Malaya J. Mat. 3(2)(2015) 143–152

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

General energy decay for nonlinear wave equation of $\phi-$Laplacian type with a delay term in the internal feedback

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Abstract

Under conditions on the delay term, using the multiplier method and general weighted integral inequalities, we study the question of asymptotic behavior of solutions for a nonlinear wave equation with $\phi-$Laplacian operator and a delay term in the internal feedback.

Keywords: Nonlinear wave equation, Time delay term, Decay rate, Multiplier method, $\phi-$Laplacian.

2010 MSC: 35B40, 35L70.

1 Introduction

It is well known that the $\phi-$Laplacian operator degenerates equations in divergence form. It has been much studied during the last years and their results is by now rather developed, especially with delay. In the classical theory of the wave equations several main parts of mathematics are joined in a fruitful way, it is very remarkable that the $\phi-$Laplace wave equation occupies a similar position, when it comes to nonlinear problems. In recent years, the PDEs with time delay effects have become an active area of research and arise in many applied problems.

In this paper we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation

$$\begin{cases} \left(|u'|^{l-2}u'\right)' - \Delta_\phi u + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers and the initial data (u_0, u_1, f_0) belong to a suitable space. The operator Δ_ϕ is defined by

$$\Delta_\phi = \sum_{i=1}^{n} \partial_{x_i}(\phi(|\partial_{x_i}|^2)\partial_{x_i}). \tag{1.2}$$

For $\phi \sim 1$, when g is linear, it is well known that if $\mu_2 = 0$, that is, in the absence of a delay, the energy of problem (1.1) exponentially decays to zero (see for instance [5, 6, 12, 18]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system (1.1) becomes unstable (see for instance [8]). In [8], the authors showed that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [19, 20, 21]). In [19] the authors examined

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the problem (P) with $\phi \sim 1$ and determined suitable relations between μ_1 and μ_2, for which stability or, alternatively, instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution will be unstable if $\mu_2 \geq \mu_1$. The main approach used in [19], is an observability inequality obtained by means of a Carleman estimate. The same results were shown if both the damping and the delay act in the boundary domain. We also recall the result by Xu, Yung and Li in [21], where the authors proved the same result as in [19] for the one-dimension space by adopting the spectral analysis approach.

When g is nonlinear and in the case $\mu_2 = 0, \phi \sim 1$, the problem of existence and energy decay have been previously studied by several authors (see [1, 3, 11, 12, 13]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of a global solution depends on the growth near zero of $g(s)$ as it was proved in [11, 12, 13, 17].

In this article, we use some technique from [3] to give energy decay estimates of solutions to the problem (1.1) for a nonlinear damping and a delay term in the $\phi-$Laplace type. We use the multiplier method and some properties of convex functions. These arguments of convexity were introduced and developed in [4, 7, 13, 14, 15], and used by Liu and Zuazua [16], Eller et al. [9] and Alabau-Boussouira [1].

2 Preliminaries and Notations

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. We use familiar function spaces $W_0^{m,\Phi}$, where the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ si colled an N-function, in the sense of Definition 2.1 given in [3, pp 6-8].

We use the following hypotheses:

(hyp1) $g : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^0(\mathbb{R})$ such that there exist ϵ_1 (sufficiently small), $c_1,\ c_2, c_3, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H' > 0$ and $H' = 0$ on $]0, \epsilon_1]$), such that

$$c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p \quad \text{if } \ |s| \geq \epsilon_1, \tag{2.3}$$

$$|s|^l + |g|^{(p+1)/p}(s) \leq H^{-1}(sg(s)) \quad \text{if } |s| \leq \epsilon_1, \tag{2.4}$$

with p satisfying

$$l-1 \leq p \leq \frac{n+2}{n-2}, \text{ if } n > 2$$
$$l-1 \leq p < \infty, \text{ if } n \leq 2$$

$$|g'(s)| \leq c_3, \tag{2.5}$$

$$\alpha_1\, sg(s) \leq G(s) \leq \alpha_2\, sg(s), \tag{2.6}$$

where

$$G(s) = \int_0^s g(r)\, dr$$

(hyp2) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $C^1(]0,+\infty[) \cap C(]0,+\infty[)$ satisfying $\phi(s) > 0$ on $]0,+\infty[$ and ϕ is non decreasing.

(hyp3)

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{2.7}$$

We first state some lemmas which will be needed later.

Lemma 2.1 (Sobolev–Poincaré's inequality). *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2, ..., p)$ or $2 \leq q \leq pn/(n-p)$ $(n \geq p+1)$. Then there is a constant $c_* = c_*(\Omega, q, p)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_p \quad \textit{for} \quad u \in W_0^{1,p}(\Omega).$$

The case $p = q = 2$ gives the known Poincare's inequality.

Lemma 2.2 ([9, 10]). *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing differentiable function and $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ a convex and increasing function such that $\Psi(0)=0$. Assume that*

$$\int_s^T \Psi(E(t))\,dt \le E(s) \quad \forall 0\le s\le T.$$

Then E satisfies the following estimate:

$$E(t)\le \psi^{-1}\left(h(t)+\psi(E(0))\right) \quad \forall t\ge 0, \tag{2.8}$$

where $\psi(t)=\int_t^1 \frac{1}{\Psi(s)}\,ds$ for $t>0$, $h(t)=0$ for $0\le t\le \frac{E(0)}{\Psi(E(0))}$, and

$$h^{-1}(t)=t+\frac{\psi^{-1}\left(t+\psi(E(0))\right)}{\Psi\left(\psi^{-1}\left(t+\psi(E(0))\right)\right)} \quad \forall t\ge \frac{E(0)}{\Psi(E(0))}.$$

We introduce as in [19] the new variable

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\quad x\in\Omega,\quad \rho\in(0,1),\quad t>0. \tag{2.9}$$

Then we have

$$\tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{2.10}$$

Therefore problem (1.1) is equivalent to:

$$\begin{cases} \left(|u'|^{l-2}u'\right)'-\Delta_\phi u(x,t)+\mu_1 g(u'(x,t))+\mu_2 g(z(x,1,t))=0 & \text{in } \Omega\times]0,+\infty[,\\ \tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times]0,1[\times]0,+\infty[,\\ u(x,t)=0 & \text{on } \partial\Omega\times[0,+\infty[,\\ z(x,0,t)=u'(x,t) & \text{on } \Omega\times[0,+\infty[,\\ u(x,0)=u_0(x)\ u'(x,0)=u_1(x) & \text{in } \Omega\\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times]0,1[. \end{cases} \tag{2.11}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{2.12}$$

The energy of u at time t of the problem (2.11) is defined by

$$E(t)=\frac{l-1}{l}\|u'(t)\|_l^l+\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}u|^2)dx+\xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx. \tag{2.13}$$

where $\tilde{\phi}(s)=\frac{1}{2}\int_0^s\phi(t)dt$. We give an explicit formula for the derivative of the energy.

Lemma 2.3. *Let (u,z) be a solution of the problem (2.11). Then, the energy functional defined by (2.13) satisfies*

$$\begin{aligned} E'(t) &\le -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx\\ &- \left(\frac{\xi}{\tau}\alpha_1-\mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx\\ &\le 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.11) by u', integrating over Ω, we get

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{(l-1)}{l}\|u'\|_l^l+\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}u|^2)\right)dx\\ &+ \mu_1\int_\Omega u'g(u')\,dx+\mu_2\int_\Omega u'g(z(x,1,t))dx. \end{aligned} \tag{2.15}$$

We multiply the second equation in (2.11) by $\xi g(z)$ and integrate the result over $\Omega\times(0,1)$ to obtain

$$\begin{aligned}\xi\int_\Omega\int_0^1 z' g(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx\\ &= -\frac{\xi}{\tau}\int_\Omega\left(G(z(x,1,t))-G(z(x,0,t))\right)dx. \qquad (2.16)\end{aligned}$$

Then

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\frac{\xi}{\tau}\int_\Omega G(u')\,dx. \qquad (2.17)$$

From (2.15), (2.17) and using the Young inequality we get

$$\begin{aligned}E'(t) &= -\left(\mu_1-\frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx\\ &- \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx-\mu_2\int_\Omega u'(t)g(z(x,1,t))\,dx. \qquad (2.18)\end{aligned}$$

Let us denote G^* to be the conjugate function of the convex function G, i.e., $G^*(s)=\sup_{t\in\mathbb{R}^+}(st-G(t))$. Then G^* is the Legendre transform of G which is given by (see [2], [4], [7], [14], [15], [17])

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)] \quad \forall s\geq 0, \qquad (2.19)$$

and satisfies the following inequality

$$st\leq G^*(s)+G(t) \quad \forall s,t\geq 0. \qquad (2.20)$$

Then by the definition of G we get

$$G^*(s)=sg^{-1}(s)-G(g^{-1}(s)).$$

Hence

$$\begin{aligned}G^*(g(z(x,1,t))) &= z(x,1,t)g(z(x,1,t))-G(z(x,1,t))\\ &\leq (1-\alpha_1)z(x,1,t)g(z(x,1,t)). \qquad (2.21)\end{aligned}$$

Making use of (2.18), (2.20) and (2.21), we have

$$\begin{aligned}E'(t) &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx\\ &+ \mu_2\int_\Omega\left(G(u')+G^*(g(z(x,1,t)))\right)dx\\ &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx\\ &+ \mu_2\int_\Omega G^*(g(z(x,1,t)))\,dx. \qquad (2.22)\end{aligned}$$

Using (2.6) and (2.12), we obtain

$$\begin{aligned}E'(t) &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx\\ &- \left(\frac{\xi}{\tau}\alpha_1-\mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx\\ &\leq 0.\end{aligned}$$

3 Main result

Our main result reads as.

Theorem 3.1. *Let $(u_0, u_1, f_0) \in W^{2,\Phi} \cap W_0^{1,\Phi} \times W_0^{1,l}(\Omega) \times W_0^{1,l}(\Omega; W^{1,l}(0,1))$ and assume that the hypotheses (hyp1)–(hyp3) hold. Then, for some constants ω, ϵ_0 we have*

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right) \quad \forall t > 0, \tag{3.23}$$

where $\psi(t) = \int_t^1 \frac{1}{\omega\varphi(\tau)} d\tau$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))}$,

$$h^{-1}(t) = t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\omega\varphi\left(\psi^{-1}\left(t + \psi(E(0))\right)\right)} \quad \forall t > 0,$$

$$\varphi(s) = \{s \text{ if } H \text{ is linear on } [0, \epsilon_1], sH'(\epsilon_0 s) \text{ if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon_1].\}$$

Proof. Multiplying the first equation of (2.11) by $\frac{\varphi(E)}{E}u$, we obtain for all $0 \leq S \leq T$,

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega u \left(\left(|u'|^{l-2}u' \right)' - \Delta_\phi u + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t)) \right) dx\, dt \\
&= \left[\frac{\varphi(E)}{E} \int_\Omega u|u'|^{l-2}u' dx \right]_S^T - \int_S^T \left(\frac{\varphi(E)}{E} \right)' \int_\Omega u|u'|^{l-2}u' dxdt \\
&- \int_S^T \frac{\varphi(E)}{E} \int_\Omega u'^l dxdt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega (\sum_{i=1}^n \phi(|\partial_{x_i} u|^2)|\partial_{x_i} u|^2 dxdt \\
&+ \mu_1 \int_S^T \frac{\varphi(E)}{E} \int_\Omega ug(u')\, dx\, dt + \mu_2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega ug(z(x,1,t))\, dxdt
\end{aligned}$$

Similarly, we multiply the second equation of (2.11) by $\frac{\varphi(E)}{E} e^{-2\tau\rho} g(z(x,\rho,t))$, we have

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega \int_0^1 e^{-2\tau\rho} g(z)(\tau z' + z_\rho)\, dxd\rho dt \\
&= \left[\frac{\varphi(E)}{E} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(z)\, dxd\rho \right]_S^T \\
&- \tau \int_S^T \left(\frac{\varphi(E)}{E} \right)' \int_\Omega \int_0^1 e^{-2\tau\rho} G(z)\, dxd\rho dt \\
&+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega \int_0^1 \left(\frac{\partial}{\partial\rho}(e^{-2\tau\rho} G(z)) + 2\tau e^{-2\tau\rho} G(z) \right) dxd\rho dt \\
&= \left[\frac{\varphi(E)}{E} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(z)\, dxd\rho \right]_S^T \\
&- \tau \int_S^T \left(\frac{\varphi(E)}{E} \right)' \int_\Omega \int_0^1 e^{-2\tau\rho} G(z)\, dxd\rho dt \\
&+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega (e^{-2\tau} G(z(x,1,t)) - G(z(x,0,t)))\, dxdt \\
&+ 2\tau \int_S^T \frac{\varphi(E)}{E} \int_0^1 \int_\Omega e^{-2\tau\rho} G(z)\, dxd\rho dt.
\end{aligned}$$

We have by (hyp2), $s\phi(s) \geq 2\tilde{\phi}(s)$, (note that $\tilde{\phi}$ is convex and defines a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$), summing to obtain, for $A=2\min\{1,\tau e^{-2\tau}/2\xi\}$

$$\begin{aligned}
A\int_S^T\varphi(E)\,dt &\leq -\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T+\int_S^T(\frac{\varphi(E)}{E})'\int_\Omega u|u'|^{l-2}u'dxdt\\
&+\frac{3l-2}{l}\int_S^T\frac{\varphi(E)}{E}\int_\Omega u'^l\,dxdt-\mu_1\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(u')\,dx\,dt\\
&-\mu_2\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(z(x,1,t))\,dxdt-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1\tau e^{-2\tau\rho}G(z)dxd\rho\right]_S^T\\
&+\tau\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)dxd\rho dt\\
&-\int_S^T\frac{\varphi(E)}{E}\int_\Omega(e^{-2\tau}G(z(x,1,t))-G(z(x,0,t)))\,dxdt. \qquad (3.24)
\end{aligned}$$

Using Lemma 2.1, since E is non-increasing, using the Holder, Cauchy–Schwartz, Poincare and Young's inequalities with exponents $\frac{l}{l-1}, l$, to get

$$\begin{aligned}
|\int_\Omega u|u'|^{l-2}u'dx| &\leq (\int_\Omega|u|^l dx)^{1/l}(\int_\Omega|u'|^l dx)^{(l-1)/l}\\
&\leq c(\int_\Omega|\nabla u|^2dx)^{1/2}E^{(l-1)/l}(t)\\
&\leq cE^{(l-1)/l}(t)(\sum_{i=1}^n\tilde{\phi}^{-1}(\int_\Omega\sum_{i=1}^n\tilde{\phi}(|\partial_{x_i}|^2)dx))^{1/2}\\
&\leq cE^{(l-1)/l}(t)(\tilde{\phi}^{-1}(E(t)))^{1/2} \qquad (3.25)
\end{aligned}$$

For $l\geq 2$, $\tilde{\phi}^{-1}$ is non decreasing and φ is convex, increasing and of class $C^1(]0,+\infty[)$ such that $\varphi(0)=0$ (then $s\to s^{(l-1)/l}, s\to\tilde{\phi}^{-1}(s)$ and $s\to\frac{\varphi(s)}{s}$ are non decreasing), we deduce that

$$\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega u(S)|u'(S)|^{l-2}u'(S)dx\\
&-\frac{\varphi(E(T))}{E(T)}\int_\Omega u(T)|u'(T)|^{l-2}u'(T)dx\\
&\leq C\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\left|\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega u|u'|^{l-2}u'\,dxdt\right| &\leq c\int_S^T\left|\left(\frac{\varphi(E)}{E}\right)'\right|E\,dt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,S))\,dxd\rho,\\
&-\frac{\varphi(E(T))}{E(T)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,T))\,dxd\rho\\
&\leq C\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T\left(\left(\frac{\varphi(E)}{E}\right)'\right)\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt &\leq c\int_S^T\left(-\left(\frac{\varphi(E)}{E}\right)'\right)Edt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T\frac{\varphi(E)}{E}\int_\Omega e^{-2\tau}G((x,1,t))\,dxdt &\leq c\int_S^T\frac{\varphi(E)}{E}(-E')\,dt\\
&\leq c\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_\Omega G(z(x,0,t))dxdt &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega G(u'(x,t))\,dxdt \\
&\leq c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \\
&\leq c\varphi(E(S)),
\end{aligned}$$

We conclude

$$\begin{aligned}
A\int_S^T \varphi(E)dt &\leq c\varphi(E(S)) + \mu_1\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(u')|\,dx\,dt \\
&+ \frac{3l-2}{l}\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^l dxdt + \mu_2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(z(x,1,t))|\,dxdt. \qquad (3.26)
\end{aligned}$$

In order to apply the results of Lemma 2.2, we estimate the terms of the right-hand side of (3.26) .

We distinguish two cases.

1. H **is linear on** $[0,\epsilon_1]$. We have $c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p$ for all $s \in \mathbb{R}$, and then, using (2.6) and noting that $s \mapsto \frac{\varphi(E(s))}{E(s)}$ is non-increasing,

$$\frac{3l-2}{l}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u'|^l dxdt \leq c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt \leq c\varphi(E(S)),$$

Using the Poincaré, Young inequalities and the energy inequality from Lemma 2.3, we obtain, for all $\epsilon > 0$,

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(u')|dxdt &\leq \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^{p+1}dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^{1+\frac{1}{p}}(u')dxdt \\
&\leq \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt \\
&\leq \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(z(x,1,t))|dxdt &\leq \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^{p+1}dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^{1+\frac{1}{p}}(z(x,1,t))dxdt \\
&\leq \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega z(x,1,t)g(z(x,1,t))dxdt \\
&\leq \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\varphi(E(S)).
\end{aligned}$$

Inserting these two inequalities into (3.26), choosing $\epsilon > 0$ small enough, we deduce that

$$\int_S^T \varphi(E(t))dt \leq c\varphi(E(S)).$$

Using Lemma 2.2 for E in the particular case where $\varphi(s) = s$, we deduce from (2.8) that

$$E(t) \leq ce^{-\omega t}.$$

2. $H'(0) = 0$ **and** $H'' > 0$ **on** $]0,\epsilon_1]$. For all $t \geq 0$, we consider the following partition of Ω

$$\begin{aligned}
\Omega_t^1 &= \{x \in \Omega : |u'| \geq \epsilon_1\}, & \Omega_t^2 &= \{x \in \Omega : |u'| \leq \epsilon_1\}, \\
\tilde{\Omega}_t^1 &= \{x \in \Omega : |z(x,1,t)| \geq \epsilon_1\}, & \tilde{\Omega}_t^2 &= \{x \in \Omega : |z(x,1,t)| \leq \epsilon_1\}.
\end{aligned}$$

Using (2.3), (2.6) and the fact that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$c\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^1}(|u'|^l + g^{(p+1)/p}(u'))dxdt \leq c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt \leq c\varphi(E(S)).$$

On the other hand, since H is convex and increasing, H^{-1} is concave and increasing. Therefore (2.4) and the reversed Jensen's inequality for a concave function imply that

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+g^{(p+1)/p}(u'))\,dxdt &\le \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}H^{-1}(u'g(u'))\,dxdt\\ &\le \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)dt. \qquad (3.27)\end{aligned}$$

Let us assume H^* to be the conjugate function of the convex function H, i.e., $H^*(s)=\sup_{t\in\mathbb{R}^+}(st-H(t))$. Then H^* is the Legendre transform of H, which is given by (see Arnold [2, pp. 61–64] and [4, 7, 14, 15])

$$H^*(s)=s(H')^{-1}(s)-H[(H')^{-1}(s)] \quad \forall s\ge 0 \qquad (3.28)$$

and satisfies the following inequality

$$st\le H^*(s)+H(t) \quad \forall s,t\ge 0. \qquad (3.29)$$

Due to our choice $\varphi(s)=sH'(\epsilon_0 s)$, we have

$$H^*\left(\frac{\varphi(s)}{s}\right)=\epsilon_0 sH'(\epsilon_0 s)-H(\epsilon_0 s)\le \epsilon_0\varphi(s). \qquad (3.30)$$

Making use of (3.27), (3.29) and (3.30), we have

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l+g^{(p+1)/p}(u'))\,dxdt &\le c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt+c\int_S^T\int_\Omega u'g(u')dt\\ &\le \epsilon_0\int_S^T\varphi(E)dt+cE(S),\end{aligned}$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\tilde\Omega_t^2}g^{(p+1)/p}(z(x,1,t))\,dxdt &\le \int_S^T \frac{\varphi(E)}{E}\int_{\tilde\Omega_t^2}H^{-1}(z(x,1,t)g(z(x,1,t)))\,dxdt\\ &\le \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega z(x,1,t)g(z(x,1,t))dx\Big)dt\\ &\le c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt+c\int_S^T\int_\Omega z(x,1,t)g(z(x,1,t))dt\\ &\le \epsilon_0\int_S^T\varphi(E)dt+cE(S). \qquad (3.31)\end{aligned}$$

Then, choosing $\epsilon_0>0$ small enough and using (3.26), we obtain in both cases

$$\begin{aligned}\int_S^{+\infty}\varphi(E(t))dt &\le c\Big(E(S)+\varphi(E(S))\Big)\\ &\le c\left(1+\frac{\varphi(E(S))}{E(S)}\right)E(S)\\ &\le cE(S) \quad \forall S\ge 0. \qquad (3.32)\end{aligned}$$

Using Lemma 2.2 in the particular case where $\Psi(s)=\omega\varphi(s)$, we deduce from (2.8) our estimate (3.23). The proof of Theorem 3.1 is now complete. □

Acknowledgments

The author would like to thank the referee for his/her careful reading of the proofs.

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Received: May 12, 2014; *Accepted*: December 22, 2014

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