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A new generalized vector-valued paranormed sequence space using modulus function

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Abstract

In this paper we introduce a new generalized vector-valued paranormed sequence spaces $N_p(E_k, \triangle_u^m, f, s)$ using modulus function f, where $p = (p_k)$ is a bounded sequence of positive real numbers such that $\inf_k p_k > 0$, (E_k, q_k) is a sequence of seminormed spaces with $E_{k+1} \subseteq E_k$ for each $k \in N$ and $s \geq 0$. We prove results regarding completeness, K-space, normality, inclusion relation are derived. These are more general than those of Ruckle [7], Maddox [5], Ozturk and Bilgin [6], Sahiner [8], Atlin *et al.* [1] and Srivastava and Kumar [9].

Keywords: Modulus function, paranormed space, normal sequence space, difference sequence space.

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1 Introduction

Let ω denote the space of all complex sequences. Kizmaz [4] studied the sequence space

$$X(\triangle) = \{x = (x_k) : \triangle x \in X\}, \text{ for } X = l_\infty, c, c_0,$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})$ and shown that these sequence spaces are Banach spaces with the norm

$$||x||_\triangle = |x|_1 + ||\triangle x||_\infty,\ x \in X(\triangle).$$

The sequence spaces $X(\triangle^m) = \{x = (x_k) :\ \triangle^m x \in X\}$ for $X = l_\infty, c$ and c_0 are introduced by Et. and Colak [2]. These sequence spaces are BK-spaces with norm

$$||x||_\triangle = \sum_{i=0}^{m} |x_i| + ||\triangle^m x||_\infty, x \in X(\triangle^m) \text{ where } m \in N.$$

Tripathy and Esi [10] introduced the difference operator $\triangle_u,\ u \geq 1$ and defined the sequence spaces

$$X(\triangle_u) = \{x = (x_k) : \triangle_u x \in X)\} \text{ for } X = l_\infty, c \text{ and } c_0 \text{ and } \triangle_u x = (\triangle_u x_k) = (x_k - x_{k+u}).$$

They proved that the above sequence spaces are Banach spaces and BK spaces with respect to the norm

$$\| x \|_{\triangle_u} = \sum_{r=1}^{u} | x_r | + \| \triangle^u x \|_\infty .$$

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Ruckle [7] constructed the sequence spaces $L(f) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} f(\mid x_k \mid) < \infty\}$ using the idea of Modulus function f. He proved that $L(f)$ is BK space. Maddox [5] introduced the class of sequences which are strongly Cesaro summable with respect to the modulus function by

$$w_0(f) = \{x = (x_k) \in \omega : \frac{1}{n}\sum_{k=1}^{n} f(\mid x_k \mid) \longrightarrow 0 \ , as \ n \longrightarrow \infty\}.$$

Ozturk and Bligin [6] generalized the sequence spaces as

$$w_0(f, P) = \{x = (x_k) \in \omega : \frac{1}{n}\sum_{k=1}^{n} [f(\mid x_k \mid)]^{p_k} \longrightarrow 0 \ , as \ n \longrightarrow \infty\}.$$

where $p = (p_k)$ is a bounded sequence of positive real numbers.

Sahiner [8] introduced the sequence spaces

$$B_g(p, f, q, s) = \left\{x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} k^{-s}[f(q(\triangle x_k))]^{p_k} < \infty, \ s \geq 0\right\},$$

and

$$B_g(p, f^r, q, s) = \left\{x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} k^{-s}[f^r(q(\triangle x_k))]^{p_k} < \infty, \ s \geq 0\right\},$$

where $r \in N$ and (X, q) is a seminormed complex linear space.

Altin *etal*. [1] generalized the sequence space $B_g(p, f, q, s)$ as

$$l(\triangle^m, f, p, q, s) = \left\{x = (x_k) \in \omega(X) : \frac{1}{n}\sum_{k=1}^{\infty} k^{-s}[f(q(\triangle^m x_k))]^{p_k} < \infty, \ s \geq 0\right\}.$$

Srivastave and Kumar [9] introduced a new vector valued sequence space $N_p(E_k, \triangle^m, f, s)$ where

$$N_p(E_k, \triangle^m, f, s) = \{x = (x_k) \in \omega(E_k) : (\mid v_k \mid^{-(s/p_k)} f(q_k(\triangle^m x_k))) \in N_p, \ s \geq 0\},$$

where (E_k, q_k) is a sequence of seminormed spaces such that $E_{k+1} \subseteq E_k$ for each $k \in N$, $w(E_k) = \{x = (x_k) : \ x_k \in E_k, \ \text{for each } k \in N\}, v = (v_k)$ is a sequence of real complex numbers such that $1 \leq\mid v_k \mid< \infty$ for each $k \in N$ and N_p is normal AK sequence space with absolutely monotonic paranorm g_{N_p}.

Let $u, m \geq 0$ be fixed integers then we introduce the following new type of Generalized paranormed vector valued sequence space which unifies some earlier cases as particular cases:

$$N_p(E_k, \triangle^m_u, f, s) = \{x = (x_k) \in \omega(E_k) : (\mid v_k \mid^{-(s/p_k)} f(q_k(\triangle^m_u x_k))) \in N_p, \ s > 0\},$$

where $p = (p_k)$ is a bounded sequence of positive real numbers such that $\inf_k p_k > 0$ and

$$\triangle^m_u x_k = \sum_{\nu=0}^{m} (-1)^{\nu} \begin{pmatrix} m \\ v \end{pmatrix} x_k + u\nu, \ \text{ for all} k \in N.$$

1.1.1 Particular Cases:

(i) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = l_1$, (where $p_k = 1$ for each $k \in N$), space $N_p(E_k, \triangle^m_u, f, s)$ reduces to $L(f)$ of Ruckle [7].

(ii) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = \omega_0$, (where $p_k = 1$ for each $k \in N$), space $N_p(E_k, \triangle^m_u, f, s)$ reduces to $\omega_0(f)$ of Maddox [5].

(iii) For $E_k = C$ for each $k \in N$, $m = 0$, $u = 1$, $s = 0$ and $N_p = \omega_0(p)$, space, $N_p(E_k, \triangle^m_u, f, s)$ reduces to $\omega_0(f, p)$ of Ozturk and Bilgin [6].

(iv) For $u = 1$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $N_p(E_k, \triangle^m, f, s)$ of Srivastave and Kumar [9].

(v) For $E_k = X$, for each $k \in N$, $v_k = k$, $m = 1$ and $u = 1$ and $N_p = l_p$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $B_g(p, f, q, s)$ of Sahiner [8].

(vi) For $E_k = X$, for each $k \in N$, $v_k = k$, $u = 1$ and $N_p = l_p$, the space $N_p(E_k, \triangle_u^m, f, s)$ reduces to $l(\triangle^m, f, q, s)$ of Altin *etal*. [1].

Thus study of the space $N_p(E_k, \triangle_u^m, f, s)$ gives a unified approach to many of the earlier known spaces.

2. Some Definitions and Lemmas

Definition 2.1[3]. A sequence space X is called normal space if $x = (x_k) \in X$ and $|\lambda_k| \leq 1$ for each $k \epsilon N$. This implies $\lambda x = (\lambda_k x_k) \epsilon X$.
For example, $l(p), c_0(p), \omega(p)$ are normal space.

Definition 2.2[3]. A sequence space X is called K space if the co-ordinate function $p_k : X \to K$ given by $p_k(x) = x_k$ is continuous for each $k \in N$.

Definition 2.3. A complete metric space is called Frechet space. An FK-space is a Frechet space with continuous co-ordinates.

Definition 2.4[9]. An FK-space X is said to be AK-space if $\Phi \subset X$ and $\{\delta^n\}$ is a basis for X, i.e., for each $x, x^{[n]} \to x$, where $x^{[n]}$ denotes the nth section of x. For example, $l(p), c_0(p), \omega(p)$ are AK-spaces.

Definition 2.5[3]. A paranorm g on a normal sequence space X is said to be absolutely monotone if

$$x = (x_k), y = (y_k) \epsilon X \text{ and } |x_k| \leq |y_k| \text{ for each } k \epsilon N \Longrightarrow g(x) \leq g(y).$$

Lemma 2.1[8]. If f is a modulus function, then f^r is also modulus function for each $r \epsilon N$, where $f^r = f \circ f \circ f \circ \cdots \circ f$($r$-times composition of f with itself).

Lemma 2.2[5]. There is a modulus function f such that $f(xy) \leq f(x) + f(y)$ for $x, y \geq 0$.

Lemma 2.3[5]. Let f_1 and f_2 be modulus functions and $0 < \delta < 1$. If $f_1(t) > \delta$ for $t \in [0, \infty)$, then

$$(f_2 \circ f_1)(t) < \left(\frac{2 f_2(1)}{\delta}\right) f_1(t).$$

3. Results on Sequence Space $N_p(E_K, \triangle_u^m, f, s)$.

Theorem:. $N_p(E_K, \triangle_u^m, f, s)$ is a linear space.

Proof. It is easy to show that $N_p(E_K, \triangle_u^m, f, s)$ is a linear space. So we omit proof.

Lemma 3.1. Let (E_k, q_k) be a sequence of seminormed spaces, and N_p be normal AK-sequence space with absolutely monotone paranaorm g_{N_p}. Then function defined by

$$\tilde{f}_n : [0, \infty) \to [0, \infty),\ \tilde{f}_n(t) = g_{N_p}[\sum_{k=1}^{n} |v_k|^{-(s/p_k)} f(t q_k(\triangle_u^m x_k) e_k]$$

is continuous function of t for each positive integer n, where $x = (x_k) \epsilon N_p(E_K, \triangle_u^m, f, s)$ and (e_k) is unit vector basis of N_p.

Proof. We define function $g_k : [0, \infty) \to N_p$ by

$$g_k(t) = |v_k|^{-(s/p_k)} f(t q_k(\triangle_u^m x_k) e_k.$$

Let $t_i \to 0$ as $i \to \infty$. Then for each $k = 1, 2, 3, ..., n$;

$$g_k(t_i) = |v_k|^{-(s/p_k)} f(t_i q_k(\triangle_u^m x_k))e_k \to (0, 0, \cdots) \ \ as \ \ i \to \infty.$$

Therefore,

$$\sum_{k=1}^{n} g_k(t_i) = \sum_{k=1}^{n} |v_k|^{-(s/p_k)} f(t_i q_k(\triangle_u^m x_k)e_k \to (0, 0, \cdots) \ \ as \ \ i \to \infty.$$

But paranorm g_{N_p} is continuous function, it follows that

$$g_{N_p}[\sum_{k=1}^{n} g_k(t_i)] \to 0 \text{as} i \to \infty.$$

Hence function $\tilde{f}_n$ is continuous function of t for each positive integer n.

Theorem 3.2. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed space with paranorm

$$g(x) = \sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)))\right], where \ \ x \in N_p(E_K, \triangle_u^m, f, s).$$

Proof: By definition of $g, g(x) \geq 0$ for any $x = (x_k) \in N_p(E_K, \triangle_u^m, f, s)$. It is clear that $g(0) = 0$, $g(x) = g(-x)$ and $g(x+y) \leq g(x) + g(y)$ for any $x, y \in N_p(E_K, \triangle_u^m, f, s)$. It is left to prove the continuity of scaler multiplication under g. Suppose $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$ and $\alpha_n \to \alpha$ as $n \to \infty$ in C. We have to show that $g(\alpha_n x^n - \alpha x) \to 0$ as $n \to \infty$. Consider

$$g(\alpha_n x^n - \alpha x) = \sum_{i=1}^{m} f(q_i(\alpha_n x_i^n - \alpha x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(\alpha_n x_k^n - \alpha x_k))))\right]$$

$$= \sum_{i=1}^{m} f(q_i(\alpha_n x_i^n - \alpha_n x_i + \alpha_n x_i - \alpha x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(\alpha_n x_k^n - \alpha_n x_k + \alpha_n x_k - \alpha x_k))))\right]$$

$$\leq \sum_{i=1}^{m} f(|\alpha_n| q_i(x_i^n - x_i) + |\alpha_n - \alpha| q_i(x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n| q_k(\triangle_u^m(x_k^n - x_k)) + |\alpha_n - \alpha| q_k(\triangle_u^m x_k))))\right].$$

This gives,

$$g(\alpha_n x^n - \alpha x) \leq M\left(\sum_{i=1}^{m} f(q_i(x_i^n - x_i) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k)))\right)\right)$$

$$+ \sum_{i=1}^{m} f(|\alpha_n - \alpha| q_i(x_i)) + g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m x_k)))\right],$$

where $M = \sup_n(1 + [|\alpha_n|])$, this gives,

$$g(\alpha_n x^n - \alpha x) \leq M g(x^n - x) + \sum_{i=1}^{m} f(|\alpha_n - \alpha| q_i(x_i))$$

$$+ g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m x_k)))\right]. \tag{3.1}$$

First and second expressions of R.H.S in (3.1) tend to zero as $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$ and $\alpha_n \to \alpha$ as $n \to \infty$. We must only show that

$$g_{N_p}\left[(|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k^n - x_k)))\right] \to 0 \text{as} n \to \infty.$$

Since $(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))\epsilon N_p$ is AK-sequence space, therefore

$$g_{N_p}\left[(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))) - \sum_{k=1}^{m} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)e_k\right] \to 0 \ \ asm \to \infty.$$

That is $g_{N_p}\left[\sum_{k=m+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k\right] \to 0$ as $m \to \infty$.
Therefore, for every $\epsilon > 0$ there exists a positive integer m_0 such that

$$g_{N_p}\left[\sum_{k=m+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))e_k\right] < \epsilon/2, \quad \text{for all } m \geq m_0.$$

In particular

$$g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))e_k\right] < \epsilon/2. \tag{3.2}$$

Since $\alpha_n \to \alpha$ as $n \to \infty$, therefore, for $\epsilon = 1$, there exists a positive integer n_0' such that $|\alpha_n - \alpha| < 1$ for all $n \geq n_0'$. Consequently

$$\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k$$
$$\leq \sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k \text{ for all } n \geq n_0'.$$

But g_{N_p} is monotone paranorm, it follows that for all $n \geq n_0'$.

$$g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right]$$
$$\leq g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k))e_k\right].$$

Using inequality (3.2), for all $n \geq n_0'$

$$g_{N_p}\left[\sum_{k=m_0+1}^{\infty} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right] \leq \epsilon/2. \tag{3.3}$$

By Lemma 3.1, function

$$\tilde{f}_{m_0}(t) = g_{N_p}\left[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(tq_k(\triangle_u^m(x_k))e_k\right], \ \ t \geq 0$$

is continuous function of t. Hence there exists $\delta \in (0, 1)$ such that

$$\tilde{f}_{m_0}(t) < \epsilon/2, \text{ whenever } t < \infty.$$

Again, since $\alpha_n \to \alpha$ as $n \to \infty$, therefore for $\delta \in (0, 1)$, there exist a positive integer n_0'' such that

$$|\alpha_n - \alpha| < \delta \text{ for all } n \geq n_0'' \text{ we have } \tilde{f}_{m_0}(|\alpha_n - \alpha|) < \epsilon/2, \text{ for all } n \geq n_0''$$

that is

$$g_{N_p}\left[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right] < \epsilon/2 \text{ for all } n \geq n_0'' \tag{3.4}$$

We take $n_0 = \max(n_0', n_0'')$. Using inequality (3.3)and (3.4), we have

$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))\right]$$
$$\leq g_{N_p}\left[\sum_{k=1}^{m_0} |v_k|^{-(s/p_k)} f(|\alpha_n - \alpha| q_k(\triangle_u^m(x_k))e_k\right]$$

$$+ g_{N_p}\left[\sum_{k=m_0+1}^{\infty}|v_k|^{-(s/p_k)}f(|\alpha_n-\alpha|q_k(\triangle_u^m(x_k))e_k\right]$$
$$< \epsilon/2+\epsilon/2=\epsilon, \text{ for all } n\geq n_0.$$

From inequality (3.1), $g(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$. Hence $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed sequence space.

Remark 3.1. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is not totally paranormed space.

Let $g(x) = 0 \Longrightarrow \sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k)))\right] = 0 \Longrightarrow q_i(x_i) = 0 \;\; foreach \;\; i = 1, 2, ..., m$ and

$$g_{N_p}\left[|v_k|^{-(s/p_k)}f(q_k(\triangle_u^m(x_k))\right]=0.$$

But

$$q_i(x_i)=0$$

does not mean $x_i = 0$ as q_i is seminorm on E_i. Hence g is not total paranormed on space $N_p(E_K, \triangle_u^m, f, s)$.

Theorem 3.3. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a K-space if N_p is a K-space.

Proof. We have to show the coordinate function $P_k : N_p(E_K, \triangle_u^m, f, s) \to E_k$ given by $P_k(x) = x_k$, where $x \in N_p(E_K, \triangle_u^m, f, s)$ is continuous for each $k \in N$.
Let (x^n) be any sequence in $N_p(E_K, \triangle_u^m, f, s)$ such that $x^n \to 0$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$. That is

$$\sum_{i=1}^{mu} f(q_i(x_i^n)) + g_{N_p}\left[|v_k|^{-(s/p_k)}f(q_k(\triangle_u^m(x_k^n)))\right] \to 0 \text{ as } n\to\infty.$$

This means that

$$f(q_i(x_i^n)) \to 0 \text{ as } n\to\infty \text{ for each } i=1,2,...,m,$$

and

$$g_{N_p}\left[|v_k|^{-(s/p_k)}f(q_k(\triangle_u^m(x_k^n)))\right] \to 0 \text{ as } n\to\infty \qquad (3.5).$$

Since N_p is a K-space, therefore for each k

$$|v_k|^{-(s/p_k)}f(q_k(\triangle_u^m(x_k^n)) \to 0 \text{ as } n\to\infty,$$

that is $f(q_k(\triangle_u^m(x_k^n)) \to 0$ as $n \to \infty$. Thus for any $\delta > 0$, there exist $n_0 \in N$ such that $f(q_k(\triangle_u^m(x_k^n)) < \delta$ for all $n \geq n_0$. Let $\delta = f(\epsilon)$, where $\epsilon > 0$. Then

$$f(q_k(\triangle_u^m(x_k^n)) < f(\epsilon) \text{ for all } n\geq n_0 \to q_k(\triangle_u^m(x_k^n) < \epsilon \text{ for all } n\geq n_0.$$

This shows that for each $k, \triangle_u^m(x_k^n) \to 0$ in E_k as $n \to \infty$. By condition (3.5), $f(q_i(x_i^n)) \to 0$ as $n \to \infty$ for each $i = 1, 2, ..., m$. But f is modulus function, it follows that $x_i^n \to 0$ in E_i as $n \to \infty$ for each $i = 1, 2, ..., m$. Now $x_i^n \to 0$ in E_i as $n \to \infty$ for each $i = 1, 2, ..., m$ and $\triangle_u^m x_k^n \to 0$ in E_i as $n \to \infty$ for each $k \in N$. This implies that $x_k^n \to 0$ in E_k as $n \to \infty$ for each $k \in N$. Thus, coordinate wise function P_k is continuous for each $k \in N$. Hence $N_p(E_K, \triangle_u^m, f, s)$ is a K-space.

Theorem 3.4. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a complete paranormed space under the paranorm g defined by

$$g(x)=\sum_{i=1}^{m} f(q_i(x_i)) + g_{N_p}\left[|v_k|^{-(s/p_k)}f(q_k(\triangle_u^m(x_k)))\right], \text{where } x\in N_p(E_K,\triangle_u^m,f,s),$$

if N_p is a K-space and (E_k, q_k) is a sequence of complete seminormed spaces.

Proof. Clearly $N_p(E_K, \triangle_u^m, f, s)$ is a paranormed space under g. To show that it is complete, Let $(x^n) = ((x_k^n)_k)$ be a Cauchy sequence in $N_p(E_K, \triangle_u^m, f, s)$.Then $g(x^n - x^t) \to 0$ as $n, t \to \infty$.

That is

$$\sum_{i=1}^{m} f(q_i(x_i^n - x_i^t)) + g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right] \to 0 \text{ as } n,t \to \infty.$$

This means that

$$f(q_i(x_i^n - x_i^t) \to 0 \text{ as } n,t \to \infty \text{ for each } i = 1,2,...,m,$$

and

$$g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right] \to 0 \text{ as } n,t \to \infty. \quad (3.6)$$

Since N_p is a K-space, therefore for each k,

$$|v_k|^{-(s/p_k)} f\left(q_k(\triangle_u^m(x_k^n - x_k^t))\right) \to 0 \text{ as } n,t \to \infty.$$

that is

$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) \to 0 \text{ as } n,t \to \infty.$$

Thus for any δ positive, there exists $n_0 \in N$ such that

$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) < \delta \text{ for all } n,t \geq n_0.$$

Let $\delta = f(\epsilon)$, where $\epsilon > 0$. Then

$$f(q_k(\triangle_u^m(x_k^n - x_k^t))) < f(\epsilon) \text{ for all } n,t \geq n_0.$$

This implies

$$(q_k(\triangle_u^m(x_k^n - x_k^t)) < \epsilon \text{ for all } n,t \geq n_0.$$

This shows that for each k, $(\triangle_u^m(x_k^n)$ is a Cauchy sequence in E_k. By condition (3.6), $f(q_i(x_i^n - x_i^t)) \to 0$ as $n,t \to \infty$, for each $i = 1,2,...,m$. But f is a modulus function, it follows that (x_i^n) is Cauchy sequence in E_i for each $i = 1,2,...,m$.

Now (x_i^n) is Cauchy sequence in E_i for each $i = 1,2,...,m$ and $(\triangle_u^m x_k^n)$ is Cauchy sequence in E_k for each $k \in N$. This implies that x_k^n is a cauchy sequence in E_k for each $k \in N$. Since each E_k is complete, so sequence (x_k^n) is convergent for each $k \in N$. Let $\lim_n x_k^N = x_k$ for each $k \in N$. Since (x^n) is Cauchy sequence therefore for each $\epsilon > 0$, there exists n_0 such that $g(x^n - x^t) < \epsilon$ for all $n,t \geq n_0$. So we have

$$\lim_t \sum_{i=1}^{m} f(q_i(x_i^n - x_i^t)) = \sum_{i=1}^{m} f(q_i(x_i^n - x_i)) < \epsilon$$

and

$$\lim_t g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k^t)))\right]$$

$$= g_{N_p}\left[|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k)))\right] < \epsilon \text{ for all } n \geq n_0.$$

This implies that $g(x^n - x) < 2\epsilon$ for all $n \geq n_0$ that is $x^n \to x$ as $n \to \infty$ in $N_p(E_K, \triangle_u^m, f, s)$.

Next we will show that $x \in N_p(E_K, \triangle_u^m, f, s)$. Let $a_k^n = |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k))$. Then for each $k, a_k^n \to 0$ as $n \to \infty$, since f is continuous function. We choose δ_k^n with $0 < \delta_k^n < 1$ such that $a_k^n < \delta_k^n |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n))$. But $(|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n))) \in N_p$ for each n. so for each $n, a^n = (a_k^n) \in N_p$. Again,

$$\begin{aligned} |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)) &= |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k - x_k^n))) \\ &\leq |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m(x_k^n - x_k))) + |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)) \\ &< (1 + \delta_k^n)|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)). \end{aligned}$$

This implies,

$$|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k)) \leq M_n |v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k^n)),$$

where $M_n = \sup_k(\delta_k^n + 1)$.
But N_p is normal sequence space, it follows that $|v_k|^{-(s/p_k)} f(q_k(\triangle_u^m x_k))) \in N_p$, that is $x \in N_p(E_K, \triangle_u^m, f, s)$.

Hence $N_p(E_K, \triangle_u^m, f, s)$ is a complete paranormed space.

Theorem 3.5. Let f, f_1, f_2 be modulus functions, (E_k, q_k) be a sequence of seminormed spaces and $s, s_1, s_2 \geq 0$.Then

$$(i) N_p(E_K, \triangle_u^m, f_1, s) \cap N_p(E_K, \triangle_u^m, f_2, s) \subseteq N_p(E_K, \triangle_u^m, f_1 + f_2, s),$$
$$(ii) N_p(E_K, \triangle_u^m, f, s_1) \subseteq N_p(E_K, \triangle_u^m, f, s_2), \;\; if \;\; s_1 \leq s_2$$

and

$$(iii) N_p(E_K, \triangle_u^m, f_1, s) \subseteq N_p(E_K, \triangle_u^m, f_2 \circ f_1, s), \;\; if \;\; (|v_k|^{-(s/p_k)}) \in N_p.$$

Proof. It is easy to prove (i) and (ii) part of the above theorem. So consider the third one,

(iii) Let $x \in N_p(E_K, \triangle_u^m, f_1, s)$. Then $(|v_k|^{-(s/p_k)} f_1(q_k(\triangle_u^m x_k))) \in N_p$. We choose δ such that $\delta \in (0,1)$ and define sets
$G_1 = \{k \in N : f_1(q_k(\triangle_u^m x_k)) \leq \delta\}$ and $G_2 = \{k \in N : f_1(q_k(\triangle_u^m x_k)) > \delta\}$.
If $k \in G_1$, then $(|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} f_2(\delta)$. Again if $k \in G_2$ then by Lemma 2.3

$$|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} \left(\frac{2f_2(1)}{\delta}\right) f_1(q_k(\triangle_u^m x_k)).$$

Therefore for any $k \in G_1 \cup G_2 = N$,

$$|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k)) < |v_k|^{-(s/p_k)} f_2(\delta) + \left(\frac{2f_2(1)}{\delta}\right) |v_k|^{-(s/p_k)} f_1(q_k(\triangle_u^m x_k))$$

Above inequality is true for each $k \in N$. But N_p is normal sequence space and $(|v_k|^{-(s/p_k)}) \in N_p$, it follows that $(|v_k|^{-(s/p_k)}(f_2 \circ f_1)(q_k(\triangle_u^m x_k))) \in N_p$, that is $x \in N_p(E_K, \triangle_u^m, f_2 \circ f_1, s)$.

Theorem 3.6. Sequence space $N_p(E_K, \triangle_u^m, f, s)$ is a normal space if $m - 0$ and $u = 1$.
Proof. Let $x \in N_p(E_K, \triangle^0, f, s)$. Then $(|v_k|^{-(s/p_k)} f(q_k(x_k))) \in N_p$. Again, let $\lambda = (\lambda_k)$ be a sequence of scalars such that $|\lambda_k| \leq 1$ for each $k \in N$. We have

$$q_k(\lambda_k x_k) = |\lambda_k| q_k(x_k) \leq q_k(x_k) \text{ implies } |v_k|^{-(s/p_k)} f(q_k(\lambda_k x_k) \leq |v_k|^{-(s/p_k)} f(q_k(x_k).$$

But N_p is normal space, it follows that $|v_k|^{-(s/p_k)} f(q_k(\lambda_k x_k) \in N_p$. That is, $\lambda x \in N_p(E_k, \triangle^0, f, s)$. Hence $N_p(E_K, \triangle^0, f, s)$ is a normal space.

Remark 3.2. Above theorem does not hold for any $m, u \in N$.

To show that the space $N_p(E_K, \triangle_u^m, f, s)$ is not normal in general, consider the following example. Let $E_k = C$ for each $k \in N$, $f(x) = x$, $q_k(x) = |x_k|$, $m = 2$, $u = 1$, $s = 0$ and $N_p = l_1$ (where $p_k = 1$ for each $k \in N$). Then $x = (x_k) = (K) \in N_p(E_K, \triangle_u^m, f, s)$. But $\lambda x \in N_p(E_k, \triangle^m, f, s)$, where $\lambda(-1^k)$ for each $k \in N$.

References

[1] Y. Altin, M. Isik and R. Colak, A New system of sequence space defined by modulus, *Studia Univ. Babes-Bolyai Mathematica* **53(2)**, 2008 3-13.

[2] M. Et. and R. Colak, On some generalized difference sequence spaces, *Soochow J. Math.* **21(4)**, 1995 377-386.

[3] P. K. Kamthan and M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York, Basel, 981.

[4] H. Kizmaz, On certain sequence spaces, *Can. Math. Bull.*, **24(2)**,(1981)169-176.

[5] I. J. Maddox Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc.*, **(100)**,(1986)161-166.

[6] E. Ozturk and T. Bilgin, Strongly summable sequence spaces defined by a modulus, *Ind. J.Pure Appl. Math.*, **25(6)**,(1994)621-625.

[7] W. H. Ruckle, F. K. spaces in which the sequence of coordinate vectors is bounded, *Can. J. Math.*, 25(5), (1973)973-978.

[8] A. Sahiner, Some new paranormed spaces defined by modulus function, *Ind. J.Pure Appl. Math.*, **33(12)**,(2002)1877-1888.

[9] P. D. Srivastava and S. Kumar, Generalised vector valued paranormed sequence spaces using modulus function *Appl. Math. and Comput.*, **215**, (2010)4110-4118.

[10] B. C. Tripathy and A. Esi, A new type of difference sequence space, *Int. J. Sci. Technol.*, **(1)**(2006), 11-14.

[11] A. Wilansky, Summability Through Functional Analysis, vol. 85, North-Holland Mathemtics Studies, Amsterdam, Netherlands, 1984.

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