Malaya J. Mat. 3(1)(2015) 62–85

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Nonlinear $\mathcal{D}$-set contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations

Bapurao C. Dhage*

Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India.

Abstract

In this paper the author introduces the notion of partially nonlinear $\mathcal{D}$-set-contraction mappings in a partially ordered normed linear space and prove some hybrid fixed point theorems under certain mixed conditions from algebra, analysis and topology. The applications of abstract results presented here are given to perturbed nonlinear hybrid functional integral equations for proving the existence as well as global attractivity of the comparable solutions under certain monotonic conditions. The abstract theory developed in this paper is also useful to develop the algorithms for the solutions of some nonlinear problems of analysis and allied areas of mathematics.

Keywords: Partial measure of noncompactness; Partially nonlinear $\mathcal{D}$-set-contraction mappings; Fixed points; Functional integral equation; Existence theorem; Attractivity of solutions.

2010 MSC: 45G10, 45M99, 47H09, 47H10.
©2012 MJM. All rights reserved.

1 Introduction

The topological methods or fixed point theorems involving the compactness arguments are useful tools to prove the existence of the solutions of some nonlinear equations but these, do not give any computational scheme or algorithm for solving such problems. See Burton [4], Burton and Zhang [5], Krasnoselskii [20] and the references therein. However if we combine the compactness (or its generalizations in terms of measure of noncompactness) with some algebraic arguments then it is possible to develop an algorithm for solutions of the nonlinear problems under some suitable conditions. The work along this line is of great interest and which is the main motivation of the present paper. Here, we combine the topological arguments with some order related hypotheses to prove some hybrid fixed point theorems for the mappings in partially ordered spaces and apply newly developed abstract results to obtain an algorithm for the solutions of a certain nonlinear functional integral equation under some mixed compactness and monotonic conditions. The results of this paper seem to be new in the literature.

The topological concept of compactness is very much useful in the development of nonlinear analysis to derive some far reaching conclusions. It is interesting to measure numerically the degree of noncompactness of sets in a normed linear space in terms of certain characteristics of the compactness property. The Kuratowskii [21] and Hausdorff [18] measures of noncompactness are well-known tools for discussing different aspects of the theory of nonlinear equations in the literature. However, an axiomatic way of approach of the measures of noncompactness is sometimes useful in the study of various qualitative properties of the dynamical systems in nonlinear analysis (cf. Appell [1] and Banas and Goebel [2]). In this article, we follow the axiomatic way of approach to define a new partial measure of noncompactness in a partially ordered normed linear space and which is subsequently exploited to derive some interesting consequences. We continue the study presented in Dhage [11], Nieto and Lopez [22] and Ran and Reurings [23] and prove some new hybrid fixed point theorems (FPTs) for partially condensing mappings in a partially ordered complete normed linear space and apply our abstract results to a certain nonlinear hybrid functional integral equation for proving the existence as well

*E-mail address: bcdhage@gmail.com

as global attractivity results on the unbounded intervals of real line. The abstract theory developed here is useful to develop the algorithms for the solutions of some nonlinear problems of analysis and allied areas of mathematics and mathematical sciences.

2 Partially Ordered Linear Spaces

Let E be a real vector or linear space. We introduce a partial order $\preceq$ in E as follows. A relation $\preceq$ in E is said to be a partial order if it satisfies the following properties: For any $a, b, c, d \in E$ and $\lambda \in \mathbb{R}$,

1. Reflexivity: $a \preceq a$ for all $a \in E$,
2. Antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$,
3. Transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$, and
4. Order linearity: $a \preceq b$ and $c \preceq d \Longrightarrow a + c \preceq b + d$; and $a \preceq b \Longrightarrow \lambda a \preceq \lambda b$ for $\lambda \geq 0$.

The linear space E together with a partial order $\preceq$ becomes a **partially ordered linear or vector space**. Two elements x and y in a partially ordered linear space E are called **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. We introduce a norm $\|\cdot\|$ in a partially ordered linear space E so that E becomes now a **partially ordered normed linear space**. If E is complete with respect to the metric d defined through the above norm, then it is called a **partially ordered complete normed linear space**. We frequently need the concept of regularity of E in what follows. It is known that E is **regular** if $\{x_n\}$ is a nondecteasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The details of an ordered Banach space and operator theoretic techniques are given in Heikkilä and Lakshmikantham [19] and Dhage [11] and Carl and Hekkilä [6] and the references therein.

The following definitions have been introduced in Dhage [11] and are frequently used in the subsequent part of this paper.

Definition 2.1. *A subset S of a partially ordered normed linear space E is called* **partially bounded** *if every chain in S is bounded. S is called* **uniformly partially bounded** *if all chains in S are bounded with a unique bound.*

It is noted that every bounded set S in a partially ordered normed linear space E is uniformly partially bounded and every uniformly partially bounded set S is partially bounded, however the reverse implication may not hold.

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called a monotone nonincreasing if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. A monotone mapping $\mathcal{T}$ is one which is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

The following terminologies may be found in any book on nonlinear operators, equations and applications.

An operator $\mathcal{T}$ on a normed linear space E into itself is called **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called a **completely continuous** on E. The details of completely continuous operators on Banach spaces appear in Zeidler [24].

Definition 2.4. *A mapping $T : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is a bounded subset of E for all totally ordered sets or chains C in E. Finally, $\mathcal{T}$ is called* **uniformly partially bounded** *if $\mathcal{T}(C)$ is a bounded by a unique constant for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. Finally, if $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **a partially completely continuous** *on E.*

Remark 2.1. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.5 (Dhage [11]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has compatibility property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ possesses compatibility property w.r.t. the usual componentwise order relation and the usual standard norm in it.

The following applicable hybrid fixed point theorem in partially ordered normed linear spaces is proved in Dhage [11].

Theorem 2.1 (Dhage [11]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered linear space and suppose that the norm in E is such that E is a complete normed linear space. Let $\mathcal{T} : E \to E$ be a nondecreasing, partially compact and continuous mapping. Further if the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible and if there is an element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*.*

Remark 2.2 (Dhage [12]). The assertion of above Theorem 2.1 also remains true if we replace the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in E with the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain C of E. The later condition holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

We note that Theorem 2.1 is very much useful for proving the existence theorems for several dynamical systems in nonlinear analysis modeled on nonlinear differential and integral equations (cf. Dhage and Dhage [14]). Here, in the following section we generalize above hybrid fixed point theorem under weaker conditions via partial measure of noncompactness and apply it to obtain the existence of the solutions of a certain nonlinear functional integral equation in a constructive way.

3 Nonlinear $\mathcal{D}$-set-contraction Mappings

Assume that $(E, ||\cdot||)$ is an infinite dimensional partially ordered Banach space with zero element θ. If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C} = C \cup C'$. The set $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C \in \overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\epsilon > 0$ there exists a $c \in C$ such that $d(c, z) < \epsilon$ and $x \leq z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way.

In what follows, we denote by $\mathcal{P}_{cl}(E)$, $\mathcal{P}_{bd}(E)$, $\mathcal{P}_{rcp}(E)$, $\mathcal{P}_{ch}(E)$, $\mathcal{P}_{bd,ch}(E)$, $\mathcal{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. Now we introduce the concept of a partial measure of noncompactness in E on the lines of usual classical theory.

Definition 3.6. *A mapping $\mu^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0, \infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following conditions:*

1^o $\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathcal{P}_{rcp,ch}(E)$,

2^o $\mu^p(\overline{C}) = \mu^p(C)$

3^o *μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leq \mu^p(C_2)$*

4^o *If $\{C_n\}$ is a sequence of closed chains from $\mathcal{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \mu^p(C_n) = 0$, then the intersection set $\overline{C}_\infty = \bigcap_{n=1}^\infty C_n$ is nonempty.*

The family of sets described in 1^o is said to be the *kernel of the partial measure of noncompactness* μ^p and is defined as

$$\ker \mu^p = \{C \in \mathcal{P}_{bd,ch}(E) \mid \mu^p(C) = 0\}.$$

Clearly, $\ker \mu^p \subset \mathcal{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family $\ker \mu^p$. In fact, since $\mu^p(C_\infty) \le \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu^p$. This simple observation will be essential in our further investigations.

The partial measure μ^p of noncompactness is called **sublinear** if it satisfies

5^o $\mu^p(C_1 + C_2) \le \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathcal{P}_{bd,ch}(E)$, and

6^o $\mu^p(\lambda C) = |\lambda| \mu^p(C)$ for $\lambda \in \mathbb{R}$.

Again, μ^p is said to satisfy **maximum property** if

7^o $\mu^p\Big(C_1 \bigcup C_2\Big) = \max\{\mu^p(C_1), \mu^p(C_2)\}$.

Finally, μ^p is said to be **full** if

8^o $\ker \mu^p = \mathcal{P}_{rcp,ch}(E)$.

Example 3.1. Define two functions $\alpha^p, \beta^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+$ by

$$\alpha^p(C) = \inf\left\{ r > 0 \;\Big|\; C = \bigcup_{i=1}^{n} C_i,\ \operatorname{diam}(C_i) \le r\, \forall\, i \right\},$$

where $C \in \mathcal{P}_{bd,ch}(E)$ and $\operatorname{diam}(C_i) = \sup\{\|x - y\| : x, y \in C_i\}$, and

$$\beta^p(C) = \inf\left\{ r > 0 \;\Big|\; C \subset \bigcup_{i=1}^{n} \mathcal{B}(x_i, r) \ \text{ for some } x_i \in E \right\},$$

where $\mathcal{B}(x_i, r) = \{x \in E : \|x_i - x\| < r\}$. It is easy to prove that α^p and β^p are partial measures of noncompactness called respectively the partial Kuratowskii and partial ball or Hausdorff measures of noncompactness in E.

The above two partially Kuratowskii and Hausdorff measures of noncompactness α^p and β^p are sublinear, full and enjoy the maximum property in E. The verification of this claim is similar to classical Kuratowskii and Hausdorff measures of noncompactness given in Appell [1] and Banas and Goebel [2]. So we omit the details.

Definition 3.7. *A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathcal{D}$**-function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.*

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
\psi(r) &= k\,r, \quad \text{for some constant } k > 0,\\
\psi(r) &= \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
\psi(r) &= \tan^{-1} r,\\
\psi(r) &= e^r - 1,\\
\psi(r) &= \tanh r,\\
\psi(r) &= \sinh r,\\
\psi(r) &= \log(1 + r), \quad \text{and}\\
\psi(r) &= r - \log(1 + r).
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations. See Dhage [7, 8, 9, 10, 11, 12] and the references therein.

Remark 3.3. *If ϕ, ψ $\mathbb{R}_+ \to \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$. The set of all $\mathcal{D}$-functions is denoted by $\mathfrak{D}$.*

Definition 3.8. *A nondecreasing mapping $\mathcal{T} : E \to E$ is called* **partially nonlinear $\mathcal{D}$-set-Lipschitz** *if there exists a $\mathcal{D}$-function ψ such that*

$$\mu^p(\mathcal{T}C) \leq \psi(\mu^p(C))$$

for all bounded chain C in E. $\mathcal{T}$ is called **partially k-set-Lipschitz** *if $\psi(r) = k\,r$, $k > 0$. $\mathcal{T}$ is called* **partially k-set-contraction** *if it is a partially k-set-Lipschitz with $k < 1$. Finally, $\mathcal{T}$ is called a* **partially nonlinear $\mathcal{D}$-set-contraction** *in E if it is a partially nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.*

The following lemma (see Dhage [11, page 159]) is frequently used in the analytical fixed point theory of metric spaces. See also Dhage [7, 8] and the references cited therein.

Lemma 3.1 (Dhage [11]). *If φ is a $\mathcal{D}$-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$ and vice-versa.*

Our main result of this section is as follows.

Theorem 3.2. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. Define a sequence $\{x_n\}$ of points in S by

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

Since $\mathcal{T}$ is nondecreasing and $x_0 \preceq \mathcal{T}x_0$, we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{3.2}$$

Denote

$$C_n = \{x_n, x_{n+1}, \ldots\}$$

for $n = 0, 1, 2, \ldots$. By the construction, each C_n is a bounded chain in S and

$$C_n = T(C_{n-1}), \quad n = 1, 2, \ldots.$$

From the definition of C_n's, it follows that

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots,$$

and so

$$\overline{C_0} \supset \overline{C_1} \supset \cdots \supset \overline{C_n} \supset \cdots. \tag{3.3}$$

Therefore, by nondecreasing nature of μ^p we obtain

$$\begin{aligned} \mu^p(C_n) &= \mu^p(\mathcal{T}(C_{n-1})) \\ &\leq \psi(\mu^p(C_{n-1})) \\ &\leq \psi^2(\mu^p(C_{n-2})) \\ &\vdots \\ &\leq \psi^n(\mu^p(C_0)). \end{aligned} \tag{3.4}$$

Taking the limit superior as $n \to \infty$ in the above equality (3.4), in view of Lemma 3.1 we obtain,

$$\mu^p(\overline{C_n}) = \lim_{n\to\infty} \mu^p(C_n) \leq \limsup_{n\to\infty} \psi^n(\mu^p(C_0)) = \lim_{n\to\infty} \psi^n(\mu^p(C_0)) = 0. \tag{3.5}$$

Hence, by condition (4^o) of μ^p,

$$\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n \neq \emptyset \quad \text{and} \quad C_\infty \in \mathcal{P}_{rcp,ch}(E).$$

From (3.5) it follows that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\mu^p(C_n) < \epsilon \quad \forall\, n \geq n_0.$$

This shows that $\overline{C}_{n_0}$ and consequently $\overline{C}_0$ is a compact chain in E. Hence, $\{x_n\}$ has a convergent subsequence. Further since the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of S, the whole sequence $\{x_n\} = \{\mathcal{T}^n x_0\}$ is convergent and converges monotonically to a point, say $x^* \in \overline{C}_0$. Since the ordered space space E is regular, we have that $x_n \leq x^*$. Finally, from the partial continuity of $\mathcal{T}$, we get

$$\mathcal{T}x^* = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

This completes the proof. □

Remark 3.4. The regularity of E and the partial continuity of $\mathcal{T}$ in above Theorem 3.2 may be replaced with the continuity of the operator $\mathcal{T}$ on E.

Remark 3.5. If the set $\mathcal{F}_{\mathcal{T}}$ of solutions to the above operator equation $\mathcal{T}x = x$ is a chain, then all the solutions belonging to $\mathcal{F}_{\mathcal{T}}$ are comparable. Further, if $\mu^p(\mathcal{F}_{\mathcal{T}}) > 0$, then $\mu^p(\mathcal{F}_{\mathcal{T}})) = \mu^p(\mathcal{T}(\mathcal{F}_{\mathcal{T}})) \leq \psi(\mu^p(\mathcal{F}_{\mathcal{T}})) < \mu^p(\mathcal{F}_{\mathcal{T}})$ which is a contradiction. Consequently, $\mathcal{F}_{\mathcal{T}} \in \ker \mu^p$. This simple fact has been utilized in the study of different qualitative properties of the comparable solutions of the dynamic systems under consideration.

Remark 3.6. Suppose that the order relation $\preceq$ is introduced in E with the help of an order cone $\mathcal{K}$ which is a non-empty closed set $\mathcal{K}$ in E satisfying (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda\,\mathcal{K} \subseteq \mathcal{K}$ and (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{0\}$ (cf. [19]). Then the order relation $\preceq$ in E is defined as $x \preceq y \iff y - x \in \mathcal{K}$. The element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T}x_0$ in above Theorem 3.2 is called a lower solution of the operator equation $x = \mathcal{T}x$. If the operator equation $x = \mathcal{T}x$ has more than one lower solution and set of all these lower solutions are comparable, then the corresponding set $\mathcal{F}(Q)$ of solutions to the above operator equation is a chain and hence all solutions in $\mathcal{F}(Q)$ are comparable. To see this, let x_0 and y_0 be any two lower solutions of the above operator equation such that $x_0 \preceq y_0$ and let x^* and y^* respectively be the corresponding solutions under the conditions of Theorem 3.2. Then, by definition of $\preceq$, one has $y_0 - x_0 \in \mathcal{K}$ and from monotone nondecreasing nature of $\mathcal{T}$ it follows that $\mathcal{T}^n y_0 - \mathcal{T}^n x_0 \in \mathcal{K}$. Since $\mathcal{K}$ is closed, we have that $y^* - x^* \in \mathcal{K}$ or $x^* \preceq y^*$.

Theorem 3.3. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction mapping. If there exists an element $x_0 \in S$ such that $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is similar to Theorem 3.2 and so we omit the details. □

Observe that Theorems 3.2 and 3.3 improve and generalize the hybrid measure theoretic fixed point theorems of Dhage [11, 12], Dhage and Dhage [15] and Dhage *et.al.* [16] under weaker compatibility condition. As a consequence of Theorems 3.2 and 3.3, we derive some interesting corollaries.

Corollary 3.1. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

We remark that Corollary 3.1 is very much useful for proving the existence results in the theory of nonlinear differential and integral equations. See Dhage [12] and the references therein. Before giving a further generalization of Theorem 3.2, we state a useful definition.

Definition 3.9. *A nondecreasing mapping $\mathcal{T} : E \to E$ is called partially condensing if for any bounded chain C in E, $\mu^p(\mathcal{T}(C)) < \mu^p(C)$ for $\mu^p(C) > 0$.*

We remark that every partially compact and partially nonlinear $\mathcal{D}$-set-contraction mappings are partially condensing, however the reverse implications may not hold.

Remark 3.7. It is clear tat every partially k-set-contraction is a partially nonlinear $\mathcal{D}$-set-contraction and every partially nonlinear $\mathcal{D}$-set-contraction is partially condensing, however, the converse implications may not be true. Actually, it is very difficult to prove practically a selfmaping of a normed linear space is partially condensing and we rarely come across a mapping of this kind. But the mappings with nonlinear $\mathcal{D}$-set-contraction and k-$\mathcal{D}$-set-contraction are easily available in the literature.

Theorem 3.4. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is standard and hence we omit the details. □

Corollary 3.2. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially compact mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

We remark that the hybrid fixed point theorems proved above improve and generalize the hybrid fixed point theorems of Dhage and Dhage [14] and Dhage *et.al.* [16] under weaker compatibility condition.

Remark 3.8. We note that the proof of Theorems 3.2, 3.3 and 3.4 do not make any use of the linear structure of underlined space E, and therefore, Theorems 3.2, 3.3 and 3.4 also remain true in the setting of a partially ordered complete metric space E.

In view of above Remark 3.8, the slight generalizations of Theorems 3.2, 3.3 and 3.4 are as follows.

Theorem 3.5. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Corollary 3.3. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially nonlinear $\mathcal{D}$-set-contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Corollary 3.4. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete metric space $(E, \preceq, d)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing, partially continuous and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Again, the regularity of E in above Theorem 3.5 and Corollaries 3.3 and 3.4 may also be replaced with a stronger condition of continuity of the operator $\mathcal{T}$ on E. The following hybrid fixed point theorems are employed for proving the existence and uniqueness of the solutions of nonlinear equations. Before stating these results, we consider the following definition in what follows.

Definition 3.10. *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{3.6}$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k. If $k < 1$, $\mathcal{T}$ is called a partially contraction with contraction constant k. Finally, $\mathcal{T}$ is called partially nonlinear $\mathcal{D}$-contraction if it is a partially nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.

Lemma 3.2. *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered complete normed linear space. If $\mathcal{T} : E \to E$ is a nondecreasing and partially nonlinear $\mathcal{D}$-Lipschitz mapping, then for any bounded chain C in E,*

$$\alpha^p(\mathcal{T}C) \leq \psi(\alpha^p(C)) \tag{3.7}$$

where α^p is a partial Kurotowskii measure of noncompactness and ψ is a associated $\mathcal{D}$-function of $\mathcal{T}$ on E.

Proof. The proof is similar to standard result for usual nonlinear $\mathcal{D}$-Lipschitz mapping with the classical Kurotowskii measure of noncomactness α in the Banach space E. We omit the details. □

Corollary 3.5 (Dhage [11]). *Let S be a non-empty, closed and partially bounded subset of the partially ordered complete metric space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{T} : S \to S$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Proof. by Lemma 3.2, $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. We simply show that the order relation $\preceq$ and the metric d are compatible in every compact chain C of S. Let C be arbitrary compact chain in S. Assume that $\{x_n\}$ is monotone nondecreasing or monotone nonincreasing and that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent. Now it can be shown as in Dhage [11] that $\{x_n\}$ is a Cauchy sequence in S. Hence the whole sequence $\{x_n\}$ is convergent and converges to a point in S. Consequently, $\preceq$ and the metric d are compatible in C. Now the desired conclusion follows by an application of Theorem 3.2. This completes the proof. □

We notice that Corollary 3.5 includes a well-known hybrid fixed point theorems of Nieto and Lopez [22] and Dhage [11] for partially contraction and monotone mappings in a partially ordered complete metric space.

4 FPTs of Krasnoselskii and Dhage Type

The study of hybrid fixed oint theorems for the sum of two operators is attributed to Krasnoselskii [20] whereas the study involving the product of two operators in Banach algebra is attributed to Dhage [7]. Again the study of fixed point theorems in a Banach algebras involving the sum as well as product of operators is credited to Dhage [11]. Here, we prove the analogous results for the sum and the product of operators in a partially ordered complete normed linear space which are useful in applications to perturbed nonlinear differential and integral equations for proving the existence and attractivity of the solutions under weaker mixed partially compact, partially Lipschitz and monotonic conditions (cf. Dhage *et.al.* [17]).

4.1 FPTs of Krasnoselskii type

Theorem 4.6. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is partially completely continuous,

(c) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and

(d) there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x + \mathcal{B}x = x \tag{4.8}$$

has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,...; converges monotonically to x^*.

Proof. Define a mapping $\mathcal{T}$ on S by

$$\mathcal{T}x = \mathcal{A}x + \mathcal{B}x.$$

By hypothesis (c), $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing, $\mathcal{T}$ is nondecreasing on S. From the partial continuity of the operators $\mathcal{A}$ and $\mathcal{B}$ and the continuity of the binary composition addition, it follows that the operator $\mathcal{T}$ is partially continuous on S. Again, by hypothesis (c), $x_0 \preceq \mathcal{T}x_0$. Next, we show that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Let C be a chain in S. Then by definition of $\mathcal{T}$, we have

$$\mathcal{T}(C) \subseteq \mathcal{A}(C) + \mathcal{B}(C).$$

Since $\mathcal{T}$ is nondecreasing and partially continuous, $\mathcal{T}(C)$ is again a chain in S which is bounded. By properties (5^o) and (8^o), α^p is subadditive and full. As a result,

$$\alpha^p(\mathcal{T}C) \leq \alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{B}(C)) \leq \psi(\alpha^p(C))$$

which shows that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Next, the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S, so the desired conclusion follows by an application of Theorem 3.2. This completes the proof. □

Theorem 4.7. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is partially completely continuous,

(c) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and

(d) there exists an element $x_0 \in S$ such that $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation (4.8) *has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,…; converges monotonically to x^*.*

4.2 FPTs of Dhage type

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{4.9}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in the normed linear algebra E. Then the following lemma is immediate and is useful in the hybrid fixed point theory in Banach algebras and applications.

Lemma 4.3 (Dhage [9]). *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 4.11. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

For any two chains C_1 and C_2 in E, denote

$$C_1C_2 = \{x \in E \mid x = c_1c_2,\ c_1 \in C_1 \text{ and } c_2 \in C_2\}.$$

Then we have the following lemma.

Lemma 4.4. *If C_1 and C_2 are two bounded chains in a partially ordered normed linear algebra E, then*

$$\alpha^p(C_1C_2) \leq \|C_2\|\alpha^p(C_1) + \|C_1\|\alpha^p(C_2) \tag{4.10}$$

where $\|C\| = \sup\{\|c\| \mid c \in C\}$.

Theorem 4.8. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear algebra $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A}, \mathcal{B} : S \to \mathcal{K}$ and $\mathcal{C} : E \to E$ be three nondecreasing operators such that*

(a) *$\mathcal{A}$ and $\mathcal{C}$ are partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,*

(b) *$\mathcal{B}$ is partially continuous and compact,*

(d) *$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x \in S$ for all $x \in S$,*

(c) *$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \|\mathcal{B}(S)\|$, and*

(e) *there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{4.11}$$

has a solution x^* in S and the sequence $\{x_n\}$ of successive approximations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n + \mathcal{C}x_n$, n=0,1,...; converges monotonically to x^*.

Proof. Suppose that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$. Define a mapping $\mathcal{T}$ on S by

$$\mathcal{T}x = \mathcal{A}x\,\mathcal{B}x + \mathcal{C}x.$$

By hypothesis (c), $\mathcal{T}$ defines a mapping $\mathcal{T} : S \to S$. Since $\mathcal{A}$ and $\mathcal{B}$ are positive and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are nondecreasing, T is nondecreasing on S. From the partial continuity of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and the continuity of the binary compositions addition and multiplication, it follows that the operator $\mathcal{T}$ is partially continuous on S. Again, by hypothesis (d), $x_0 \preceq \mathcal{T}x_0$. Next, we show that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Let C be a chain in S. Then by definition of $\mathcal{T}$, we have

$$\mathcal{T}(C) \subseteq \mathcal{A}(C)\,\mathcal{B}(C) + \mathcal{C}(C).$$

Since $\mathcal{T}$ is nondecreasing and partially continuous, $\mathcal{T}(C)$ is again a chain in S. By properties (5^o) and (8^o), α^p is a subadditive and full partial measure of noncompactness. As a result, we have

$$\begin{aligned} \alpha^p(\mathcal{T}C) &\leq \|\mathcal{A}(C)\|\alpha^p(\mathcal{B}(C)) + \|\mathcal{B}(C)\|\alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{C}(C)) \\ &\leq \|\mathcal{B}(E)\|\alpha^p(\mathcal{A}(C)) + \alpha^p(\mathcal{C}(C)) \\ &\leq M\psi_{\mathcal{A}}(\alpha^p(C)) + \psi_{\mathcal{C}}(\alpha^p(C)) \\ &= \psi(\alpha^p(C)), \end{aligned} \tag{4.12}$$

where $\psi(r) = M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$ and the constant M exists in view of the fact that $\mathcal{B}$ is compact operator on S. This shows that $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-set-contraction on S. Next, the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S, so the desired conclusion follows by an application of Theorem 3.2. Similarly, if $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$, then using Theorem 3.3 it can be proved that $\mathcal{T}$ has a fixed point. This completes the proof. □

Remark 4.9. If we take $\psi_{\mathcal{A}}(r) = \frac{L_1 r}{K+r}$ and $\psi_{\mathcal{C}}(r) = L_2\, r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\frac{L_1 M}{K+r} + L_2 < 1$ for each real number $r > 0$. Similarly, if $\psi_{\mathcal{A}}(r) = L_1\, r$, and $\psi_{\mathcal{C}}(r) = \frac{L_2 r}{K+r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $M\,L_1 + \frac{L_2 M}{K+r} < 1$ for each real number $r > 0$.

Corollary 4.6. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear algebra $(E, \preceq, \|\cdot\|)$ such that $S \cap \mathcal{K} \neq \emptyset$ and the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of S. Let $\mathcal{A}, \mathcal{B} : S \to \mathcal{K}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*

(b) *$\mathcal{B}$ is partially continuous and compact,*

(c) *$\mathcal{A}x\,\mathcal{B}x \in S$ for all $x \in S$,*

(d) *$M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \|\mathcal{B}(S)\|$, and*

(e) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x = x \tag{4.13}$$

has a positive solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$, n=0,1,...; converges monotonically to x^*.

Remark 4.10. The hypotheses (b) and (c) of Theorem 4.8 may be replaced with the weaker hypotheses as follows:

(b′) $\mathcal{B}$ is partially continuous and uniformly partially compact, and

(c′) $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \in \mathcal{P}_{ch}(S)\}$.

The proof of Theorem 4.8 under these new hypotheses is essentially the same as that given in the theorem. Similarly, the conclusion of Corollary 4.6 also remains true under the corresponding changes in the hypotheses (b) and (c) thereof. We mention that Theorem 4.8 and Corollary 4.6 are useful in the study of quadratic nonlinear differential and integral equations for discussing the qualitative aspects of the solutions.

5 Functional Integral Equations

In this section, we are going to prove a result on the existence and uniform global attractivity of the solutions of a nonlinear functional integral equation. Our investigations will be carried out in the Banach space of real functions which are defined, continuous and bounded on the right half real axis $\mathbb{R}_+$. The integral equation in question has rather general form and contains as particular cases a few of other functional equations and nonlinear integral equations of Volterra type. The main tool used in our considerations is the technique of partial measures of noncompactness and the measure theoretic hybrid fixed point result established in Theorem 3.2. The partial measure of noncompactness used in this paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize the comparable solutions in terms of uniform global ultimate attractivity. This assertion means that all the possible comparable solutions of the functional integral equation in question are globally uniformly attractive in the sense of notion defined in the following section.

5.1 Notation, Definitions and Auxiliary facts

As mentioned earlier, our considerations will be placed in the function space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. We place the HFIE (3.1) in the space $E = BC(\mathbb{R}_+, \mathbb{R})$. Define a norm $\|\cdot\|$ and the order relation $\le$ in E by

$$\|x\| = \sup\{|x(t)| : t \ge 0\}. \tag{5.14}$$

and

$$x \le y \iff x(t) \le y(t) \tag{5.15}$$

for all $t \in \mathbb{R}_+$. Clearly, E is a partially ordered Banach space with respect to the above norm $\|\cdot\|$ and the order relation $\le$. The following lemma follows immediately by an application of Arzellá-Ascoli theorem.

Lemma 5.5. *Let $\big(BC(\mathbb{R}_+, \mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (5.14) and (5.15) respectively. Then the norm $\|\cdot\|$ and the order relation $\le$ are compatible in every partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$.*

Proof. Let S be a partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots \tag{$*$}$$

for each $t \in \mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}$ is convergent and converges to the point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n(t)\}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

For our purposes we introduce a partial measure of noncompactness which is a handy tool of the partial Hausdorff measure of noncompactness in the study of the solutions of certain nonlinear integral equations. To define this partial measure, let us fix a nonempty and bounded chain X of the space $BC(\mathbb{R}_+,\mathbb{R})$ and a positive real number T. For $x \in X$ and $\epsilon \geq 0$ denote by $\omega^T(x,\epsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$ defined by

$$\omega^T(x,\epsilon) = \sup\{|x(t)-x(s)| : \ t,s \in [0,T], \ |t-s| \leq \epsilon\} .$$

Next, let us put

$$\omega^T(X,\epsilon) = \sup\{\omega^T(x,\epsilon) : \ x \in X\} ,$$

$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X,\epsilon) ,$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X) .$$

The partial Hausdorff measure of noncompactness β^p in the function space $C([0,T],\mathbb{R})$ of continuous real-valued functions defined on a closed and bounded interval $[0,T]$ in $\mathbb{R}$, is very much useful in the applications to nonlinear differential and integral equations and it can be shown that

$$\beta^p(X) = \frac{1}{2}\omega_0^T(X)$$

for all bounded chain X in $C([0,T],\mathbb{R})$. The proof of this fact follows from the arguments that given in Banas and Goebel [2] and the references therein. Similarly, ω_0 is a handy tool of partial measure of noncompactness in the ordered Banach space $BC(\mathbb{R}_+,\mathbb{R})$ useful for some practical applications to nonlinear differential and integral equations.

Now, for a fixed number $t \in \mathbb{R}_+$ and a fixed bounded chain X in $BC(\mathbb{R}_+,\mathbb{R})$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Let

$$\delta_a(X(t)) = |X(t)| = \sup\{|x(t)| : \ x \in X\} ,$$

$$\delta_a^T(X(t)) = \sup_{t\geq T} \delta_a(X(t)) = \sup_{t\geq T} |X(t)|$$

and

$$\delta_a(X) = \lim_{T\to\infty} \delta_a^T(X(t)) = \limsup_{t\to\infty} \ |X(t)|.$$

Again, for a fixed real number c, denote

$$X(t)-c = \{x(t)-c \ : \ x \in X\}$$

and

$$\delta_b(X(t)) = |X(t)-c| = \sup\{|x(t)-c| : \ x \in X\} .$$

Define

$$\delta_b^T(X(t)) = \sup_{t\geq T} \delta_b(X(t)) = \sup_{t\geq T} |X(t)-c|$$

and

$$\delta_b(X) = \lim_{T\to\infty} \delta_b^T(X(t)) = \limsup_{t\to\infty} \ |X(t)-c|.$$

Similarly, let

$$\delta_c(X(t)) = \operatorname{diam} X(t) = \sup\{|x(t)-y(t)| : \ x,y \in X\} ,$$

$$\delta_c^T(X(t)) = \sup_{t\geq T} \delta(X(t)) = \sup_{t\geq T} \operatorname{diam} X(t)$$

and

$$\delta_c(X) = \lim_{T\to\infty} \delta^T(X(t)) = \limsup_{t\to\infty} \text{ diam } X(t).$$

The details of the functions δ_a, δ_b and δ_c appear in Dhage [12]. Finally, let us consider the functions μ_a^p, μ_b^p and μ_c^p defined on the family of bounded chains in $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$\mu_a^p(X) = \max\left\{\omega_0(X)\,,\, \delta_a(X)\right\}\,, \tag{5.16}$$

$$\mu_b^p(X) = \max\left\{\omega_0(X)\,,\, \delta_b(X)\right\} \tag{5.17}$$

and

$$\mu_c^p(X) = \max\left\{\omega_0(X)\,,\, \delta_c(X)\right\}\,. \tag{5.18}$$

It can be shown that the function μ_a^p, μ_b^p and μ_c^p are partially measures of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The components ω_0 and δ_a are called the characteristic values of the partial measure of noncompactness μ_a^p. Similarly, ω_0, δ_b and ω_0, δ_c are respectively the characteristic values of the partial measures of noncompactness μ_b^p and μ_c^p in $BC(\mathbb{R}_+, \mathbb{R})$.

Remark 5.11. The kernel ker μ_c^p consists of nonempty and bounded chains X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by the functions from X tends to zero at infinity. Similarly, the kernels ker μ_a^p, and ker μ_b^p consist of nonempty and bounded chains X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X around the lines respectively $x(t) = c$ and $x(t) = 0$ tends to zero at infinity. These particular characteristics of ker μ_a^p, ker μ_b^p and ker μ_c^p have been useful in establishing the global attractivity and global asymptotic stability of the comparable solutions of the considered functional integral equations.

In order to introduce further concepts used in the paper, let us assume that Ω is a nonempty chain of the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+, \mathbb{R})$.

Consider the operator equation of the form

$$x(t) = Qx(t),\ t \in \mathbb{R}_+. \tag{5.19}$$

Definition 5.12. We say that comparable solutions of the equation (5.19) are **locally attractive** if there exists a ball $\overline{\mathcal{B}}(x_0, r)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary comparable solutions $x = x(t)$ and $y = y(t)$ of the equation (5.19) belonging to $\overline{\mathcal{B}}(x_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} \left[x(t) - y(t)\right] = 0\,. \tag{5.20}$$

In the case when limit (5.20) is uniform with respect to the set $\overline{\mathcal{B}}(x_0, r) \cap \Omega$, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{5.21}$$

for all $x, y \in \overline{\mathcal{B}}(x_0, r) \cap \Omega$ being the comparable solutions of (5.19) and for $t \ge T$, we will say that the comparable solutions of the operator equation (5.19) are **uniformly locally ultimately attractive** defined on $\mathbb{R}_+$.

Definition 5.13. A solution $x = x(t)$ of equation (5.19) is said to be **globally ultimately attractive** if (5.20) holds for each comparable solution $y = y(t)$ of (5.19) on $\mathbb{R}_+$. Other words we may say that the comparable solutions of the equation (5.19) are globally ultimately attractive if for arbitrary comparable solutions $x(t)$ and $y(t)$ of (5.19) the condition (5.20) is satisfied. In the case when condition (5.20) is satisfied uniformly with respect to the set $\overline{\mathcal{B}}(x_0, r) \cap \Omega$, i.e. if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality (5.21) is satisfied for all $x, y \in \Omega$ being the comparable solutions of (5.19) and for $t \ge T$, we will say that the comparable solutions of the equation (5.19) are **uniformly globally ultimately attractive** on $\mathbb{R}_+$.

Let us mention that the concept of asymptotic stability may be found in Banas and Dhage [3] and references therein whereas the concept of global attractivity of solutions is introduced in Dhage [9] and proved attractivity results for certain nonlinear integral equations. We mention that the present approach is constructive and different from that given in the above stated papers.

5.2 Integral Equation and Attractivity Result

Now, we will investigate the following nonlinear hybrid functional integral equation (in short HFIE)

$$x(t) = f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \tag{5.22}$$

for all $t \in \mathbb{R}_+$, where the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, 2$, are continuous.

By a *solution* of the HFIE (5.22) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (5.22), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

Observe that the above integral equation (5.22) has been discussed in Dhage [9] under strong Lipschitz condition for the attractivity of solutions and includes several classes of functional, integral and functional-integral equations considered in the literature (cf. [1, 3, 9] and references therein). Let us also mention that the functional integral equation considered in [3, 9] is a special case of the equation (5.22), where $\alpha_1(t) = \alpha_2(t) = \beta(t) = \gamma(t) = t$.

The equation (5.22) will be considered under the following assumptions:

(H_1) The functions $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\alpha_1(t) \geq t$ and $\alpha_2(t) \geq t$ for all $t \in \mathbb{R}_+$.

(H_2) The function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $F(t) = |f(t, 0, 0)|$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup_{t \geq 0} F(t)$.

(H_3) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \frac{L \max\{x_1 - y_1\,,\, x_2 - y_2\}}{K + \max\{x_1 - y_1\,,\, x_2 - y_2\}}$$

for all $t \in \mathbb{R}_+$ and $(x, x_2), (y_1, y_2) \in v \times \mathbb{R}$ with $x_1 \geq y_1$ and $x_2 \geq y_2$. Moreover, $L \leq K$.

(H_4) $g(t, s, x, y)$ is nondecreasing in x and y for each $t, s \in \mathbb{R}_+$.

(H_5) There exists an element $u \in C(J, \mathbb{R})$ such that

$$u(t) \leq f(t, u(\alpha_1(t)), u(\alpha_2(t)) + \int_{t_0}^{\beta(t))} g(t, s, u(\gamma_1(s)), u(\gamma_2(s)))\, ds$$

for all $t \in \mathbb{R}_+$.

(H_6) There exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x, y)| \leq a(t)b(s)$$

for $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s) ds = 0\,.$$

(H_7) There exists a real number c such that $f(t, c, c) = c$ for all $t \in \mathbb{R}_+$.

The hypotheses (H_1)-(H_2) and (H_4), (H_6) have been widely used in the literature in the theory of nonlinear differential and integral equations. The special case of hypothesis (H_3) with $L < K$ is considered in Nieto and Lopez [22]. Now we formulate the main existence result for the integral equation (5.22) under above mentioned natural conditions.

Theorem 5.9. *Assume that the hypotheses (H$_1$) through (H$_6$) hold. Then the functional integral equation (5.22) has at least one solution x^* in the space $BC(\mathbb{R}_+, \mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_n(t) &= f(t, x_{n-1}(\alpha_1(t)), x_{n-1}(\alpha_2(t))) \\ &\quad + \int_0^{\beta(t)} g(t, s, x_{n-1}(\gamma_1(s)), x_{n-1}(\gamma_2(s)))\, ds, \ \ t \in \mathbb{R}_+, \end{aligned} \tag{5.23}$$

for each $n \in \mathbb{N}$ with $x_0 = u$ converges monotonically to x^*. Moreover, the comparable solutions of the equation (5.22) are uniformly globally ultimately attractive on $\mathbb{R}_+$.

Proof. We seek the solutions of the HFIE (5.22) in the space $BC(\mathbb{R}_+, \mathbb{R})$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Set $E = BC(\mathbb{R}_+, \mathbb{R})$. Then, in view of Lemma 5.5, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operator Q defined on the space $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$Qx(t) = f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds, \ \ t \in \mathbb{R}_+. \tag{5.24}$$

Observe that in view of our assumptions, for any function $x \in E$ the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q : E \to E$. Let $x_0 = u$ and define an open ball $\mathcal{B}(x_0, r)$ in E, where $r = \|x_0\| + L + F_0 + V$. We show that Q satisfies all the conditions of Theorem 3.2 on $S = \overline{\mathcal{B}}(x_0, r)$. This will be achieved in a series of following steps:

Step I: *Q is a nondecreasing on S.*

Let $x, y \in S$ be such that $x \leq y$. Then by hypothesis (H$_3$)-(H$_4$), we obtain

$$\begin{aligned} Qx(t) &= f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \\ &\leq f(t, y(\alpha_1(t)), y(\alpha_1(t)) + \int_0^{\beta(t)} g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))\, ds \\ &= Qy(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on S.

Step II: *Q maps a closed and partially bounded set S into itself.*

Let X be a chain in S and let $x \in X$. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v(t) = \lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)\, ds \tag{5.25}$$

is continuous and in view of hypothesis (H$_6$), the number $V = \sup_{t\geq 0} v(t)$ exists. Moreover if $x \geq 0$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned} |Qx(t)| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t)))| + \int_0^{\beta(t)} |g(t, s, x(s))|\, ds \\ &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, 0, 0)| \\ &\qquad + |f(t, 0, 0)| + a(t) \int_0^{\beta(t)} b(s)\, ds \\ &\leq \frac{L \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}} + F(t) + v(t) \\ &\leq \frac{L\|x\|}{K + \|x\|} + F_0 + V \\ &= L + F_0 + V \end{aligned} \tag{5.26}$$

Similarly, if $x \leq 0$, then it can be shown that $|Qx(t)| \leq L + F_0 + V$ for all $t \in \mathbb{R}_+$. Taking the suremum over t, we obtain $||Qx|| \leq L + F_0 + V$ for all $x \in X$. This means that the operator Q transforms any chain X into a bounded chain in E. Moreover, we have

$$\|x_0 - Qx\| \leq \|x_0\| + \|Qx\| \leq \|x_0\| + L + F_0 + V$$

for all $x \in X$. More precisely, we infer that the operator Q transforms every chain X in $\overline{\mathcal{B}}(x_0, r)$ into the chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$, where $r = \|x_0\| + L + F_0 + V$. As a result, Q defines a mapping $Q : \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r)) \to \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r))$ and so Q maps a closed and partially bounded set $S = \overline{\mathcal{B}}(x_0, r)$ into itself. Moreover, in view of Lemma 5.5, every compact chain in S possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Step III: *Q is a partially continuous on S.*

Now we show that the operator Q is partially continuous on S. To do this, let X be a chain in S and let us fix an arbitrary $\epsilon > 0$ and take $x, y \in X$ such that $x \geq y$ and $||x - y|| \leq \epsilon$. Then we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq \big|f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \left| \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right. \\
&\quad \left. - \int_0^{\beta(t)} g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))\, ds \right| \\
&\leq \big|f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \int_0^{\beta(t)} |g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\, ds \\
&\quad + \int_0^{\beta(t)} |g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\, ds \\
&\leq \frac{L \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\
&\quad + 2a(t) \int_0^{\beta(t)} b(s)\, ds \\
&\leq \frac{L\|x - y\|}{K + \|x - y\|} + 2v(t) \\
&< \epsilon + 2\, v(t).
\end{aligned}$$

Hence, by virtue of hypothesis (H_6) we infer that there exists $T > 0$ such tha $v(t) \leq \frac{\epsilon}{2}$ for $t \geq T$. Thus, for $t \geq T$ we derive that

$$|Qx(t) - Qy(t)| < 2\epsilon\,. \tag{5.27}$$

Further, let us assume that $t \in [0, T]$. Then, evaluating similarly as above we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, y(\alpha_1(t)), y(\alpha_2(t)))\big| \\
&\quad + \int_0^{\beta(t)} [|g(t, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|]\, ds \\
&\leq \frac{L \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\
&\quad + \int_0^{\beta_T} \big[|g(t, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t, s, y(\gamma_1(s)), y(\gamma_2(s)))|\big]\, ds \\
&< \epsilon + \beta_T\, \omega_r^T(g, \epsilon)\,,
\end{aligned} \tag{5.28}$$

where we have denoted

$$\beta_T = \sup\{\beta(t) : \ t \in [0, T]\}\,,$$

and

$$\begin{aligned}
\omega_r^T(g, \epsilon) = \sup \big\{ &|g(t, s, x, y) - g(t, s, w, z)| : \\
&t, s \in [0, T],\ x, y, w, z \in [-r, r],\ |x - w| \leq \epsilon,\ |y - z| \leq \epsilon \big\}\,.
\end{aligned}$$

Obviously, in view of the continuity of β, we have that $\beta_T < \infty$. Moreover, from uniform continuity of the function $g(t, s, x, y)$ on the compact $[0, T] \times [0, T] \times [-r, r] \times [-r, r]$ we derive that $\omega_r^T(g, \epsilon) \to 0$ as $\epsilon \to 0$. Now, linking (5.27), (5.28) and the above established facts we conclude that the operator Q maps partially continuously the closed ball $\overline{\mathcal{B}}(x_0, r)$ into itself.

Step IV: *Q is a nonlinear $\mathcal{D}$-set-contraction w.r.t. characteristic value* ω_0.

Further on let us take a bounded chain X in S with bound $r > 0$, i.e., the chain X belonging to the ball $\mathcal{B}(\theta, r)$. Next, fix arbitrarily $T > 0$ and $\epsilon > 0$. Let us choose $x \in X$ and $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| \le \epsilon$. Without loss of generality we may assume that $x(\alpha_1(t_1)) \ge x(\alpha_1(t_2))$ and $x(\alpha_2(t_1)) \ge x(\alpha_2(t_2))$. Then, taking into account our assumptions, we get:

$$\begin{aligned}
|Qx(t_1) - Qx(t_2)| &\le \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_1(t_1))) - f(t_2, x(\alpha(t_2)), x(\alpha_2(t_1)))\big| \\
&\quad + \left| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right. \\
&\qquad \left. - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right| \\
&\le \big|f(t_1, x(\alpha(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \left| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right. \\
&\qquad \left. - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right| \\
&\quad + \left| \int_0^{\beta(t_1)} g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right. \\
&\qquad \left. - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds \right| \\
&\le \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \int_0^{\beta(t_1)} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + \left| \int_{\beta(t_2)}^{\beta(t_1)} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \right| \\
&\le \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2)))\big| \\
&\quad + \int_0^{\beta_T} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + G_T^r\, |v(t_1) - v(t_2)|, \qquad (5.29)
\end{aligned}$$

where

$$G_T^r = \sup\{|g(t, s, x, y)| : \ t \in [0, T],\ s \in [0, \beta_T],\ x, y \in [-r, r]\}$$

which does exist in view of continuity of the function g on compact $[0, T] \times [0, \beta_T] \times [-r, r] \times [-r, r]$.

Now combining (5.28) and (5.29) we obtain,

$$\begin{aligned}
|Qx(t_2) - Qx(t_1)| &\le \big|f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) - f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1)))\big| \\
&\quad + \frac{L \max\{|x(\alpha_1(t_1)) - x(\alpha_2(t_2))|, |x(\alpha_1(t_1)) - x(\alpha_2(t_2))|\}}{K + \max\{|x(\alpha_1(t_1)) - x(\alpha_2(t_2))|, |x(\alpha_1(t_1)) - x(\alpha_2(t_2))|\}} \\
&\quad + \int_0^{\beta_T} |g(t_1, s, x(\gamma_1(s)), x(\gamma_2(s))) - g(t_2, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\
&\quad + G_T^r\, |v(t_1) - v(t_2)| \\
&\le \frac{L \max\left\{\omega^T\big(x, \omega^T(\alpha_1, \epsilon)\big), \omega^T\big(x, \omega^T(\alpha_2, \epsilon)\big)\right\}}{K + \max\left\{\omega^T\big(x, \omega^T(\alpha_1, \epsilon)\big), \omega^T\big(x, \omega^T(\alpha_2, \epsilon)\big)\right\}} + \omega_r^T(f, \epsilon) \\
&\quad + \int_0^{\beta_T} \omega_r^T(g, \epsilon)\, ds + G_T^r\, \omega^T(v, \epsilon), \qquad (5.30)
\end{aligned}$$

where we have denoted

$$\omega^T(\alpha_1,\epsilon) = \sup\{|\alpha_1(t_2) - \alpha_1(t_1)| : \ t_1, t_2 \in [0,T], \ |t_2 - t_1| \leq \epsilon\} \ ,$$

$$\omega^T(\alpha_2,\epsilon) = \sup\{|\alpha_2(t_2) - \alpha_2(t_1)| : \ t_1, t_2 \in [0,T], \ |t_2 - t_1| \leq \epsilon\} \ ,$$

$$\omega^T(v,\epsilon) = \sup\{|v(t_2) - v(t_1)| : \ t_1, t_2 \in [0,T], \ |t_2 - t_1| \leq \epsilon\} \ ,$$

$$\begin{aligned}\omega_r^T(f,\epsilon) \quad &= \quad \sup\{|f(t_2,x,y) - f(t_1,x,y)| : \ t_1, t_2 \in [0,T], \\ &\qquad |t_2 - t_1| \leq \epsilon, \ x,y \in [-r,r]\} \ ,\end{aligned}$$

and

$$\begin{aligned}\omega_r^T(g,\epsilon) = \sup\big\{&|g(t_2,s,x,y) - g(t_1,s,x,y)| : \ t_1,t_2 \in [0,T], \ |t_2 - t_1| \leq \epsilon, \\ &s \in [0,\beta_T], \ x,y \in [-r,r]\big\} \ .\end{aligned}$$

From the above estimate we derive the following one:

$$\begin{aligned}\omega^T(Q(X),\epsilon) \leq \frac{L \max\left\{\omega^T\left(x,\omega^T(\alpha_1,\epsilon)\right),\omega^T\left(x,\omega^T(\alpha_2,\epsilon)\right)\right\}}{K + \max\left\{\omega^T\left(x,\omega^T(\alpha_1,\epsilon)\right),\omega^T\left(x,\omega^T(\alpha_2,\epsilon)\right)\right\}} + \omega_r^T(f,\epsilon) \\ + \int_0^{\beta_T} \omega_r^T(g,\epsilon)\, ds + G_T^r\, \omega^T(v,\epsilon). \qquad (5.31)\end{aligned}$$

Observe that $\omega_r^T(f,\epsilon) \to 0$ and $\omega_r^T(g,\epsilon) \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions f and g on the sets $[0,T] \times [-r,r] \times [-r,r]$ and $[0,T] \times [0,\beta_T] \times [-r,r] \times [-r,r]$ respectively. Moreover, from the uniform continuity of α_1, α_2, v on $[0,T]$, it follows that $\omega^T(\alpha_1,\epsilon) \to 0$, $\omega^T(\alpha_2,\epsilon) \to 0$, $\omega^T(v,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, linking the established facts with the estimate (5.31) we get

$$\omega_0^T(Q(X)) \leq \frac{L\omega_0^T(X)}{K + \omega_0^T(X)} \ .$$

Consequently, we obtain

$$\omega_0(Q(X)) \leq \frac{L\ \omega_0(X)}{K + \omega_0(X)} \ . \qquad (5.32)$$

Step V: *Q is a nonlinear $\mathcal{D}$-set-contraction w.r.t. the characteristic value δ_c.*

Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x, y \in X$ with $x \geq y$, we deduce the following estimate (cf. the estimate (5.28)):

$$\begin{aligned}|(Qx)(t) - (Qy)(t)| &\leq |f(t,x(\alpha_1(t)),x(\alpha_2(t))) - f(t,y(\alpha_1(t)),y(\alpha_2(t)))| \\ &\quad + 2\left(a(t)\int_0^{\beta(t)} b(s)\, ds\right) \\ &\leq \frac{L\max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{K + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\ &\quad + 2v(t).\end{aligned}$$

From the above inequality it follows that

$$\operatorname{diam}(QX(t)) \leq \frac{L\ \max\{\operatorname{diam}(X(\alpha_1(t))), \operatorname{diam}(X(\alpha_2(t)))\}}{K + \max\{\operatorname{diam}(X(\alpha_1(t))), \operatorname{diam}(X(\alpha_2(t)))\}} + v(t)$$

for each $t \in \mathbb{R}_+$. Therefore, taking the limit superior over $t \to \infty$, we obtain

$$
\begin{aligned}
\delta_c(QX) &= \limsup_{t\to\infty} \operatorname{diam}\left(Q(X(t))\right) \\
&\leq \frac{L \max\left\{\limsup\limits_{t\to\infty} \operatorname{diam}\left(X(\alpha_1(t))\right), \limsup\limits_{t\to\infty} \operatorname{diam}\left(X(\alpha_2(t))\right)\right\}}{K + \max\left\{\limsup\limits_{t\to\infty} \operatorname{diam}\left(X(\alpha_1(t))\right), \limsup\limits_{t\to\infty} \operatorname{diam}\left(X(\alpha_2(t))\right)\right\}} \\
&\leq \frac{L \limsup_{t\to\infty} \operatorname{diam}\left(X(t)\right)}{K + \limsup_{t\to\infty} \operatorname{diam}\left(X(t)\right)} \\
&= \frac{L\delta_c(X)}{K+\delta_c(X)} \qquad (5.33)
\end{aligned}
$$

Step VI: *Q is a partially nonlinear $\mathcal{D}$-set-contraction on S.*

Further, using the measure of noncompactness μ_c^p defined by the formula (5.18) and keeping in mind the estimates (5.32) and (5.33), we obtain

$$
\begin{aligned}
\mu_c^p(QX) &= \max\left\{\omega_0(QX)\,,\,\delta_c(QX)\right\} \\
&\leq \max\left\{\frac{L\,\omega_0(X)}{K+\omega_0(X)}\,,\,\frac{L\delta_c(X)}{K+\delta_c(X)}\right\} \\
&\leq \frac{L\,\max\left\{\omega_0(X)\,,\,\delta_c(X)\right\}}{K+\max\left\{\omega_0(X)\,,\,\delta_c(X)\right\}}\,. \\
&= \frac{L\mu_c^p(X)}{K+\mu_c^p(X)}
\end{aligned}
$$

for all chains X in S. Since $L \leq K$, the operator Q is a partially nonlinear $\mathcal{D}$-set-contraction on S with $\mathcal{D}$-function $\psi(r) = \dfrac{Lr}{K+r}$. Again, by hypothesis ($H_5$), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the HFIE (5.22) defined on $\mathbb{R}_+$.

Thus Q satisfies all the conditions of Theorem 3.2 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in S. Obviously x^* is a solution of the functional integral equation (5.22) and the sequence $\{x_n\}$ of successive approximations defined by

$$
\begin{aligned}
x_n(t) \quad &= \quad f(t, x_{n-1}(\alpha_1(t)), x_{n-1}(\alpha_2(t))) \\
&\qquad + \int_0^{\beta(t)} g(t,s,x_{n-1}(\gamma_1(s)), x_{n-1}(\gamma_2(s)))\,ds, \;\; t\in\mathbb{R}_+,
\end{aligned}
$$

for each $n \in \mathbb{N}$ converges monotonically to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0,r)$ we infer that the set $\mathcal{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathcal{B}}(x_0,r)$. If the set $\mathcal{F}(Q)$ contains all comparable solutions of the equation (5.22), then we conclude from Remark 3.5 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_c^p. Now, taking into account the description of sets belonging to ker μ_c^p (given in Subsection 5.1) we deduce that all the comparable solutions of the equation (5.22) are uniformly globally ultimately attractive on $\mathbb{R}_+$. This completes the proof. □

Theorem 5.10. *Assume that the hypotheses (H_1) through (H_7) hold. Then the functional HFIE (5.22) has at least one solution x^* in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by* (5.22) *converges monotonically to x^*. Moreover, the comparable solutions of the equation (5.21) are uniformly globally ultimately attractive and asymptotically stable to the line $x(t) = c$ defined on $\mathbb{R}_+$.*

Proof. As in Theorem 5.9, we seek the solutions of the HFIE (5.22) in the Banach space $E = BC(\mathbb{R}_+,\mathbb{R})$. Define the closed and bounded set $S = \overline{\mathcal{B}}(x_0,r)$, where $r = \|x_0\| + L + F_0 + V$ and define the operator Q on S into itself by (5.24). Then proceeding as in the Step IV of the proof of Theorem 5.9 it can be proved that Q is a nonlinear $\mathcal{D}$-set-contraction with respect to the characteristic value ω_c with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{K+r}$, i.e., the inequality (5.32) holds on E.

Next, we show that Q is a partially nonlinear $\mathcal{D}$-set-contraction with respect to the characteristic value δ_b. Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x \in X$ with $x \geq c$ on $\mathbb{R}_+$, we deduce the following estimate:

$$\begin{aligned} |(Qx)(t) - c| &\leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t, c, c)| \\ &\quad + \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))|\, ds \\ &\leq \frac{L \max\{|x(\alpha_1(t)) - c|, |x(\alpha_2(t)) - c|\}}{K + \max\{|x(\alpha_1(t)) - c|, |x(\alpha_2(t)) - c|\}} + v(t). \end{aligned}$$

From the above inequality it follows that

$$|QX(t) - c| \leq \frac{L \max\{|X(\alpha_1(t)) - c|, |X(\alpha_2(t)) - c|\}}{K + \max\{|X(\alpha_1(t)) - c|, |X(\alpha_2(t)) - c|\}} + v(t)$$

for each $t \in \mathbb{R}_+$. Therefore, taking the limit superior over $t \to \infty$, we obtain

$$\begin{aligned} \delta_b(QX) &= \limsup_{t\to\infty} |Q(X(t)) - c| \\ &\leq \frac{L \max\left\{ \limsup\limits_{t\to\infty} |X(\alpha_1(t)) - c|, \limsup\limits_{t\to\infty} |X(\alpha_2(t)) - c| \right\}}{K + \max\left\{ \limsup\limits_{t\to\infty} |X(\alpha_1(t)) - c|, \limsup\limits_{t\to\infty} |X(\alpha_2(t)) - c| \right\}} \\ &\leq \frac{L \limsup_{t\to\infty} |X(t) - c|}{K + \limsup_{t\to\infty} |X(t) - c|} \\ &= \frac{L\, \delta_b(X)}{K + \delta_b(X)}. \end{aligned} \tag{5.34}$$

Further, using the partial measure of noncompactness μ_b^p defined by the formula (5.17) and keeping in mind the estimates (5.32) and (5.34), we obtain

$$\begin{aligned} \mu_b^p(QX) &= \max\left\{\omega_0(QX)\,,\, \delta_b(QX)\right\} \\ &\leq \max\left\{ \frac{L\, \omega_0(X)}{K + \omega_0(X)}\,,\, \frac{L\delta_b(X)}{K + \delta_b(X)} \right\} \\ &\leq \frac{L \max\left\{\omega_0(X)\,,\, \delta_b(X)\right\}}{K + \max\left\{\omega_0(X)\,,\, \delta_b(X)\right\}}\,. \\ &= \frac{L\mu_b^p(X)}{K + \mu_b^p(X)} \end{aligned}$$

for all chains X in S. Since $L \leq K$, the operator Q is a partially nonlinear $\mathcal{D}$-set-contraction on S with $\mathcal{D}$-function $\psi(r) = \frac{Lr}{K+r}$. Again, by hypothesis (H_5), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the HFIE (5.22) defined on $\mathbb{R}_+$. The rest of the proof is similar to Theorem 5.9 and now we conclude from Remark 3.5 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_b^p. Now, taking into account the description of sets belonging to ker μ_b^p (given in Section 5.1) we deduce that the equation (5.22) has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by (5.23) converges monotonically to x^*. Moreover, all the comparable solutions of the equation (5.22) are uniformly globally ultimately asymptotically stable to the line $x(t) = c$ on $\mathbb{R}_+$. This completes the proof. □

If $c = 0$ in hypothesis (H_7), we obtain the following existence result concerning the asymptotic stability of the comparable solutions defined on $\mathbb{R}_+$.

Theorem 5.11. *Assume that the hypotheses (H$_1$) through (H$_7$) hold with $c = 0$. Then the functional HFIE (5.22) has at least one solution x^* in the space $BC(\mathbb{R}_+, \mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by* (5.23) *converges monotonically to x^*. Moreover, the comparable solutions of the equation (5.22) are uniformly globally ultimately asymptotically stable to* 0 *defined on $\mathbb{R}_+$.*

Remark 5.12. We remark that if a nonlinear hybrid integral equation (5.22) has more than one lower solution, then they are comparable in view of the fact that E is a lattice. In consequence, it may have a number of comparable lower solutions. The case of upper solution is similar. Furthermore, the order relation $\leq$ in $C(\mathbb{R}_+, \mathbb{R})$ is same as the order relation induced by the order cone

$$\mathcal{K} = \big\{x \in C(\mathbb{R}_+, \mathbb{R}) \mid x(t) \geq 0 \ \text{ for all } t \in \mathbb{R}_+\big\}$$

in $C(\mathbb{R}_+, \mathbb{R})$. Hence, by virtue of Remark 3.5 the integral equation (5.22) has a number of comparable solutions defined on $\mathbb{R}_+$. As a result, under the given conditions of Theorem 5.9 all the comparable solutions of the nonlinear functional integral equation (5.22) are uniformly globally ultimately attractive on $\mathbb{R}_+$

Remark 5.13. The conclusion of Theorems 5.9, 5.10 and 5.11 remains true if we replace the hypothesis (H$_5$) with the following one:

(H$'_5$) There exists an element $v \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$v(t) \geq f(t, v(\alpha_1(t)), v(\alpha_2(t)) + \int_{t_0}^{\beta(t))} g(t, s, v(\gamma_1(s)), v(\gamma_2(s))) \, ds$$

for all $t \in \mathbb{R}_+$.

The proof under this new hypothesis is similar to Theorems 5.9 and 5.10 and now, the desired conclusion follows by an application of Theorem 3.3.

Remark 5.14. The conclusion of Theorems 5.9, 5.10 and 5.11 also remains true if we replace the hypothesis (H$_5$) with the following one:

(H$'_3$) There exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \phi\big(\max\{x_1 - y_1 \,,\, x_2 - y_2\}\big)$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$. Moreover, $\phi(r) < r$ for $r > 0$.

Example 5.2. Consider the linearly perturbed nonlinear hybrid functional integral equation,

$$x(t) = \tan^{-1} x(2t) + \int_0^{3t} \frac{1}{t^2+1} \, g(s, x(s/2)) \, ds \tag{5.35}$$

for all $t \in \mathbb{R}_+$, where $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$g(t,x) = \begin{cases} 1, & \text{if} \quad x \leq 0, \\ x^2 + 1, & \text{if} \quad 0 < x \leq 1, \\ \dfrac{4x}{x+1}, & \text{if} \quad x > 1. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.2 are satisfied by the functions involved in HFIE (5.35). Here, $\alpha(t) = 2t$, $\beta(t) = 3t$ and $\gamma(t) = t/2$ and so, α, β, γ are continuous on $\mathbb{R}_+$ into itself and $\alpha(t) \geq t$ for all $t \in \mathbb{R}_+$. Thus, hypothesis (H$_0$) is satisfied. Again, $f(t,x) = \tan^{-1} x$ so that f is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and nondecreasing in x for each $t \in \mathbb{R}_+$. The kernel $k(t,s)$ is given by $k(t,s) = \frac{1}{t^2+1}$. Obviously k is continuous and nonnegative function on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H$_2$) holds. Next, $g(t,x)$ is defines a continuous and nondecreasing function in x for each $t \in \mathbb{R}_+$. Moreover, $f(t,0) = 0$. So the hypotheses (H$_1$), (H$_2$), (H$_4$) and (H$_5$) are held.

Next, we show that f satisfies hypothesis (H$'_3$) on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x \geq y$. Then,

$$0 \leq f(t,x) - f(t,y) = \tan^{-1} x - \tan^{-1} y = \frac{1}{1+\xi^2}(x-y)$$

for all $y < \xi < x$, and so hypothesis (H_3') is satisfied with $\dfrac{r}{1+\xi^2}$, $0 < \xi < r$.

Furthermore, $g(t,x) \leq 4$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore,

$$v(t) = \int_0^{3t} \frac{1}{t^2+1} \cdot 4\, ds = \frac{12t}{t^2+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{12t}{t^2+1} = 0.$$

Finally, it is easy to prove that $u \equiv 0$ is a lower solution of the HFIE (5.35) defined on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 5.9 are satisfied and by a direct application, we conclude that the HFIE (5.35) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined

$$x_{n+1}(t) = \tan^{-1} x_n(2t) + \int_0^{3t} \frac{1}{t^2+1}\, g(s, x_n(s/2))\, ds$$

converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the HFIE (5.35) are uniformly asymptotically attractive and stable to zero defined on $\mathbb{R}_+$.

Example 5.3. Consider the hybrid differential equation with a linear perturbation of first type, viz.,

$$x(t) = f(t, x(2t)) + \int_0^{3t} \frac{t}{t^3+1} \tanh x(s/2)\, ds \tag{5.36}$$

for all $t \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$f(t,x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + \dfrac{x}{x+1}, & \text{if } x > 0. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.2 are satisfied by the functions involved in HFIE (5.36). Here, as before, $\alpha(t) = 2t$, $\beta(t) = 3t$ and $\gamma(t) = t/2$ and so, α, β, γ are continuous on $\mathbb{R}_+$ into itself and $\alpha(t) \geq t$ for all $t \in \mathbb{R}_+$. Thus, hypothesis (H_0) is satisfied. Again, $f(t,x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and nondecreasing in x for each $t \in \mathbb{R}_+$. The kernel $k(t,s)$ is given by $k(t,s) = \frac{t}{t^3+1}$. Obviously k is continuous and nonnegative function on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H_2) holds. Next, $g(t,x) = \tanh x$ is a continuous and nondecreasing function in x for each $t \in \mathbb{R}_+$. So the hypotheses (H_1), (H_2), (H_4) and (H_5) are held.

Now, we show that f satisfies hypothesis (H_3') on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x \geq y$. Then,

$$0 \leq f(t,x) - f(t,y) = \frac{x}{x+1} - \frac{y}{y+1} = \frac{x-y}{1+x+y+xy} \leq \frac{x-y}{1+x-y}$$

and so, the hypothesis (H_3') is satisfied with $\phi(r) = \dfrac{r}{1+r}$ for $r > 0$.

Furthermore, $|g(t,x)| = |\tanh x| \leq 1$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore,

$$v(t) = \int_0^{3t} \frac{t}{t^3+1} \cdot 1\, ds = \frac{3t^2}{t^3+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{3t^2}{t^3+1} = 0.$$

Finally, it is easy to prove that $u \equiv 0$ is a lower solution of the HFIE (5.36) defined on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 5.9 are satisfied and by a direct application, we conclude that the HFIE (5.36) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined

$$x_{n+1}(t) = f(t, x_n(2t)) + \int_0^{3t} \frac{t}{t^3+1} \tanh x_n(s/2))\, ds$$

converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the HFIE (5.36) are uniformly globally attractive on $\mathbb{R}_+$.

Remark 5.15. In this paper we have been able to weaken the Lipschitz condition to nonlinear one-sided Lipschitz condition which otherwise is considered to be a very strong condition in the existence theory of nonlinear differential and integral equations. However, we needed an additional assumption of monotonicity on the nonlinearities involved in the integral equation in order to guarantee the required characterization of attractivity of the comparable solutions.

Remark 5.16. The existence theorems proved in Section 5 may be extended with appropriate modifications to the generalized nonlinear hybrid functional integral equation

$$x(t) = f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), \ldots, \gamma_n(s)))\, ds \tag{5.37}$$

for all $t \in \mathbb{R}_+$, where $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+,\ i = 1, 2, \ldots, n,\ f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

Remark 5.17. The study of the present paper may be extended to other types of nonlinear hybrid integral equations with different linear as well as quadratic perturbations of first and second type. The detials of different types of perturbations are given in Dhage [8] and the references therein.

6 Conclusion

Observe that the main measure theoretic hybrid fixed point theorems of this paper may be applied to other nonlinear equations like hybrid causal and fractional differential, integral and integro-differential equations for proving the existence results, however unlike existence theorem for nonlinear hybrid integral equations discussed in Dhage [11] we do not require the assumption that E to be a lattice. Again the continuity of the functions $f(t,x)$ and $g(t,s,x)$ in the variable x means that they are partially continuous on $\mathbb{R}$ since $\mathbb{R}$ is a totally ordered set, and therefore, the corresponding operators defined in the proof of above theorem are partially continuous on the domains of their definition which is contrary to the case considered in Dhage [9]. The advantage of the present approach over the previous ones lies in the fact that we have been able to develop an algorithm for the solutions of the considered integral equation which otherwise is not possible via classical approach of measure of noncompactness treated in Banas and Goebel [2]. Another interesting feature of our work is that we generally need the uniqueness of the solution for predicting the behavior of the dynamic system related to the considered nonlinear functional integral equation, however with the present approach it has become possible for us to discuss the qualitative behaviour of the systems even though there exist a number of comparable solutions. Finally, while concluding this paper, we mention that the existence theorem proved in this paper for the considered nonlinear integral equation may also be proved by using the Krasnoselskii type hybrid fixed point theorem, Theorem 4.6 under weaker Carathéodory condition than continuity of the function g with appropriate modifications. Some of the results in the above mentioned direction will be reported elsewhere.

Acknowledgment

The author is thankful to the referee for pointing out some misprints and giving some suggestions for the improvement of this paper.

References

[1] J. Appell, Measures of noncompactness, condensing operators and fixed points : An application-oriented survey, *Fixed Point Theory*, **6**(2005), 157-229.

[2] J. Banas, K. Goebel, *Measures of Noncompactness in Banach Space*, in: Lecture Notes in Pure and Applied Mathematics, Vol. 60, Dekker, New York, 1980.

[3] J. Banas, B. C. Dhage, Global asymptotic stability of solutions of a functional integral equations, *Nonlinear Analysis*, **69** (2008), 1945-1952

[4] T. A. Burton, A fixed point theorem of Krasnoselskii, *Appl. Math. Lett.*, **11**(1998), 83-88.

[5] T. A. Burton, B. Zhang, Fixed points and stability of an integral equation: nonuniqueness, *Appl. Math. Letters*, **17**(2004), 839–846.

[6] S. Carl, S. Hekkilä, *Fixed Point Theory in Ordered Sets and Applications*, Springer, 2011.

[7] B. C. Dhage, Fixed point theorems in ordered Banach algebras and applications, *PanAmer. Math. J.*, **9**(4)(1999), 93-102.

[8] B. C. Dhage, Asymptotic stability of the solution of certain nonlinear functional integral equations via measures of noncompactness, *Comm. Appl. Nonlinear Anal.* **15** (2008), no. 2, 89-101.

[9] B. C. Dhage, Global attractivity results for nonlinear integral equations via a Krasnoselskii type fixed point theorem, *Nonlinear Analysis*, **70**(2009), 2485-2493.

[10] B. C. Dhage, Quadratic perturbations of boundary value problems of second order ordinary differential equations, *Differ. Equ. & Appl.*, **2**(4) (2010), 465-486.

[11] B. C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.*, **5**(2013), 155-184.

[12] B. C. Dhage, Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.*, **45**(2014), 397-426. doi:10.5556/j.tkjm.45.2014.1512.

[13] B. C. Dhage, Operator theoretic techniques in the theory of nonlinear hybrid differential equations, *Nonlinear Anal. Forum*, **20**(2015), 15-31.

[14] B. C. Dhage, S. B. Dhage, Global attractivity and stability results for comparable solutions of nonlinear fractional integral equations, *Nonlinear Studies*, **21**(2014), 255–268.

[15] B. C. Dhage, S. B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *Global J. Math. Sci.*, **3**(2014), 00-00.

[16] B. C. Dhage, S. B. Dhage, D. V. Mule, Local attractivity and stability results for hybrid functional nonlinear fractional integral equations, *Nonlinear Funct. Anal. & Appl.*, **19**(2014), 415-433.

[17] B. C. Dhage, S. B. Dhage and S. K. Ntouyas, Approximating solutions of nonlinear hybrid differential equations, *Appl. Math. Lett.*, **34** (2014), 76-80.

[18] I. T. Gokhberg, I. S. Goldstein and A. S. Markus, *Investigation of some properties of bounded linear operators in connection with their q-norm*, *Ucen zap Kishinevsk Un-an*, **29**(1957), 29-36.

[19] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[20] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Translated from the Russian, Macmillan, New York, (1964).

[21] C. Kuratowskii, Sur les espaces complete, *Fund. Math.*, **15**(1930), 301-309.

[22] J. J. Nieto and R. Rodriguez-Lopez, Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(2005), 223-239.

[23] A. C. M. Ran, M.C. R. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132**(2003), 1435–1443.

[24] E. Zeidler, *Nonlinear Functinal Analysis and its Applications: Part I*, Springer Verlag 1985.

Received: October 6, 2014; *Accepted*: November 4, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/