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# Some curvature properties of ($\kappa$, $\mu$) contact space forms

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**Abstract**

The object of the present paper is to find Ricci tensor of $(k, \mu)$ space forms. In particular we prove that a three dimensional $(k, \mu)$ space forms is $\eta-$Einstein for $\mu = \frac{1}{2}$. We also study three dimensional $(k, \mu)$ space forms with $\eta-$ parallel and cyclic parallel Ricci tensor for $\mu = \frac{1}{2}$. We also prove that every $(k, \mu)$ space forms is locally $\phi-$ symmetric.

*Keywords:* $(k, \mu)$ contact space forms, $\eta-$Einstein, $\eta-$ parallel and cyclic parallel Ricci tensor, locally $\phi-$ symmetric.

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## 1 Introduction

Now a days, a good number of contact geometers have worked on $(k, \mu)$ contact metric manifold. The notion of $(k, \mu)$ contact metric manifold was introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [2]. The notion of $(k, \mu)$ space forms was introduced by T. Koufogiorgos [8]. The Ricci curvature and the Riemannian curvature are two key objects regarding symmetry of a manifolds. The notion of local symmetry has been weakened by several authors in several ways. As a weaker version of local symmetry T. Takahashi [10] introduced the notion of local $\phi-$ symmetry in Sasakian manifolds. The notion of $\eta-$ parallel and cyclic parallel Ricci tensor was introduced in the paper [7] and [9]. In this regard we mention that $\eta-$ parallel and cyclic parallel Ricci tensor have been studied by the present authors in the paper[1]. Again $\eta-$ parallel and cyclic parallel Ricci tensor was studied by the authors in the paper [5]. The present paper is organized by the following way:
After introduction in Section 1 we give some preliminaries in Section 2. In Section 3 we study Ricci tensor of $(k, \mu)$ space forms. $\eta$-parallel, cyclic parallel Ricci tensors and Ricci operator of $(k, \mu)$ space forms of dimension three have been studied in Section 4. In Section 5 we have proved that every (2n+1) dimensional $(k, \mu)$ space forms is locally $\phi$- symmetric.

## 2 Preliminaries

A differentiable manifold $M^{2n+1}$ is said to be a contact manifold if it admits a global differentiable 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$, everywhere on $M^{2n+1}$.
Given a contact form $\eta$, one has a unique vectoe field, satisfying

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0, \tag{2.1}$$

for any vector field $X$.
It is well-known that, there exists a Riemannian metric $g$ and a (1,1) tensor field $\phi$ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.2}$$

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where $X$ and $Y$ are vector fields on $M$.
From (2.2) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

A differentiable manifold $M^{2n+1}$ equipped with the structure tensors ($\phi$, $\xi$, $\eta$, $g$) satisfying (2.3) is said to be a contact metric manifold.
On a contact metric manifold $M(\phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2}L_{\xi}\phi$, where $L$ denotes Lie differentiation. Then we may observe that $h$ is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \quad \nabla_X \xi = -\phi X - \phi h X \tag{2.4}$$

where $\nabla$ is levi-Civita connection[2].
For a contact metric manifold $M$ one may define naturally an almost complex structure on the product $M \times \mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. A Sasakian manifold is characterized by the condition

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

for all vector fields $X$ and $Y$ on the manifold [4]. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all $X, Y$ on $M$ [4].
For a contact manifold we have [3]

$$(\nabla_X h)(Y) = \{(1-k)g(X, \phi Y) + g(X, h\phi Y)\}\xi + \eta(Y)\{h(\phi X + \phi h X)\}. \tag{2.6}$$

The $(k, \mu)-$nullity distribution of a contact metric manifold $M(\phi, \xi, \eta, g)$ is a distribution [8]

$$\begin{aligned} N(k,\mu) \quad &: \quad p \to N_p(k,\mu) \\ &= \quad \{Z \in T_p(M) : R(X,Y)Z \\ &= \quad k(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)hX - g(X,Z)hY)\}, \end{aligned} \tag{2.7}$$

for any $X, Y \in T_pM$ and $\kappa, \mu \in R$. If $k = 1$, then $h = 0$ and $M$ is a Sasakian manifold [8]. Also one has $trh = 0$, $trh\phi = 0$ and $h^2 = (k-1)\phi^2$. So if the characteristic vector field $\xi$ belongs to the $(k, \mu)-$nullity distribution, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.8}$$

Moreover, if $M$ has constant $\phi-$sectional curvature $c$ then it is called a ($k$, $\mu$) space forms and is denoted by $M(c)$.
The curvature tensor of $M(c)$ is given by[8]:

$$\begin{aligned} 4R(X,Y)Z \quad &= \quad (c+3)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \quad (c+3-4k)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \quad g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &+ \quad (c-1)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &- \quad 2\{g(hX,Z)hY - g(hY,Z)hX + g(X,Z)hY \\ &- \quad 2g(Y,Z)hX - 2\eta(X)\eta(Z)hY + 2\eta(Y)\eta(Z)hX \\ &+ \quad 2g(hX,Z)Y - 2g(hY,Z)X + 2g(hY,Z)\eta(X)\xi \\ &- \quad 2g(hX,Z)\eta(Y)\xi - g(\phi hX,Z)\phi hY + g(\phi hY,Z)\phi hX\} \\ &+ \quad 4\mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY \\ &+ \quad g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi, \end{aligned} \tag{2.9}$$

for any vector fields $X, Y, Z$ on $M$.
If $k \neq 1$, then $\mu = \kappa + 1$ and $c = -2k - 1$.

**Definition 2.1.** *If an almost contact Riemannian manifold $M$ satisfies the condition $S = ag + b\eta \otimes \eta$, for some functions $a$, $b$ in $C^{\infty}(M)$ and $S$ is the Ricci tensor, then $M$ is said to be an $\eta$-Einstein manifold. If, in particular, a=0 then this manifold will be called a special type of $\eta$-Einstein manifold.*

## 3 Ricci tensor of $(\kappa, \mu)$ space forms

In this section we study Ricci tensor of $(k, \mu)$ space forms. Taking inner product on both side of (2.9) with $W$ we obtain

$$\begin{aligned}
& 4g(R(X,Y)Z,W) \\
= & (c+3)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
+ & (c+3-4k)\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
+ & g(X,Z)\eta(Y)\eta(\xi,W) - g(Y,Z)\eta(X)g(\xi,W)\} \\
+ & (c-1)\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\} \\
- & 2\{g(hX,Z)g(hY,W) - g(hY,Z)g(hX,W) + g(X,Z)g(hY,W) \\
- & 2g(Y,Z)g(hX,W) - 2\eta(X)\eta(Z)g(hY,W) + 2\eta(Y)\eta(Z)g(hX,W) \\
+ & 2g(hX,Z)g(Y,W) - 2g(hY,Z)g(X,W) + 2g(hY,Z)\eta(X)g(\xi,W) \\
- & 2g(hX,Z)\eta(Y)g(\xi,W) - g(\phi hX,Z)g(\phi hY,W) + g(\phi hY,Z)g(\phi hX,W)\} \\
+ & 4\mu\{\eta(Y)\eta(Z)g(hX,W) - \eta(X)\eta(Z)g(hY,W) \\
+ & g(hY,Z)\eta(X)g(\xi,W) - g(hX,Z)\eta(Y)g(\xi,W).
\end{aligned} \tag{3.1}$$

Putting $X = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i$, $i = 1, 2, ...., 2n+1$, we get from (3.1)

$$\begin{aligned}
& 4S(X,W) \\
= & (c+3)\{(2n+1)g(X,W) - g(X,W)\} \\
+ & (c+3-4k)\{\textstyle\sum_i \eta(X)\eta(e_i)g(e_i,W) - \sum_i \eta(e_i)\eta(e_i)g(X,W) \\
+ & \textstyle\sum_i g(X,e_i)\eta(e_i)\eta(\xi,W) - \sum_i g(e_i,e_i)\eta(X)g(\xi,W)\} \\
+ & (c-1)\{\textstyle\sum_i g(X,\phi e_i)g(\phi e_i,W) - \sum_i g(e_i,\phi e_i)g(\phi X,W) + 2\sum_i g(X,\phi e_i)g(\phi_i,W)\} \\
- & 2\textstyle\sum_i\{g(hX,e_i)g(he_i,W) - \sum_i g(he_i,e_i)g(hX,W) + \sum_i g(X,e_i)g(he_i,W) \\
- & 2\textstyle\sum_i g(e_i,e_i)g(hX,W) - 2\sum_i \eta(X)\eta(e_i)g(he_i,W) + 2\sum_i \eta(e_i)\eta(e_i)g(hX,W) \\
+ & 2\textstyle\sum_i g(hX,e_i)g(e_i,W) - 2\sum_i g(he_i,e_i)g(X,W) + 2\sum_i g(he_i,e_i)\eta(X)g(\xi,W) \\
- & 2\textstyle\sum_i g(hX,e_i)\eta(e_i)g(\xi,W) - \sum_i g(\phi hX,e_i)g(\phi he_i,W) + \sum_i g(\phi he_i,e_i)g(\phi hX,W)\} \\
+ & 4\mu\{\textstyle\sum_i \eta(e_i)\eta(e_i)g(hX,W) - \sum_i \eta(X)\eta(e_i)g(he_i,W) \\
+ & \textstyle\sum_i g(he_i,e_i)\eta(X)g(\xi,W) - \sum_i g(hX,e_i)\eta(e_i)g(\xi,W).
\end{aligned} \tag{3.2}$$

or,

$$\begin{aligned}
& 4S(X,W) \\
= & 2n(c+3)g(X,W) \\
+ & (c+3-4k)\{\eta(X)\eta(W) - (2n+1)g(X,W) \\
+ & \eta(X)\eta(W) - (2n+1)\eta(X)\eta(W)\} \\
+ & (c-1)\{g(\phi X,\phi W) + 2g(\phi X,\phi W)\} \\
- & 2\{g(hX,hW) + g(X,hW) - 2(2n+1)g(hX,W) \\
- & \eta(X)\eta(hW) + 2(2n+1)g(hX,W) + 2g(hX,W) \\
- & \eta(hX)g(\xi,W) + g(\phi hX,h\phi W)\} \\
+ & 4\mu\{(2n+1)g(hX,W) - \eta(X)\eta(hW) - \eta(hX)g(\xi,W).
\end{aligned} \tag{3.3}$$

Since $h\xi = 0$, therefore $\eta(hX) = g(hX,\xi) = g(X,h\xi) = g(X,0) = 0$.
Using above result we obtain from (3.3)

$$\begin{aligned}
& 4S(X,W) \\
= & 2n(c+3)g(X,W) \\
+ & (c+3-4k)\{(1-2n)\eta(X)\eta(W) - (2n+1)g(X,W) \\
+ & 3(c-1)\{g(\phi X,\phi W)\} \\
- & 2\{g(h^2X,W) + 3g(X,hW) + g(\phi hX,h\phi W)\} \\
+ & 4\mu\{(2n+1)g(hX,W).
\end{aligned} \tag{3.4}$$

Using $h\phi = -\phi h$, $\phi^2(X) = -X + \eta(X)\xi$, in relation (3.4) we get

$$\begin{aligned} & 4S(X,W) \\ = \ & 2n(c+3)g(X,W) \\ + \ & (c+3-4k)\{(1-2n)\eta(X)\eta(W) - (2n+1)g(X,W) \\ + \ & 3(c-1)\{g(X,W) - \eta(X)\eta(W)\} \\ - \ & 2\{g(h^2X,W) + 3g(X,hW) - g(h^2X,hW)\} \\ + \ & 4\mu(2n+1)g(hX,W). \end{aligned} \tag{3.5}$$

or,

$$\begin{aligned} & 4S(X,W) \\ = \ & (8nk+4k+2c-3)g(X,W) + (8nk+4k-2nc-6n-2c+6)\eta(X)\eta(W) \\ + \ & \{4\mu(2n+1)-6\}g(hX,W). \end{aligned} \tag{3.6}$$

If we take $\mu = \frac{1}{2}$ and $n = 1$, then (3.6) becomes

$$4S(X,W) = (12k+2c-3)g(X,W) + (12k-4c)\eta(X)\eta(W) \tag{3.7}$$

i.e.

$$S(X,W) = (3k + \frac{c}{2} - \frac{3}{4})g(X,W) + (3k-c)\eta(X)\eta(W) \tag{3.8}$$

Thus we are in a position to state the following result:

**Theorem 3.1.** *A (k, μ) space forms of dimension three is η -Einstein for* $\mu = \frac{1}{2}$.
*Again we know that* $S(X,W) = g(QX,W)$, *where Q is the Ricci operator. Thus using this in (3.8) we get*

$$QX = (3k + \frac{c}{2} - \frac{3}{4})X + (3k-c)\eta(X)\xi, \tag{3.9}$$

*where Q is the Ricci operator of (k, μ) space forms of dimension three for* $\mu = \frac{1}{2}$. *Again we have from (3.8) that*

$$r = \sum_{i=1}^{3} S(e_i, e_i) \quad = 3(6k - \frac{c}{2} - \frac{3}{4}), \tag{3.10}$$

*where r is the scalar curvature of (k, μ) space forms of dimension three for* $\mu = \frac{1}{2}$.

## 4 $\eta$-parallel, cyclic parallel Ricci tensors and Ricci operator of ($k$, $\mu$) space forms of dimension three

**Definition 4.1.** *The Ricci tensor S of (k, μ) space forms of dimension three will be called η-parallel if it satisfies,*

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{4.1}$$

*for any vector fields X, Y, Z.*
*From (3.8) we get*

$$(\nabla_W S)(X,Y) = (3k-c)\{\nabla_W \eta)(X)\eta(Y) + (\nabla_W \eta)(Y)\eta(X)\}. \tag{4.2}$$

*From above it is clear that*

$$(\nabla_X S)(\phi Y, \phi Z) = 0. \tag{4.3}$$

Now we are in a position to state the following:

**Theorem 4.1.** *The Ricci tensor of a (k, μ) space forms of dimension three is η-parallel for* $\mu = \frac{1}{2}$ .

**Definition 4.2.** *The Ricci tensor of (k, μ) space forms of dimension three will be called cyclic parallel if*

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{4.4}$$

*From* (3.8) *we get*

$$\begin{aligned} & (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = \quad & (3k-c)\{(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) \\ + \quad & (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Y \eta)(X)\eta(Z) \\ + \quad & (\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\}. \end{aligned} \tag{4.5}$$

*If we take* $X, Y, Z$ *orthogonal to* $\xi$, *then we obtain from above,*

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{4.6}$$

Now we are in a position to state the following:

**Theorem 4.2.** *The Ricci tensor of (k, μ) space forms of dimension three is cyclic parallel for* $\mu = \frac{1}{2}$ .

**Definition 4.3.** *A (k, μ) space forms of dimension three is called locally ϕ-Ricci symmetric if,*

$$\phi^2(\nabla_W Q)X = 0 \tag{4.7}$$

*, where the vector fields X and W are orthogonal to ξ. The notion of locally ϕ-Ricci symmetry was introduced by U. C. De and A. Sarkar [6].*
*Again from* (3.8) *we obtain*

$$(\nabla_W Q)X = (3k-c)\{(\nabla_W \eta)(X)\xi + \eta(X)\nabla_W \xi\} \tag{4.8}$$

*Taking X orthogonal to* $\xi$ *and applying* $\phi^2$ *on both side of above we get*

$$\phi^2(\nabla_W Q)X = 0. \tag{4.9}$$

Now we are in a position to state the following:

**Theorem 4.3.** *A (k, μ) space forms of dimension three is locally ϕ-Ricci symmetric for* $\mu = \frac{1}{2}$ .

## 5 Locally $\phi$- symmetric ($k$, $\mu$) space forms

**Definition 5.1.** *A (2n+1)-dimensional (k, μ) space forms will be called locally ϕ-symmetric if* $\phi^2(\nabla_W R)(X,Y)Z = 0$, *for any vector fields X, Y, Z and W orthogonal to ξ.*

In this connection it should be mentioned that the notion of locally $\phi$- symmetric manifolds was introduced by T. Takahashi [10] in the context of Sasakian geometry.
First, we suppose that $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$. Then relation (2.9) reduces to

$$\begin{aligned} 4R(X,Y)Z \quad = \quad & (c+3)\{g(Y,Z)X - g(X,Z)Y\} \\ + \quad & (c-1)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ - \quad & 2\{g(hX,Z)hY - g(hY,Z)hX + g(X,Z)hY \\ - \quad & 2g(Y,Z)hX + 2g(hX,Z)Y \\ - \quad & 2g(hY,Z)X - g(\phi hX,Z)\phi hY + g(\phi hY,Z)\phi hX\}. \end{aligned} \tag{5.1}$$

Differentiating (5.1) covariantly with respect to a horizontal vector field $W$ we get,

$$\begin{aligned} 4(\nabla_W R)(X,Y)Z \quad = \quad & (c-1)\{g(X,(\nabla_W \phi)Z)\phi Y + g(X,\phi Z)(\nabla_W \phi)Y \\ - \quad & g(Y,(\nabla_W \phi)Z)\phi X - g(Y,\phi Z)(\nabla_W \phi)X \\ + \quad & 2g(X,(\nabla_W \phi)Y)\phi Z + g(X,\phi Y)(\nabla_W \phi)Z\} \\ - \quad & 2\{g((\nabla_W h)X,Z)hY + g(hX,Z)(\nabla_W h)Y \\ - \quad & g((\nabla_W h)Y,Z)hX - g(hY,Z)(\nabla_W h)X \\ + \quad & g(X,Z)(\nabla_W h)Y - 2g(Y,Z)(\nabla_W h)X \\ + \quad & 2g((\nabla_W h)X,Z)Y - 2g((\nabla_W h)Y,Z)X \\ - \quad & g((\nabla_W \phi)hX,Z)\phi hY - g(\phi hX,Z)(\nabla_W \phi)hY \\ + \quad & g((\nabla_W \phi)hY,Z)\phi hX + g(\phi hY,Z)(\nabla_W \phi)hX\}. \end{aligned} \tag{5.2}$$

Again, as $X$, $Y$ are orthogonal to $\xi$, so (2.5) and (2.6) reduces to

$$(\nabla_X \phi)Y = g(X,Y)\xi, \tag{5.3}$$

$$(\nabla_X h)(Y) = \{(1-k)g(X,\phi Y) + g(X, h\phi Y)\}\xi. \tag{5.4}$$

After using (5.3) and (5.4) in (5.2) and then applying $\phi^2$ on both side we obtain

$$\phi^2(\nabla_W R)(X,Y)Z = 0. \tag{5.5}$$

Thus we are in a position to state the following result:

**Theorem 5.1.** *Every (2n+1) dimensional (k, μ) space forms is locally ϕ- symmetric.*

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