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A spline method for solving fourth order singularly perturbed boundary value problem

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Abstract

In this paper, singularly perturbed boundary value problem of fourth order ordinary differential equation with a small positive parameter multiplying with the highest derivative of the form

$$\varepsilon u^{(4)}(x) + p(x)u^{''}(x) + q(x)u(x) = r(x), \;\; 0 \leq x \leq 1,$$

$$u(0) = \gamma_0, u(1) = \gamma_1, u^{''}(0) = \eta_0, u^{''}(1) = \eta_1, 0 \leq \varepsilon \leq 1$$

is considered. We have developed a numerical technique for the above problem using parametric and polynomial septic spline method. The method is shown to have second and fourth order convergent depending on the choice of parameters involved in the method. Truncation error and boundary equations are obtained. The method is tested on an example and the results are found to be in agreement with the theoretical analysis.

Keywords: Parametric septic splines, Polynomial septic splines, Boundary value problems, Boundary equations.

2010 MSC: 65D07, 65L10, 65L11.

1 Introduction

Singular perturbation problems appear in many branches of applied mathematics, and for more than two decades quite a large number of research papers on the qualitative and quantitative analysis of these problems for both ordinary differential equations (ODEs) and partial differential equations (PDEs) have been reported in the literature. Most of the papers connected with computational aspects are confined to second order equations. But only few authors have developed numerical methods for singularly perturbed higher order differential equations. These problems are classified on the basis of how the order of original differential equation is affected if sets $\varepsilon = 0$ [8]. Here, ε is a small positive parameter multiplying with the highest derivative of the differential equation. The singularly perturbed problem is of convection-diffusion type if the order of the differential equation is reduced by 1, whereas it is called reaction-diffusion type if order is reduced by 2. The objective of the present paper is to develop a computational method to solve singularly perturbed boundary value problems of fourth order ordinary differential equations of the form:

$$\left.\begin{aligned} &\varepsilon u^{(4)}(x) + p(x)u^{''}(x) + q(x)u(x) = r(x), \;\; 0 \leq x \leq 1, \\ &u(0) = \gamma_0, u(1) = \gamma_1, u^{''}(0) = \eta_0, u^{''}(1) = \eta_1, 0 \leq \varepsilon \leq 1, \end{aligned}\right\} \tag{1.1}$$

where $p(x),\; q(x)$ and $r(x)$ are smooth, bounded, real functions $p(x) : R \to R,\; q(x) : R \to R,\; r(x) : R \to R$ satisfying the following conditions

$$-p \geq \beta > 0, 0 \geq q \geq -\gamma, \gamma > 0,$$

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$$\beta - 2\gamma \geq \eta > 0, \; for \; some \;\; \eta,$$
$$D = (0,1), \overline{D} = [0,1] \; and \; u \, \epsilon C^4(D) \bigcap C^2(\overline{D}). \tag{1.2}$$

Analytical and numerical treatment of these equations have drawn much attention of many authors [5,22-25]. The analytical treatment of singularly perturbed boundary value problems for higher order nonlinear ordinary differential equations, which have important applications in Fluid Dynamics, can be found in [1,2,6-9,13,20]. Semper [2] and Roos and Stynes [8] considered fourth order equations and applied a standard finite element method. Garland [3] has shown that uniform stability of discrete boundary value problem follows from uniform stability of the discrete initial value problem and uniform consistency of the scheme. Some results connected with the exponentially fitted higher order differences with identity expansion method [3] and defect corrections are available in the literature. Loghmani and Ahmadinia [5] have developed a numerical technique for solving singularly perturbed boundary value problems based on optimal control strategy by using B-spline functions and least square method. Also finite element method is reported in [6,7]. In [9], an iterative method is described. In [10,11,16,17,20], the authors have applied boundary value technique to find the numerical solution for singularly perturbed second order boundary value problems. Niederdrenk and Yserentant [12] considered convection diffusion type problems and derived conditions for the uniform stability of discrete and continuous problems. Feckan [13] considered higher order problems and his approach is based on the nonlinear analysis involving fixed point theory, Leray-Schauder theory etc. In [15], authors have given a brief survey on computational techniques for the different classes of singularly perturbed problems. Bawa [19] and Aziz and Khan [21] have solved second order singularly perturbed boundary value problem using spline technique. Shanthi and Ramanujam [22-25] have developed numerical methods for singularly perturbed higher order boundary value problems.
In this problem, we take $p(x) = p =$ constant and $q(x) = q =$ constant. In the present paper, parametric septic spline is described for fourth order boundary value problems. In section 2, a brief description of the method is given. Development of the boundary equations are given in section 3. In section 4, truncation error and class of methods are discussed. We established the convergence of our method in section 5 and section 6 contains the numerical results and discussions.

2 Derivation of the method

In order to develop the numerical method for approximating the solution of singularly perturbed fourth order boundary value problem, the interval $[0,1]$ is divided into N equal subintervals using the grid points $x_j = jh, \;\; j = 0(1)N$, where

$$x_0 = 0, \quad x_N = 1, \quad and \quad h = \frac{1}{N}. \tag{2.1}$$

A function $S_\Delta(x,\tau)$ of class $C^6[0,1]$ which interpolates $u(x)$ at the mesh point x_j depends on a parameter τ, and as $\tau \to 0$ it reduces to septic spline $S_\Delta(x)$ in $[0,1]$ is termed as parametric septic spline function. Since the parameter τ can occur in $S_\Delta(x)$ in many ways such a spline is not unique.
If $S_\Delta(x,\tau) = S_\Delta(x)$ is a piecewise function satisfying the following differential equation in the interval $[x_{j-1}, x_j]$

$$\begin{aligned} S_\Delta^{(6)}(x) - \tau^2 S_\Delta''(x) &= (Q_j - \tau^2 M_j)\frac{x - x_{j-1}}{h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{x_j - x}{h} \\ &= A_j z + A_{j-1}\bar{z}, \end{aligned} \tag{2.2}$$

where

$$z = \frac{x - x_{j-1}}{h}, \; \bar{z} = 1 - z, \; A_k = Q_k - \tau^2 M_k, S_\Delta''(x_k,\tau) = M_k, \;\; S_\Delta^{(6)}(x_k,\tau) = Q_k, \;\; k = j-1, j; \;\; \tau > 0,$$

then it is termed as parametric septic spline II. Solving equation (2.2), we get

$$\begin{aligned} S_\triangle(x) = {} & B_1 + B_2 x + B_3 \cosh\sqrt{\tau}x + B_4 \sinh\sqrt{\tau}x + B_5 \cos\sqrt{\tau}x + B_6 \sin\sqrt{\tau}x \\ & - \frac{1}{\tau^2}\left\{ (Q_j - \tau^2 M_j)\frac{(x - x_{j-1})^3}{6h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{(x_j - x)^3}{6h} \right\}. \end{aligned} \tag{2.3}$$

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$\left.\begin{aligned} S_\Delta(x_j) &= u_j, \quad S_\Delta(x_{j+1}) = u_{j+1},\\ S''_\Delta(x_j) &= M_j, \quad S''_\Delta(x_{j+1}) = M_{j+1},\\ S^{(4)}_\Delta(x_j) &= F_j, \quad S^{(4)}_\Delta(x_{j+1}) = F_{j+1}. \end{aligned}\right\} \tag{2.4}$$

To define spline in terms of u_j's, M_j's and F_j's, the coefficients introduced in Eq.(2.3) are calculated as

$$\begin{aligned}
B_1 &= u_{j-1} + \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) - \frac{F_{j-1}}{\tau^2}\\
&\quad - \frac{x_{j-1}}{h}\Big[(u_j - u_{j-1}) - \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) + \frac{h^2}{6\tau^2}(Q_j - \tau^2 M_j) + \frac{1}{\tau^2}(F_{j-1} - F_j)\Big],\\
B_2 &= \frac{1}{h}(u_j - u_{j-1}) + \frac{h}{6\tau^2}\Big[-(Q_{j-1} - \tau^2 M_{j-1}) + (Q_j - \tau^2 M_j)\Big] + \frac{1}{\tau^2 h}(F_{j-1} - F_j),\\
B_3 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Big[\frac{1}{2}\sinh\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) - \frac{1}{2}\sinh\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\\
&\quad - \frac{1}{\tau}\sinh\sqrt{\tau}x_{j-1}Q_j + \frac{1}{\tau}\sinh\sqrt{\tau}x_j Q_{j-1}\Big],\\
B_4 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Big[-\frac{1}{2}\cosh\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) + \frac{1}{2}\cosh\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\\
&\quad + \frac{1}{\tau}\cosh\sqrt{\tau}x_{j-1}Q_j - \frac{1}{\tau}\cosh\sqrt{\tau}x_j Q_{j-1}\Big],\\
B_5 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Big[\sin\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) - \sin\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\Big],\\
B_6 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Big[-\cos\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) + \cos\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\Big].
\end{aligned} \tag{2.5}$$

Substituting these values in (2.3), we get

$$S_\Delta(x) = zu_j + \bar{z}u_{j-1} + \frac{h^2}{6}\Big[p(z)M_j + p(\bar{z})M_{j-1}\Big] + \frac{h^4}{2}\Big[r(z)F_j + r(\bar{z})F_{j-1}\Big] + \frac{h^6}{6}\Big[q(z)Q_j + q(\bar{z})Q_{j-1}\Big], \tag{2.6}$$

where

$$\begin{aligned} p(z) = z^3 - z, \quad q(z) &= \frac{z}{\omega^4} - \frac{z^3}{\omega^4} + \frac{3\sinh\omega z}{\omega^6 \sinh\omega} - \frac{3\sin\omega z}{\omega^6 \sin\omega},\\ r(z) &= \frac{-2z}{\omega^4} + \frac{\sinh\omega z}{\omega^4\sinh\omega} + \frac{\sin\omega z}{\omega^4 \sin\omega}, \quad \omega = \sqrt{\tau}h. \end{aligned} \tag{2.7}$$

Applying the first, third and fifth derivative continuities at the knots, i.e. $S^{(\mu)}_\Delta(x_j^-) = S^{(\mu)}_\Delta(x_j^+)$, $\mu = 1, 3$ *and* 5, the following consistency relations are derived:

$$\begin{aligned}
M_{j+1} + 4M_j + M_{j-1} &= \frac{6}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + 3h^2(\alpha_2 F_{j+1} + 2\beta_2 F_j + \alpha_2 F_{j-1})\\
&\quad + h^4(\alpha_1 Q_{j+1} + 2\beta_1 Q_j + \alpha_1 Q_{j-1}), \quad j = 1(1)N-1, \qquad (2.8)\\
M_{j+1} - 2M_j + M_{j-1} &= \frac{h^2}{6}[(1 - \omega^4\alpha_1)F_{j+1} + 2(2 - \omega^4\beta_1)F_j + (1 - \omega^4\alpha_1)F_{j-1}]\\
&\quad - \frac{h^4}{2}(\alpha_2 Q_{j+1} + 2\beta_2 Q_j + \alpha_2 Q_{j-1}), \quad j = 1(1)N-1, \qquad (2.9)
\end{aligned}$$

$$h^2[(1-\omega^4\alpha_1)Q_{j+1}+2(2-\omega^4\beta_1)Q_j+(1-\omega^4\alpha_1)Q_{j-1}] = \\ 3[(\omega^4\alpha_2+2)F_{j+1}+2(\omega^4\beta_2-2)F_j+(\omega^4\alpha_2+2)F_{j-1}],\ \ j=1(1)N-1, \tag{2.10}$$

where

$$\alpha_1=\frac{1}{\omega^4}+\frac{3}{\omega^5\sinh\omega}-\frac{3}{\omega^5\sin\omega},\quad \beta_1=\frac{2}{\omega^4}-\frac{3}{\omega^5}\coth\omega+\frac{3}{\omega^5}\cot\omega,$$
$$\alpha_2=\frac{-2}{\omega^4}+\frac{1}{\omega^3\sinh\omega}+\frac{1}{\omega^3\sin\omega},\quad \beta_2=\frac{2}{\omega^4}-\frac{1}{\omega^3}\coth\omega-\frac{1}{\omega^3}\cot\omega. \tag{2.11}$$

As $\tau\to 0$ that is $\omega\to 0$ then $(\alpha_1,\beta_1,\alpha_2,\beta_2)\to(\frac{-31}{2520},\frac{-4}{315},\frac{7}{180},\frac{2}{45})$.
Using equations (2.8)-(2.10), we obtain the following scheme

$$(e_1u_{j-3}+e_2u_{j-2}+e_3u_{j-1}+e_4u_j+e_3u_{j+1}+e_2u_{j+2}+e_1u_{j+3})$$
$$=\frac{h^4}{6}(p_1F_{j-3}+p_2F_{j-2}+p_3F_{j-1}+p_4F_j+p_3F_{j+1}+p_2F_{j+2}+p_1F_{j+3}), j=3(1)N-3, \tag{2.12}$$

where the coefficients (e_1,e_2,e_3,e_4) and (p_1,p_2,p_3,p_4) of the developed scheme are given by

$$\begin{aligned}
e_1 &= 1-3\omega^4\alpha_1+3\omega^8\alpha_1^2-\omega^{12}\alpha_1^3,\\
e_2 &= 4\omega^4\alpha_1-2\omega^4\beta_1-8\omega^8\alpha_1^2+4\omega^8\alpha_1\beta_1-2\omega^{12}\alpha_1^2\beta_1,\\
e_3 &= 7(1-\omega^4\alpha_1)^3-8(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1),\\
e_4 &= 12(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1)-8(1-\omega^4\alpha_1)^3,\\
p_1 &= c_1(1-\omega^4\alpha_1)^2,\\
p_2 &= 2c_1(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+c_2(1-\omega^4\alpha_1)^2-3d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_3 &= (c_1+c_3)(1-\omega^4\alpha_1)^2+6d_1(1-\omega^4\alpha_1)(2-\omega^4\beta_2)+2c_2(1-\omega^4\alpha_1)(2-\omega^4\beta_1)\\
&\quad -3d_2(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_4 &= 2c_2(1-\omega^4\alpha_1)^2-6d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2)-6d_1(2-\omega^4\beta_1)(2-\omega^4\beta_2)\\
&\quad +2c_3(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+6d_2(1-\omega^4\alpha_1)(2-\omega^4\beta_2).
\end{aligned} \tag{2.13}$$

Also

$$\begin{aligned}
c_1 &= \frac{1}{6}\omega^8\alpha_1^2-\frac{3}{2}\omega^4\alpha_2^2-\frac{1}{3}\omega^4\alpha_1-6\alpha_1-6\alpha_2+\frac{1}{6},\\
c_2 &= \frac{2}{3}\omega^8\alpha_1^2+\frac{1}{3}\omega^8\alpha_1\beta_1-18\omega^4\alpha_1\alpha_2-3\omega^4\alpha_2\beta_2-6\omega^4\alpha_2^2-2\omega^4\alpha_1-\frac{1}{3}\omega^4\beta_1-12\alpha_1-6\beta_2+\frac{4}{3},\\
c_3 &= \frac{1}{3}\omega^8\alpha_1^2+\frac{4}{3}\omega^8\alpha_1\beta_1-36\omega^4\alpha_1\beta_2-12\omega^4\alpha_2\beta_2-3\omega^4\alpha_2^2-\frac{10}{3}\omega^4\alpha_1-\frac{4}{3}\omega^4\beta_1+36\alpha_1\\
&\quad +12\alpha_2+12\beta_2+3,\\
d_1 &= \omega^4\alpha_2\beta_1-\omega^4\alpha_1\beta_2+6\omega^4\alpha_1^2-10\alpha_1-2\alpha_2+2\beta_1+\beta_2,\\
d_2 &= 4\omega^4\alpha_2\beta_1-4\omega^4\alpha_1\beta_2+12\omega^4\alpha_1\beta_1-16\alpha_1-18\alpha_2-4\beta_1+4\beta_2.
\end{aligned} \tag{2.14}$$

As $\tau\to 0$ that is $\omega\to 0$, we have
(i)$(e_1,e_2,e_3,e_4)\longrightarrow(1,0,-9,16)$,
(ii)$(c_1,c_2,c_3,d_1,d_2)\longrightarrow\left(\frac{1}{140},\frac{17}{14},\frac{249}{70},\frac{9}{140},\frac{4}{35}\right)$,
(iii)$(p_1,p_2,p_3,p_4)\longrightarrow\left(\frac{1}{140},\frac{6}{7},\frac{1191}{140},\frac{604}{35}\right)$.
[Remarks:] For these values our scheme reduces to the polynomial septic spline for fourth order boundary value problem which is given as equation (7) in G. Akram and S. S. Siddiqi [4].
We have taken $(e_1,e_2,e_3,e_4)=(1,0,-9,16)$ in Eq. (2.12) and obtained

$$p_1(F_{j-3}+F_{j+3})+p_2(F_{j-2}+F_{j+2})+p_3(F_{j-1}+F_{j+1})+p_4F_j=\frac{6}{h^4}[(u_{j-3}+u_{j+3})-9(u_{j-1}+u_{j+1})+16u_j]. \tag{2.15}$$

Eq. (1.1) can be written in the following form by taking $p(x) = p$ and $q(x) = q$ as

$$\varepsilon F_j + pM_j + qu_j = r_j. \tag{2.16}$$

Operate Λ_x on the both side of Eq. (2.16), we get

$$\varepsilon \Lambda_x F_j + p\Lambda_x M_j + q\Lambda_x u_j = \Lambda_x r_j \tag{2.17}$$

where operator Λ_x is defined as follows for any function $'W'$ evaluated at mesh point

$$\Lambda_x W_j = p_1(W_{j-3} + W_{j+3}) + p_2(W_{j-2} + W_{j+2}) + p_3(W_{j-1} + W_{j+1}) + p_4 W_j. \tag{2.18}$$

For second derivative of u, we take relation from [Eq. (5), Ref. [4]]

$$\Lambda_x M_j = \frac{42}{h^2}[(u_{j-3} + u_{j+3}) + 24(u_{j-2} + u_{j+2}) + 15(u_{j-1} + u_{j+1}) - 80u_j]. \tag{2.19}$$

Here, $(p_1, p_2, p_3, p_4) = (1, 120, 1191, 2416)$ for second derivative. Using (2.15-2.19), we get

$$\begin{aligned}
&(6\varepsilon + 42ph^2 + qh^4 p_1)(u_{j-3} + u_{j+3}) + (1008ph^2 + qh^4 p_2)(u_{j-2} + u_{j+2}) \\
&+(-54\varepsilon + 630ph^2 + qh^4 p_3)(u_{j-1} + u_{j+1}) + (96\varepsilon - 3360ph^2 + qh^4 p_4)u_j \\
&= h^4[p_1(r_{j-3} + r_{j+3}) + p_2(r_{j-2} + r_{j+2}) + p_3(r_{j-1} + r_{j+1}) + p_4 r_j],\ j = 3(1)(N-3).
\end{aligned} \tag{2.20}$$

3 Development of boundary equations

The relation (2.20) gives $(N-5)$ algebraic equations in $(N-1)$ unknowns $u_j, j = 1(1)N-1$. We require four more equations, two at each end of range of integartion in order to have closed form solution for u_j. For the discretization of the boundary conditions, we have developed boundary equations for second and fourth order methods as follows:

3.1 Second order method

(i) $-5u_1 + 4u_2 - u_3 = -2\gamma_0 + h^2\eta_0 - \frac{11h^4}{12\varepsilon}(r_0 - q\gamma_0 - p\eta_0), j = 1,$

(ii) $\frac{52}{5}u_1 - \frac{57}{5}u_2 + \frac{28}{5}u_3 - \frac{11}{10}u_4 = \frac{7}{2}\gamma_0 - \frac{6}{5}h^2\eta_0, j = 2,$

(iii) $-\frac{11}{10}u_{N-4} + \frac{28}{5}u_{N-3} - \frac{57}{5}u_{N-2} + \frac{52}{5}u_{N-1} = \frac{7}{2}\gamma_1 - \frac{6}{5}h^2\eta_1, j = N-2,$

(iv) $-u_{N-3} + 4u_{N-2} - 5u_{N-1} = -2\gamma_1 + h^2\eta_1 - \frac{11h^4}{12\varepsilon}(r_N - q\gamma_1 - p\eta_1), j = N-1.$

3.2 Fourth order method

(i) $-\frac{1322}{35}u_1 + \frac{11066}{245}u_2 - \frac{6684}{245}u_3 + \frac{2171}{245}u_4 - \frac{302}{245}u_5 = -\frac{429}{35}\gamma_0 + \frac{204}{49}h^2\eta_0 - \frac{274h^4}{245\varepsilon}(r_0 - q\gamma_0 - p\eta_0), j = 1,$

(ii) $-\frac{1342}{97}u_1 + \frac{7213}{375}u_2 - \frac{2049}{146}u_3 + \frac{3799}{691}u_4 - \frac{11059}{12203}u_5 + \frac{12}{8551}u_6 = -\frac{9898}{2449}\gamma_0 + \frac{348}{323}h^2\eta_0, j = 2,$

(iii) $\frac{12}{8551}u_{N-6} - \frac{11059}{12203}u_{N-5} + \frac{3799}{691}u_{N-4} - \frac{2049}{146}u_{N-3} + \frac{7213}{375}u_{N-2} - \frac{1342}{97}u_{N-1} = -\frac{9898}{2449}\gamma_1 + \frac{348}{323}h^2\eta_1, j = N-2,$

(iv) $-\frac{1322}{35}u_{N-1} + \frac{11066}{245}u_{N-2} - \frac{6684}{245}u_{N-3} + \frac{2171}{245}u_{N-4} - \frac{302}{245}u_{N-5} = -\frac{429}{35}\gamma_1 + \frac{204}{49}h^2\eta_1 - \frac{274h^4}{245\varepsilon}(r_N - q\gamma_1 - p\eta_1), j = N-1.$

4 Truncation error

To obtain the local truncation error t_j, $j = 3(1)(N-3)$, associated with the scheme (2.20), substitute $r_j = \varepsilon u_j^{(4)} + pu_j'' + qu_j$ in eq. (2.20) and expanding it by Taylor series about x_j, we obtain the following local truncation error

$$\begin{aligned} t_j &= (864 - 48p_1 - 48p_2 - 48p_3 - 24p_4)\frac{\varepsilon h^4}{4!}u_j^{(4)} \\ &\quad +(8640 - 6480p_1 - 2880p_2 - 720p_3)\frac{\varepsilon h^6}{6!}u_j^{(6)} \\ &\quad +(78624 - 272160p_1 - 53760p_2 - 3360p_3)\frac{\varepsilon h^8}{8!}u_j^{(8)} - \frac{6720}{8!}ph^{10}u_j^{(8)} \\ &\quad +(708480 - 7348320p_1 - 645120p_2 - 10080p_3)\frac{\varepsilon h^{10}}{10!}u_j^{(10)} \\ &\quad +\frac{10080}{10!}ph^{12}u_j^{(10)} + O(h^{12}). \end{aligned} \tag{4.1}$$

By using the above equation and eliminating the coefficients of various powers of h, we can obtain class of the methods. For arbitrary choices of p_1, p_2, p_3 and p_4, we obtain the following methods:

4.1 Second order methods

By equating the coefficient of h^4 equal to zero in (4.1), we get second order methods. Therefore,

$$p_4 = 36 - 2p_1 - 2p_2 - 2p_3, \tag{4.2}$$

where p_1, p_2, p_3 and p_4, are arbitrary. The truncation error is given by

$$t_j = \frac{\varepsilon h^6}{6!}u_j^{(6)}(8640 - 6480p_1 - 2880p_2 - 720p_3). \tag{4.3}$$

The local truncation error at $j = 1, 2, N-2, N-1$ for second order methods is

$$t_j = \begin{cases} (\frac{-239}{360})h^6u_j^{(6)} + O(h^7), & j = 1, N-1, \\ (\frac{-119}{75})h^6u_j^{(6)} + O(h^7), & j = 2, N-2. \end{cases} \tag{4.4}$$

4.2 Fourth order methods

By equating the coefficient of h^4 and h^6 equal to zero in (4.1), we get fourth order methods. Therefore,

$$p_3 = 12 - 9p_1 - 4p_2 \textit{ and } p_4 = 12 + 16p_1 + 6p_2, \tag{4.5}$$

where p_1 and p_2 are arbitrary. The truncation error is given by

$$t_j = \frac{\varepsilon h^8}{8!}u_j^{(8)}(38304 - 241920p_1 - 40320p_2). \tag{4.6}$$

The local truncation error at $j = 1, 2, N-2, N-1$ for fourth order method is

$$t_j = \begin{cases} (\frac{-204737}{120960})h^8u_j^{(8)} + O(h^9), & j = 1, N-1, \\ (\frac{-14323}{10080})h^8u_j^{(8)} + O(h^9), & j = 2, N-2. \end{cases} \tag{4.7}$$

5 Convergence analysis

Now, we investigate the convergence analysis of the spline method described in section 2 for problem (1.1). To do so, we let, $U = (u_j)$, $\overline{U} = (\overline{u}_j)$, $V = (v_j)$, $T = (t_j)$ and $E = (e_j) = U - \overline{U}$, be N-dimensional column vectors. Here, $U, \overline{U}, T$ denotes the exact solution, the approximate solution and truncation error respectively and e_j is the discretization error for $j = 1(1)(N-1)$. Thus we can write the system (2.20) in the matrix form:

$$A\overline{U} - h^4 BR = V \ and \ A = A_1 + h^2 pA_2 + h^4 qB \tag{5.1}$$

where $A_1,\ A_2,\ B,\ R$ and V are defined by

$$A_1 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & & & \\ a_1^* & a_2^* & a_3^* & a_4^* & a_5^* & a_6^* & & \\ 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon & & \\ 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 & 6\varepsilon \\ & & 6\varepsilon & 0 & -54\varepsilon & 96\varepsilon & -54\varepsilon & 0 \\ & & a_{N-6}^* & a_{N-5}^* & a_{N-4}^* & a_{N-3}^* & a_{N-2}^* & a_{N-1}^* \\ & & & a_{N-5} & a_{N-4} & a_{N-3} & a_{N-2} & a_{N-1} \end{bmatrix}, \tag{5.2}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ 1008 & 630 & -3360 & 630 & 1008 & 42 & & \\ 42 & 1008 & 630 & -3360 & 630 & 1008 & 42 & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & 42 & 1008 & 630 & -3360 & 630 & 1008 & 42 \\ & & 42 & 1008 & 630 & -3360 & 630 & 1008 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.3}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ p_2 & p_3 & p_4 & p_3 & p_2 & p_1 & & \\ p_1 & p_2 & p_3 & p_4 & p_3 & p_2 & p_1 & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & p_1 & p_2 & p_3 & p_4 & p_3 & p_2 & p_1 \\ & & p_1 & p_2 & p_3 & p_4 & p_3 & p_2 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.4}$$

$R = [r_j]^T$ and $V = [v_j]^T$, $j = 1(1)N-1$.
Moreover,

$$v_j = \begin{cases} a_0\gamma_0 + b_1 h^2 \eta_0 + d_0 \frac{h^4}{\varepsilon}(r_0 - q\gamma_0 - p\eta_0), & j = 1, \\ a_0^*\gamma_0 + b_2 h^2 \eta_0, & j = 2, \\ -(6\varepsilon + 42ph^2 + qp_1h^4)\gamma_0 + h^4 p_1 r_0, & j = 3, \\ 0, & j = 4(1)N-4, \\ -(6\varepsilon + 42ph^2 + qp_1h^4)\gamma_1 + h^4 p_1 r_N, & j = N-3, \\ a_N^*\gamma_0 + b_{N-2}h^2\eta_1, & j = N-2, \\ a_N\gamma_1 + b_{N-1}h^2\eta_1 + d_N \frac{h^4}{\varepsilon}(r_N - q\gamma_1 - p\eta_1), & j = N-1, \end{cases} \tag{5.5}$$

where for second order method, we have

(i) $(a_0, a_1, a_2, a_3, a_4, a_5, b_1, d_0) = \left(-2, -5, 4, -1, 0, 0, 1, -\frac{11}{12}\right)$,

(ii) $(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*, b_2) = \left(\frac{7}{2}, \frac{52}{5}, -\frac{57}{5}, \frac{28}{5}, -\frac{11}{10}, 0, 0, -\frac{6}{5}\right)$,

(iii) $(a_N^*, a_{N-6}^*, a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, b_{N-2}) = \left(\frac{7}{2}, 0, 0, -\frac{11}{10}, \frac{28}{5}, -\frac{57}{5}, \frac{52}{5}, -\frac{6}{5}\right)$,

(iv) $(a_N, a_{N-5}, a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, b_{N-1}, d_N) = \left(-2, 0, 0, -1, 4, -5, 1, -\frac{11}{12}\right)$.

and for fourth order method, we have

(i) $(a_0, a_1, a_2, a_3, a_4, a_5, b_1, d_0) = \left(-\frac{429}{35}, -\frac{1322}{35}, \frac{11066}{245}, -\frac{6684}{245}, \frac{2171}{245}, -\frac{302}{245}, \frac{204}{49}, -\frac{274}{245}\right)$,

(ii) $(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*, b_2) = \left(-\frac{9898}{2449}, -\frac{1342}{97}, \frac{7213}{375}, -\frac{2049}{146}, \frac{3799}{691}, -\frac{11059}{12203}, \frac{12}{8551}, \frac{348}{323}\right)$,

(iii) $(a_N^*, a_{N-6}^*, a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, b_{N-2}) = \left(-\frac{9898}{2449}, \frac{12}{8551}, -\frac{11059}{12203}, \frac{3799}{691}, -\frac{2049}{146}, \frac{7213}{375}, -\frac{1342}{97}, \frac{348}{323}\right)$,

(iv) $(a_N, a_{N-5}, a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, b_{N-1}, d_N) = \left(-\frac{429}{35}, -\frac{302}{245}, \frac{2171}{245}, -\frac{6684}{245}, \frac{11066}{245}, -\frac{1322}{35}, \frac{204}{49}, -\frac{274}{245}\right)$.

Also, we have

$$AU - h^4 BR = T(h) + V. \tag{5.6}$$

From Eq. (5.1) and Eq. (5.6), we get

$$A(U - \overline{U}) = T(h),$$
$$AE = T(h). \tag{5.7}$$

Clearly

$$S_j = \begin{cases} a_1 + a_2 + a_3 + a_4 + a_5, & j = 1, \\ a_1^* + a_2^* + a_3^* + a_4^* + a_5^* + a_6^*, & j = 2, \\ -(6\varepsilon + 42ph^2) + q(p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = 3, \\ q(2p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = 4(1)N-4, \\ -(6\varepsilon + 42ph^2) + q(p_1 + 2p_2 + 2p_3 + p_4)h^4, & j = N-3, \\ a_{N-6}^* + a_{N-5}^* + a_{N-4}^* + a_{N-3}^* + a_{N-2}^* + a_{N-1}^*, & j = N-2, \\ a_{N-5} + a_{N-4} + a_{N-3} + a_{N-2} + a_{N-1}, & j = N-1. \end{cases} \tag{5.8}$$

We can choose h sufficiently small so that the matrix A is irreducible and monotone [18]. It follows that A^{-1} exists and its elements are non negative. Hence, from Eq. (5.7), we have

$$E = A^{-1}T(h). \tag{5.9}$$

Also, from the theory of the matrices, we have

$$\sum_{j=1}^{N-1} \overline{a}_{k,j} S_j = 1, \ k = 1(1)N-1, \tag{5.10}$$

where $\overline{a}_{k,j}$ is the $(k,j)th$ element of the matrix A^{-1}. Therefore

$$\sum_{j=1}^{N-1} \overline{a}_{k,j} \leq \frac{1}{\min_{1\leq j\leq N-1} S_0} = \frac{1}{q(2p_1 + 2p_2 + 2p_3 + p_4)h^4} \tag{5.11}$$

From Eq.(5.9) and (5.11), we have

$$e_j = \sum_{j=1}^{N-1} \overline{a}_{k,j} T_j(h), \ k = 1(1)N-1$$

and therefore

$$| e_j | \leq \frac{KT_j}{h^4}, j = 1(1)N-1,$$

where K is a constant indepent of h. It follows that
(i) For second order methods the truncation error is $\| T \| = O(h^6)$. It follows that $\| E \| = O(h^2)$.
(ii) For fourth order methods the truncation error is $\| T \| = O(h^8)$. It follows that $\| E \| = O(h^4)$.

6 Numerical results and discussion

Example 1: Consider the boundary value problem

$$\varepsilon u^{(4)}(x) - 4u''(x) - u(x) = -\frac{x(1-x)}{8} - \frac{5\varepsilon}{16} + \frac{5\varepsilon}{16}\left[\frac{e^{-\frac{2x}{\sqrt{\varepsilon}}} - e^{-\frac{2(1+x)}{\sqrt{\varepsilon}}} + e^{-\frac{2(1-x)}{\sqrt{\varepsilon}}} - e^{-\frac{2(2-x)}{\sqrt{\varepsilon}}}}{1 - e^{-\frac{4}{\sqrt{\varepsilon}}}}\right],$$

$$u(0) = u(1) = 1,\ u''(0) = u''(1) = -1.$$

We have solved this example by scheme (2.20) and have obtained approximate solution at $x = 0.001(0.001)0.009$ for the sake of comparison with references. The obtained numerical results are tabulated in table 1 and 2 for second and fourth order methods respectively. The comparison is also made in table 2 with the obatined results of [23].

Table 1. Maximum absolute errors for example 1
Second order method,$\varepsilon = 0.01, h = 0.001$

x	Present method $(p_1, p_2, p_3, p_4) = (0,0,0,36)$	Exact [23]	Errors of present method
0.001	1.0000033	1.000063	5.97(−5)
0.002	1.0000058	1.000127	1.21(−4)
0.003	1.0000074	1.000192	1.84(−4)
0.004	1.0000084	1.000258	2.50(−4)
0.005	1.0000087	1.000324	3.15(−4)
0.006	1.0000084	1.000392	3.84(−4)
0.007	1.0000074	1.000461	4.54(−4)
0.008	1.0000058	1.000530	5.24(−4)
0.009	1.0000033	1.000600	5.97(−4)

Conclusion

We have developed a numerical method for the solution of fourth order singularly pertubed boundary value problem using parametric septic spline. It is a computationally efficient method and the algorithm can easily be implemented on a computer. The method has been analysed for convergence and proved that the method is second and fourth order convergent. Also, the errors at nodal points are compared with the errors of [23] and observed to be better.

Table 2. Maximum absolute errors for example 1
Fourth order method,$\varepsilon = 0.01, h = 0.001$

x	Present method $(p_1, p_2, p_3, p_4) = (0,0,12,12)$	Exact [23]	Errors of present method	Errors [23]
0.001	1.000016	1.000063	4.70(−5)	6.19(−5)
0.002	1.000028	1.000127	9.90(−5)	1.22(−4)
0.003	1.000036	1.000192	1.56(−4)	1.82(−4)
0.004	1.000041	1.000258	2.17(−4)	2.40(−4)
0.005	1.000042	1.000324	2.82(−4)	2.97(−4)
0.006	1.000041	1.000392	3.51(−4)	3.53(−4)
0.007	1.000056	1.000461	4.05(−4)	4.08(−4)
0.008	1.000028	1.000530	5.02(−4)	4.62(−4)
0.009	1.000090	1.000600	5.10(−4)	5.15(−4)

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