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Generalization of integral inequalities of the type of Hermite-Hadamard through invexity

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Abstract

In this paper, we obtain some inequalities of Hermite-Hadamard type for functions whose derivatives absolute values are prequasiinvex function. Applications to some special means are considered.

Keywords: Hermite-Hadamard inequality; quasi-convex function; power-mean inequality; Holder's integral inequality.

2010 MSC: 26D07; 26D10; 26D15; 47A63.

1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamarad inequality, due to its rich geometrical significance and applications, which is stated as in [1]

Let $f : I \subset R \to R$ be a convex function defined on the interval $I$of real numbers and $a, b \in I$, with $a < b$.

Then f satisfies the following well-known Hermite Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

In many areas of analysis applications of Hermite-Hadamard inequality appear for different classes of functions with and without weights; see for convex functions [4,5], [7-10] [18-20].

In [5] Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in the above inequality.

Theorem 1. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|$ is convex function on $[a, b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Theorem 2. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a, b \in I^0$ with $a < b$,and If $|f'|^{p/(p-1)}$ is convex function on $[a, b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}\left[\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}\right]^{(p-1)/p}.$$

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In [11], Pearce and J. Pecaric gave an improvement and simplication of the constants in Theorem 2and consolidated this results with Theorem 1. The following is the main result from [11]:

Theorem 3. Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a,b \in I^0$ with $a < b$,and If $|f'|^q$ is convex function on $[a,b]$,for some fixed $q \geq 1$.then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

If $|f'|^q$ is concave function on $[a,b]$, for some fixed $q \geq 1$.then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|.$$

Now we recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f : [a,b] \to R$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda) y) \leq \max \{f(x), f(y)\}, \quad \forall x,y \in [a,b].$$

Clearly, any convex function is a quasi-convex function but the reverse are not true. Because there exist quasi-convex functions which are not convex, (see for example [8]).

Recently, D.A.Ion [8] obtained two inequalities of the right hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex functions, as follows:

Theorem 4. Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a,b \in I^0$ with $a < b$, and If $|f'|$ is quasi-convex function on $[a,b]$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \max \left[|f'(a)|, |f'(b)| \right].$$

Theorem 5. Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a,b \in I^0$ with $a < b$, and If $|f'|^p$ is quasi-convex function on $[a,b]$, for some fixed $p > 1$. then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} \frac{1}{b-a} \int_a^b g(x)dx - \frac{1}{b-a} \int_a^b f(x)g(x)\,dx \right|$$
$$\leq \frac{b-a}{2(p+1)^{1/p}} \left[\max \left\{ |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right\} \right]^{(p-1)/p}.$$

In [2] Alomari , Draus and Kirmaci established Hermite-Hadamard inequalities for quasi-convex functioss whose give refinements of those given above in Theorem 4 and Theorem 5.

Theorem 6. Let $f : I^0 \subset [0,\infty) \to R$ be differentiable mapping on $I^0 \quad a,b \in I^0$ with $a < b$,and If $|f'|$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8} \left[\begin{array}{c} \max \left\{ |f'(a)|, \left| f'\left(\frac{a+b}{2}\right) \right| \right\} \\ + \max \left\{ \left| f'\left(\frac{a+b}{2}\right) \right|, |f'(b)| \right\} \end{array} \right].$$

Theorem 7. Let $f : I^0 \subset [0,\infty) \to R$ be differentiable mapping on $I^0 \quad a,b \in I^0$ with $a < b$,and If $|f'|^q$ is quasi-convex on$[a,b]$, $p > 1$.then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \left(\frac{1}{(1+p)} \right)^{1/p}$$
$$\times \left[\begin{array}{c} \left(\max \left\{ \left| f'\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \\ + \left(\max \left\{ \left| f'\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \end{array} \right].$$

Theorem 8. Let $f : I^0 \subset [0,\infty) \to R$ be a differentiable mapping on I^0 such that $f' \in L([a,b])$ for $a,b \in I$ with $a < b$,If $|f'(x)|$ is quasi-convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left[\max\left\{|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}^{\frac{1}{q}} + \left\{\max\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right\}^{\frac{1}{q}}\right].$$

Alomari, Darus and Dragomir in [3] introduced the following theorems for twice differentiable quasi-convex functions which are generalizations of Theorems 3, 4 and 5.

Theorem 9. Let $f : I^0 \subseteq R \to R$ is a differentiable function on I^0 $a,b \in I^0$ with $a < b$,and If $|f''|$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\left\{\left|f''(a)\right| + \left|f''(b)\right|\right\}.$$

Theorem 10. Let $f : I^0 \subseteq R \to R$ be twice differentiable function on I^0 $a,b \in I^0$ with $a < b$,and If $|f''|^{p/(p-1)}$ is quasi-convex on $[a,b]$,then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q + |f''(b)|^q\right\}\right)^{1/q}.$$

Theorem 11. Let $f : I^0 \subseteq R \to R$ be twice differentiable function on I^0 $a,b \in I^0$ with $a < b$,and If $|f''|^q$ is quasi-convex on$[a,b]$, $q \geq 1$.then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\right)^{1/q}.$$

Let K be a closed set R^n and let $f : K \to \mathrm{R}$ and $\eta : K \times K \to \mathrm{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(.,.)$,

If $x + t\eta(y,x) \in K, \forall x,y \in K, t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 12. The function f on the invex set $K$is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq +(1-t)f(u) + tf(v), \forall u,v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse is not true.

Definition 13. The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u,v \in K, t \in [0,1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(u,v)$ but the converse does not holds, see for example [21].

In the recent paper, Noor [18] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 14. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a open preinvex function on the interval of real numbers K^0 (the interior of K^0) and $a,b \in K^0$ with $a < a+\eta(b,a)$. the following inequality holds:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [16], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Theorem 15. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \left\{ \left|f'(a)\right| + \left|f'(b)\right| \right\}.$$

Theorem 16. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned} &\left| \tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \tfrac{\eta(b,a)}{2(1+p)^{1/p}} \left\{ \tfrac{|f'(a)|^{\frac{p}{(p-1)}} + |f'(b)|^{\frac{p}{(p-1)}}}{2} \right\}^{\frac{p-1}{p}}. \end{aligned}$$

In [15] Barani, Ghazanfari and Dragomir gave similar results for quasi-preinvex functions as follows:

Theorem 17. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is quasi-preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{\eta(b,a)}{8} \sup \left\{ \left|f'(a)\right|, \left|f'(b)\right| \right\}.$$

Theorem 18. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned} &\left| \tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \tfrac{\eta(b,a)}{2(1+p)^{1/p}} \sup \left\{ |f'(a)|^{\frac{p}{(p-1)}}, |f'(b)|^{\frac{p}{(p-1)}} \right\}^{\frac{p-1}{p}}. \end{aligned}$$

The main aim of this paper is to establish new refined inequalities of the right-hand side of Hermite-Hadamard result for the class of functions whose derivatives in absolute values are quasi-preinvex. Then we give some applications for some special means of real numbers.

2 Main results

Before proceeding towards our main theorem regarding generalization of the Hermite-Hadamard inequality using prequasinvex . We begin with the following Lemma.

Lemma 1. Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$ and $a, b \in K$ with $a < a + \eta(b,a)$ suppose $f : K \to R$ is a differentiable mapping on K with $a, b \in I^0$ with $a < b, f'' \in L([a, a + \eta(b,a)])$. Then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned} &\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \\ &= \tfrac{(\eta(b,a))^2}{16} \int_0^1 \left(1 - \lambda^2\right) \left\{ f''\left(a + \left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) d\lambda + f''\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right) d\lambda \right\}. \end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 (1-\lambda^2) f''\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)d\lambda \\ &= \frac{2(1-\lambda^2)f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\bigg|_0^1 - \frac{2}{\eta(b,a)}\int_0^1 f'\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) \\ &= \frac{2(1-\lambda^2)f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\bigg|_0^1 - \frac{4}{\eta(b,a)}\left[\begin{array}{c} \frac{2\lambda f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)}{-\eta(b,a)} \\ -\frac{2}{\eta(b,a)}\int_0^1 f\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right) \end{array}\right] \\ &= -\frac{2}{\eta(b,a)}f'\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{8}{\eta(b,a)}f(a) - \frac{8}{\eta(b,a)}\int_0^1 f\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) \end{aligned}$$

Setting $x = a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)$ and $dx = \frac{-\eta(b,a)}{2}d\lambda$ which gives

$$I_1 = \frac{2}{\eta(b,a)}f'\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{8}{(\eta(b,a))^2}f(a) - \frac{16}{(\eta(b,a))^3}\int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx$$

Similarly we can show that

$$\begin{aligned} I_2 &= \int_0^1 (1-\lambda^2) f''\left(a+\left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right)d\lambda \\ &= -\frac{2}{\eta(b,a)}f'\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{8}{(\eta(b,a))^2}f(a+\eta(b,a)) - \frac{16}{(\eta(b,a))^3}\int_{a+\frac{1}{2}\eta(b,a)}^{b} f(x)\,dx \end{aligned}$$

Therefore
$$\begin{aligned} &\frac{(\eta(b,a))^2}{16}[I_1+I_2] \\ &= \frac{(\eta(b,a))^2}{16}\left[\frac{8}{(\eta(b,a))^2}(f(a)+f(a+\eta(b,a))) - \frac{16}{(\eta(b,a))^3}\int_a^b f(x)\,dx\right] \\ &= \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \end{aligned}$$

Which completes the proof.

In the following theorem, we shall propose some new upper bound for the right-hand side of Hermite-Hadamard inequality for functions whose second derivatives absolute values are prequasiinvex, which is better than the inequality had done in [3,6].

Theorem A. Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to R$ and $a, b \in K$with $a < a+\eta(b,a)$suppose $f : K \to R$ is a differentiable mapping on Ksuch that $f'' \in L([a, a+\eta(b,a)])$. If $|f''|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{24}\left[\begin{array}{l} \sup\left\{|f''(a)|, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} \\ +\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|, |f''(a+\eta(b,a))|\right\} \end{array}\right]. \end{aligned} \tag{2.1}$$

Proof. From Lemma 1, and Since $|f''|$ is prequasinvex, then we have

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left[\begin{array}{l} \int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \\ +\int_0^1 (1-\lambda^2)\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \end{array}\right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\sup\left\{|f''(a)|, \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\}d\lambda \\ &+\frac{(\eta(b,a))^2}{16}\int_0^1 (1-\lambda^2)\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|, |f''(a+\eta(b,a))|\right\}d\lambda \end{aligned}$$

$$\leq \frac{(\eta(b,a))^2}{16}\sup\left\{|f''(a)|,\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1\left(1-\lambda^2\right)d\lambda$$
$$+\frac{(\eta(b,a))^2}{16}\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,|f''(a+\eta(b,a))|\right\}\int_0^1\left(1-\lambda^2\right)d\lambda$$
$$=\frac{(\eta(b,a))^2}{24}\sup\left\{|f''(a)|,\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right\}$$
$$+\frac{(\eta(b,a))^2}{24}\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,|f''(a+\eta(b,a))|\right\}.$$

which completes the proof.

Corollary 1. Let f be defined as in Theorem A, if in addition

1. $|f''|$ is increasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{24}\left[|f''(a+\eta(b,a))|+\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\tag{2.2}$$

2. $|f''|$ is decreasing, then we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{24}\left[|f''(a)|+\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\tag{2.3}$$

Remark 2.1. we note that the inequalities (2.2) and (2.3) are two new refinements of the trapezoid inequality for quasipreinvex functions, and thus for convex functions.

Observation 1. If we take $\eta(b,a)=b-a$ in Theorem A, then inequality reduces to the [Theorem 2.1, 6]. If we take $\eta(b,a)=b-a$ in corollary 1, then (2.2) and (2.3) reduce to the related corollary of Theorem 2.1 from [6].

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following result:

Theorem B. Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\to R$ and $a,b\in K$ with $a<a+\eta(b,a)$ suppose $f:K\to R$ is a differentiable mapping on Ksuch that $f''\in L([a,a+\eta(b,a)])$. If $|f''|^p$ is preinvex on K,from some $p>1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\tag{2.4}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}}\left(\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},|f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

Where $q=p/(p-1)$.

Proof . From Lemma 1, and using the well known Holder integral inequality, we

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{16}\int_0^1\left(1-\lambda^2\right)\left|f''\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$
$$+\frac{(\eta(b,a))^2}{16}\int_0^1\left(1-\lambda^2\right)\left|f''\left(a+\left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda$$

$$\leq\frac{(\eta(b,a))^2}{16}\left(\int_0^1\left(1-\lambda^2\right)^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\int_0^1\left(1-\lambda^2\right)^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(a+\left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}}$$

$$\leq \frac{(\eta(b,a))^2}{16}\left(\int_0^1 \left(1-\lambda^2\right)^p\right)^{\frac{1}{p}}\left(\int_0^1 \sup\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left|\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\int_0^1 \left(1-\lambda^2\right)^p\right)^{\frac{1}{p}}\left(\int_0^1 \sup\left\{\left|f''\left|\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right|^{\frac{p}{p-1}},|f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}}$$
$$=\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)}\right]^{\frac{1}{p}}\left(\sup\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$
$$+\frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)}\right]^{\frac{1}{p}}\left(\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},|f''(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

Which completes the proof.

Corollary 2. Let f be defined as in Theorem B, if in addition

1. $|f''|^{p/p-1}$ is increasing, then we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)}\right]^{\frac{1}{p}} \\ &\times\left[|f''(a+\eta(b,a))|+\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.5}$$

2. $|f''|^{p/p-1}$ is decreasing, then we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left[\frac{\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)}\right]^{\frac{1}{p}} \\ &\times\left[|f''(a)|+\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.6}$$

Observation 2. If we take $\eta(b,a)=b-a$ in Theorem B, then inequality reduces to the [Theorem 2.2, 6]. If we take $\eta(b,a)=b-a$ in corollary 2, then (2.5) and (2.6) reduce to the related corollary of Theorem 2.2 from [6].

Theorem C. Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to R$and $a,b\in K$with $a<a+\eta(b,a)$suppose $f:K\to R$is a differentiable mapping on Ksuch that $f''\in L([a,a+\eta(b,a)])$. If $|f''|^q$ is preinvex on K, $q\geq 1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left(\sup\left\{|f''(a)|^q,\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(\eta(b,a))^2}{16}\left(\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q,|f''(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}.\end{aligned} \tag{2.7}$$

Proof . Suppose that$q\geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\int_0^1\left(1-\lambda^2\right)\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \\ &+\frac{(\eta(b,a))^2}{16}\int_0^1\left(1-\lambda^2\right)\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \\ &\leq \frac{(\eta(b,a))^2}{16}\left(\int_0^1\left(1-\lambda^2\right)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda\right)^{\frac{1}{q}} \\ &+\frac{(\eta(b,a))^2}{16}\left(\int_0^1\left(1-\lambda^2\right)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\end{aligned}$$

Since $|f''|^q$ is quasi-preinvexity, we have

$$\left|f''\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(|f''(a)|^q, \left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right)$$

$$\left|f''\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, |f''(a+\eta(b,a))|^q\right)$$

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left(\sup\left\{|f''(a)|^q, \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(\eta(b,a))^2}{16}\left(\sup\left\{\left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q, |f''(a+\eta(b,a))|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Which completes the proof.

Corollary 3. Let f be defined as in Theorem C, if in addition

1. $|f''|^{p/p-1}$ is increasing, then we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left[|f''(a+\eta(b,a))| + \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.8}$$

2. $|f''|^{p/p-1}$ is decreasing, then we have

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{(\eta(b,a))^2}{16}\left[|f''(a)| + \left|f''\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned} \tag{2.9}$$

Observation 3. If we take $\eta(b,a) = b-a$ in Theorem C, then inequality reduces to the [Theorem 2.3, 6]. If we take $\eta(b,a) = b-a$ in corollary 3, then (2.8) and (2.9) reduce to the related corollary of Theorem 2.3 from [6].]

3 Application to some special means

In what follows we give certain generalization of some notions for a positive valued function of a positive variable.

Definition 3[14]. A function $M : R \to R$, is called a mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x,y)$,for all $a > 0$,
2. Symmetry: $M(x,y) = M(x,y)$,
3. Reflexivity: $M(x,x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) = M(x',y')$,
5. Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We consider some means for arbitrary positive real numbers a, b(see for instance [14]).

We now consider the applications of our theorem to the special means.

The Arithmetic Mean;

$$A := A(a,b) = \frac{a+b}{2}$$

The Geometic Mean;

$$G := G(a,b) = \sqrt{ab}$$

The Power Mean;

$$P_r := P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \quad , \quad r \geq 1,$$

The Indentric Mean:

$$I = I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right) & , \quad if a \neq b \\ a \ , & if a = b \end{cases}$$

The Harmonic Mean:

$$H := H(a,b) = \frac{2ab}{a+b} \quad ,$$

The Logarithmic Mean:

$$L = L(a,b) = \frac{a-b}{\ln|a| - \ln|b|}, \quad |a| \neq |b|$$

The *p*- Logarithmic Mean:

$$L_p \equiv L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right], \quad a \neq b$$

$p \in \Re \setminus \{-1, 0\}$: $a, b > 0$.

It is well known that L_P is monotonic nondecreasing over $p \in$ R with $L_{-1} := L$and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Now let a and b be positive real numbers such that $a < b$. consider the function $a < b$. $M : M(b,a) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to R$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

$\eta(b,a) = M(b,a)$in (2.1), (2.4) and (2.7), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{24} \left[\begin{array}{l} \sup\left\{|f''(a)|, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|\right\} \\ +\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|, |f''(a+M(b,a))|\right\} \end{array}\right]. \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{16} \left(\frac{\sqrt{x}}{2}\right)^{1/p} \left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \\ &\times \left(\sup\left\{|f''(a)|^{\frac{p}{p-1}}, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \\ &+\frac{(M(b,a))^2}{16} \left(\frac{\sqrt{x}}{2}\right)^{1/p} \left[\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}\right]^{\frac{1}{p}} \\ &\times \left(\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}, |f''(a+M(a,b))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right| \\ &\leq \frac{(M(b,a))^2}{16} \left(\sup\left\{|f''(a)|^q, \left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(M(b,a))^2}{16} \left(\sup\left\{\left|f''\left(a+\frac{1}{2}M(b,a)\right)\right|^q, |f''(a+M(b,a))|^q\right\}\right)^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

For $q \geq 1$.Letting $M = A, G, P_r, I, H, L, L_p$ in (3.10), (3.11) and (3.12), we can get the required inequalities, and the details are left to the interested reader.

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