u(x) - u(x^\lambda)}(u(x) - u(x^\lambda)) \\ &\geq \Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x))\end{aligned}$$

where

$$C_\lambda(x) = \frac{f(|x|, u(x), \nabla u(x)) - f(|x|, u(x^\lambda), \nabla u(x))}{u(x) - u(x^\lambda)}$$
$$= \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt.$$

□

Before the next lemma we shall define, $B_0 = \{x \in R^n : |x| < R_0\}$ and $\bar{B_0} = \{x \in R^n : |x| \leq R_0\}$.

Lemma 3.2. *Let $\lambda > 0$, If $V_\lambda > 0$ on $\partial\Sigma_\lambda \cap \bar{B_0}$ then $\lambda \in \Lambda$.*

Proof. Let $\lambda > 0$. Suppose $V_\lambda > 0$ on $\partial\Sigma_\lambda \cap \bar{B_0}$ then from lemma 3.1 and assumption we have

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \leq 0 \text{ in } \Sigma_\lambda \setminus \bar{B_0}$$
$$V_\lambda(x) \geq 0 \text{ on } \partial(\Sigma_\lambda \setminus \bar{B_0})$$

As $U(r) = sup\{u(x) : |x| \geq r\}$ and $\Phi(r) = sup\{f_u(|x|, u(x), \nabla u(x))\}$
$U(r)$ is nonincreasing,
$0 < u(x) + t(u(x^\lambda) - u(x)) \leq u(|x|), \ \ 0 \leq t \leq 1.$
Then by lemma 3.1,

$$C_\lambda(x) = \int_0^1 f_u(|x|, u(x) + t(u(x^\lambda) - u(x)), \nabla u(x))dt$$
$$\leq \int_0^1 f_u(|x|, U(x), \nabla u(x))dt$$
$$\leq \int_0^1 \Phi(|x|)dt \leq \Phi(|x|) \text{ in } \Sigma_\lambda.$$

From

$$\Delta w + \phi(|x|)w \geq 0 \text{ in } |x| \geq R_0$$
$$\lim_{|x|\to\infty} \frac{u(|x|)}{w(x)} = 0.$$

The positive function w satisfies

$$\Delta w + \phi(|x|)w \leq 0 \text{ in } \Sigma_\lambda \setminus \bar{B_0}$$

and

$$\frac{V_\lambda(x)}{w(x)} \leq \frac{U(|x|)}{w(x)} \to 0 \text{ in } x \in \Sigma_\lambda \setminus \bar{B_0} \text{ as } |x| \to \infty.$$

Hence by maximum principle, $V_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \bar{B_0}$.
By assumption $V_\lambda(x) > 0$ in Σ_λ. Therefore $\lambda \in \Lambda$. □

Lemma 3.3. *If $\lambda \in \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$ on T_λ*

Proof. Let $\lambda \in \Lambda$. Hence $\lambda > 0$. By lemma 3.1

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \leq 0 \text{ in } \Sigma_\lambda$$
$$V_\lambda(x) \geq 0 \text{ on } \partial(\Sigma_\lambda).$$

On T_λ we have,

$$u(x) = u(x^\lambda).$$

Hence $V_\lambda(x) = 0$ on T_λ.
By Hopf boundary lemma, $\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ. Therefore $\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ □

Now we shall prove the main theorem 3.2

Proof. Since $u(x)$ is positive solution of (1.1) such that

$$\lim_{|x|\to\infty} u(x) = 0,$$

then we can find $R_1 > R_0$ such that

$$\max\{u(x) : |x| > R_1\} < \min\{u(x) : |x| \le R_0\}$$

where R_0 is constant. We shall prove the theorem in following three steps.
Step-I: Define $\bar{B_0} = \{x \in R^n : |x| < R_0\}$. Clearly $\bar{B_0} \subset \Sigma_{\bar{\lambda}}$. Let $\lambda \ge R_1$. Also $V_\lambda(x) \ge 0$ in $\bar{B_0}$. Therefore $V_\lambda(x) \ge 0$ in $\Sigma_\lambda \cap \bar{B_0}$. Hence $\lambda \in \Lambda$. Thus we can conclude that $[R_1, \infty) \subset \Lambda$.
Step-II: To prove,If $\lambda_0 \in \Lambda$ then there exist $\epsilon > 0$ such that $(\lambda_0 - \epsilon, \lambda_0) \subset \Lambda$. We use contradiction method to prove this. Suppose there exist increasing sequence $\{\lambda_i\}, i = 1, 2, 3, ...$ such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$ then by contradiction to lemma 3.2 we have a sequence $\{x_i\}, i = 1, 2, 3, ...$ such that $x_i \in \Sigma_{\lambda_i} \cap \bar{B_0}$ and $V_{\lambda_i}(x_i) \le 0$. It has a subsequence which converges to $x_0 \in \bar{\Sigma_{\lambda_0}} \cap \bar{B_0}$. Then $V_{\lambda_0}(x_0) \le 0$ but in Σ_{λ_0} we have $V_{\lambda_0}(x_0) > 0$, therefore $x_0 \in T_{\lambda_0}$.
Using mean value theorem we can find y_i satisfying

$$\frac{\partial u}{\partial x_i}(y_i) \ge 0$$

on the line segment joining $x_i \to x_i^{\lambda_i}$ for each $i = 1, 2, 3, ...$ also $y_i \to x_0$ as $i \to \infty$. So $\frac{\partial u}{\partial x_1}(x_0) \ge 0$. But by lemma 3.3 we have

$$\frac{\partial u}{\partial x_i}(x_0) \le 0.$$

This is a contradiction.Hence the step-II is proved.
Thus if $\lambda_0 \in \Lambda$ then there exist $\epsilon > 0$ such that $(\lambda_0 - \epsilon, \lambda_0) \subset \Lambda$.
Step-III: In this step we shall prove that either statement (A) or statement (B) happens.
(A)$V_{\lambda_1}(x) = 0$ for some $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$
or
(B)$V_{\lambda_1}(x) > 0$ in Σ_{λ_0} and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_0$
We have $\lambda_1 = \inf\{\lambda > 0 | (\lambda, \infty) \subset \Lambda\}$. Therefore $\lambda_1 > 0$ or $\lambda_1 = 0$ be the two distinct cases.
Case - 1: If $\lambda_1 > 0$, since u is continuous function, $V_{\lambda_1} \ge 0$ in Σ_{λ_1}. Therefore by lemma 3.1

$$\Delta V_\lambda(x) + C_\lambda(x) V_\lambda(x) \le 0 \ \ in \ \ \Sigma_{\lambda_1}$$

By strong maximum principle we have either $V_\lambda(x) \ge 0$ or $V_\lambda(x) = 0$ in Σ_{λ_1}. If $V_\lambda(x) = 0$, then statement (A) occurs, i.e. $u(x) = u(x^{\lambda_1})$ holds and by lemma 3.3 $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$. Now suppose $V_{\lambda_1}(x) > 0$ in Σ_{λ_1}, then $\lambda_1 \in \Lambda$ As we have already proved, there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon, \lambda_1) \subset \Lambda$. This contradicts to the fact that λ_1 is infimum. Therefore $V_{\lambda_1}(x) = 0$ in Σ_λ. This implies (A) holds.
Case - 2: $\lambda_1 = 0$
Since $u(x)$ is continuous we have $V_{\lambda_1}(x) > 0$ in Σ_0.
Therefore $V_{\lambda_1}(x) = u(x) - u(x^0) \ge 0$ in Σ_0. Therefore $u(x) \ge u(x^0)$ in Σ_0.
By lemma 3.3 $\frac{\partial u}{\partial x_1} < 0$ on T_λ for$\lambda > 0$. Thus (B) holds. If statement (B) occurs in step- III, we can repeat the previous steps I-III for the negative x_1- direction to conclude that either u is symmetric in the x_1- direction about some plane $x_1 = \lambda_1 < 0$ or $u(x) \le u(x^0)$ in Σ_0. Thus $u(x) = u(x^0)$ in Σ_0. Thus u must be radially symmetric in x_1- direction about some plane and strictly decreasing away from the plane. Since we can place x_1- axis along any direction we conclude that $u(x)$ is radially symmetric in every direction about some plane. Therefore u is radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$. □

Corollary 3.1. *Assume that $f_u(r, u, \nabla u) \le 0$ for $r \ge r_0, 0 \le u \le u_0$ with some constants $r_0 \ge 0$ and $u_0 \ge 0$. Let u be the positive solution of* (1.1). *Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for* $r = |x - x_0| > 0$.

Remark 3.1. *. If gradient term is absent in f, related results have been obtained in [4, 7, 9].*

Proof. We see that the function U defined by (3.1) satisfies $U(r) \to 0$ as $|x| \to \infty$.

Take $R_0 > r_0$ so large that $U(r) < u_0$ for $r \geq R_0$. Define w as $w(x) \equiv 1$ on $|x| > R_0$, then w satisfies (3.4). Since $\Phi(r) = \max\{\frac{\partial f}{\partial u}(|x|, u(x), \nabla u(x)) | 0 \leq u(x) \leq U(r)\} \leq 0$ for $r \geq R_0$ we have (3.3). Thus all the conditions of theorem are satisfied so we can apply the theorem for conclusion. □

For simplicity we consider the equation of the form

$$\Delta u + \phi(|x|) f(u(x), \nabla u(x)) = 0 \text{ in } R^n \tag{3.7}$$

with the assumption that $\phi \in C[0, \infty)$ satisfies $\phi(r) \geq 0$ for $r \geq 0$ and $\phi(r)$ is nonincreasing in $r > 0$, and that $f \in C^1[0, \infty)$ with $f(u, \nabla u)) > 0$ for $u > 0$.

Corollary 3.2. *In equation* (3.7)*,suppose that*

1. *f is continuous in all of its variables and Lipschitz in u*
2. $f(|x|, u, p_1, p_2, p_3, ..., p_{i-1}, -p_i, p_{i+1}, ..., p_n) = f(|x|, u, p_1, p_2, p_3, ..., p_n)$ *for all* $1 \leq i \leq n$
3. *f is nonincreasing in $|x|$, for each fixed $u \geq 0$*

we furthermore assume that $\phi \not\equiv 0$ and

$$\int_0^\infty r\phi(r)dr < \infty. \tag{3.8}$$

Let u be positive solution of (3.7) *, satisfying $u(x) \to c$ as $|x| \to \infty$ for some constant $C \geq 0$, then u must be radially symmetric about the origin and $u_r < 0$ for $r > 0$.*

Remark 3.2. *. If gradient term is absent in f, related results have been obtained by [4, 7, 9].*

Proof. Consider $V(x) = u(x) - C$. Then we have $\nabla V(x) = \nabla u(x)$ and hence $\Delta V(x) = \Delta u(x)$. Then V satisfies

$$\Delta V(x) + \phi(|x|)h(V) = 0 \text{ in } R^n$$
$$V(x) \to 0 \text{ as } |x| \to \infty$$

where $h(V) = f(V + C, \nabla V)$

Since $-\Delta V = \phi h \geq 0$, we have $V > 0$ in R^n, by the maximum principle. We apply theorem 3.2 to the problem (3.7). We define U and Φ as $U(r) = sup\{V(x) : |x| \geq r\}$ and $\Phi(r) = sup\{\phi(r)h'(s) : 0 < s \leq U(r)\}$ respectively. Since $\Phi(r) \leq M\phi(r)$ for some constant $M > 0$. and (3.8) holds. Then there exist a positive function w on $|x| > R_0$ for some $R_0 > 0$ satisfying

$$\Delta w(x) + \phi(|x|)w(x) = 0 \text{ and}$$
$$w(x) \to 0 \text{ as } |x| \to \infty$$

Then w satisfies the conditions of the theorem 3.2. Therefore theorem 3.2 can be applied to conclude the assertion. □

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