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A note on Civin-Yood Theorem for locally C*-algebras

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Abstract

In the present note we establish Civin-Yood Theorem for locally C^*-algebras, i.e. we show that if A be a locally C^*-algebra, and J be its closed Jordan ideal, then J is as well a closed two-sided *-ideals in A.

Keywords: C^*-algebras, locally C^*-algebras, projective limit of projective family of C^*-algebras.

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1 Introduction

Let A be a C^*-algebra, and J be a closed Jordan ideal in A. In 1965 in their paper [2] Civin and Yood proved among other things that J is a two-sided *-ideal in A.

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [8]. The Hausdorff projective limits of projective families of C^*-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [4]. We will follow Inoue [6] in the usage of the name **locally C^*-algebras** for these algebras.

The purpose of the present notes is to extend the aforementioned result of Civin and Yood from [2] to locally C^*-algebras.

2 Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutory algebra) is an algebra A over $\mathbb{C}$ with an involution

$$^*: A \rightarrow A,$$

such that

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

A seminorm $\|.\|$ on a *-algebra A is a C^*-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \, \|b\| \, ,$$

and satisfies the C^*-condition, i.e.

$$\|a^* a\| = \|a\|^2 \, ,$$

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for every $a, b \in A$. Note that the C^*-condition alone implies that $\|.\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [4]).

When a seminorm $\|.\|$ on a *-algebra A is a C^*-norm, and A is complete in in the topology generated by this norm, A is called a C^***-algebra**.

A topological *-algebra is a *-algebra A equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra A, one puts $N(A)$ for the set of continuous C^*-seminorms on A. One can see that $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|.\|_\alpha, \|.\|_\beta\} \in N(A)$$

for every $\|.\|_\alpha, \|.\|_\beta \in N(A)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological *-algebra A, and $\|.\|_\alpha \in N(A)$, $\alpha \in \Lambda$,

$$\ker \|.\|_\alpha = \{a \in A : \|a\|_\alpha = 0\}$$

is a *-ideal in A, and $\|.\|_\alpha$ induces a C^*-norm (we as well denote it by $\|.\|_\alpha$) on the quotient $A_\alpha = A/\ker\|.\|_\alpha$, and A_α is automatically complete in the topology generated by the norm $\|.\|_\alpha$, thus is a C^*-algebra (see [4] for details). Each pair $\|.\|_\alpha, \|.\|_\beta \in N(A)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism

$$g_\alpha^\beta : A_\beta \to A_\alpha.$$

Let, again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " $\preceq$ ". Let

$$\{A_\alpha, \alpha \in \Lambda\}$$

be a family of C^*-algebras, and g_α^β be, for

$$\alpha \preceq \beta,$$

the continuous linear *-mappings

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,$$

so that

$$g_\alpha^\alpha(x_\alpha) = x_\alpha,$$

for all $\alpha \in \Lambda$, and

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma,$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let A be a *-subalgebra of the direct product algebra

$$_{\alpha\in\Lambda} A_\alpha,$$

so that for its elements

$$x_\alpha = g_\alpha^\beta(x_\beta),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_\alpha \in A_\alpha,$$

and

$$x_\beta \in A_\beta.$$

The *-algebra A constructed above is called a **Hausdorff projective limit** of the projective family

$$\{A_\alpha, \alpha \in \Lambda\},$$

relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\underleftarrow{\lim} A_\alpha,$$

$\alpha \in \Lambda$, and is called the Arens-Michael decomposition of A.

It is well known (see, for example [11]) that for each $x \in A$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : A \longrightarrow A_\beta,$$

defined by

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

A topological *-algebra (A, τ) over $\mathbb{C}$ is called a **locally C^*-algebra** if there exists a projective family of C^*-algebras

$$\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$A \cong \underleftarrow{\lim} A_\alpha,$$

$\alpha \in \Lambda$, i.e. A is topologically *-isomorphic to a projective limit of a projective family of C^*-algebras, i.e. there exits its Arens-Michael decomposition of A composed entirely of C^*-algebras.

A topological *-algebra (A, τ) over $\mathbb{C}$ is a locally C^*-algebra iff A is a complete Hausdorff topological *-algebra in which the topology τ is generated by a saturated separating family $\mathcal{F}$ of C^*-seminorms (see [4] for details).

Every C^*-algebra is a locally C^*-algebra.

A closed *-subalgebra of a locally C^*-algebra is a locally C^*-algebra.

The product ${}_{\alpha \in \Lambda} A_\alpha$ of C^*-algebras A_α, with the product topology, is a locally C^*-algebra.

Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally C^*-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see [7] for details).

Let A be a locally C^*-algebra. Then an element $a \in A$ is called **bounded**, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|.\|_\alpha \in N(A)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is well-known that for each locally C^*-algebra A, its set $b(A)$ of bounded elements of A is a locally C^*-subalgebra, which is a C^*-algebra in the norm $\|.\|_\infty$, such that it is dense in A in its topology (see for example [4]).

3 Civin-Yood Theorem for locally C*-algebras

Let us recall that a subspace J of an associative algebra A is called a **Jordan ideal** of A, if for each $a \in J$ and $b \in A$,

$$\frac{ab + ba}{2} = a \circ b \in J,$$

where the multiplication $a \circ b$ thus defined is called symmetric (see [5] for details).

Now we are ready to present the main theorem of the current notes.

Theorem 3.1. *Let (A, τ_A) be a locally C^*-algebra, and (J, τ_J) be a closed Jordan ideals in A, such that*

$$\tau_J = \tau_A|J.$$

Then (J, τ_J) is a closed two-sided *-ideal of A.

Proof. Let now (A, τ_A) be a locally C^*-algebra, and let

$$A = \varprojlim A_\alpha,$$

$\alpha \in \Lambda$, be its Arens-Michael decomposition into a projective limit of a projective family of C^*-algebras $A_\alpha, \alpha \in \Lambda$, built using the family of seminorms $\|.\|_\alpha, \alpha \in \Lambda$, that defines the topology τ_A. Let

$$\pi_\alpha : A \to A_\alpha,$$

$\alpha \in \Lambda$, be a projection from A onto A_α, for each $\alpha \in \Lambda$. Each π_α is an surjective *-homomorphism from A onto $A_\alpha, \alpha \in \Lambda$. Let

$$g_\alpha^\beta : A_\beta \to A_\alpha,$$

be a surjective *-homomorphism from A_β onto A_α, for each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$. Such family g_α^β, $\alpha, \beta \in \Lambda$ does exist because the family $A_\alpha, \alpha \in \Lambda$ is projective. Let

$$J_\alpha = \pi_\alpha(J),$$

for each $\alpha \in \Lambda$. One can see now that

$$g_\alpha^\beta(J_\beta) = J_\alpha,$$

because

$$\pi_\alpha = g_\alpha^\beta \circ \pi_\beta,$$

for all $\alpha \preceq \beta, \alpha, \beta \in \Lambda$.

From the fact that J is a closed in τ_J topology subspace of A it follows that J_α is a closed in $\|.\|_\alpha$ subspace of A_α for all $\alpha \in \Lambda$.

We show now that J_α is a Jordan ideal of A_α for each $\alpha \in \Lambda$. In fact, let $a_\alpha \in J_\alpha$, and $b_\alpha \in A_\alpha$ be arbitrary, and $\alpha \in \Lambda$. We select arbitrary $a \in \pi_\alpha^{-1}(a_\alpha)$ which is obviously in J, and $b \in \pi_\alpha^{-1}(b_\alpha)$, which is obviously in A. Because J is a Jordan ideal of A it follows that

$$a \circ b = \frac{ab + ba}{2} \in J.$$

One can see that

$$\pi_\alpha(a) = a_\alpha \text{ and } \pi_\alpha(b) = b_\alpha.$$

Thus,

$$\begin{aligned} J_\alpha \quad &\ni \quad \pi_\alpha(a \circ b) = \pi_\alpha(\frac{ab + ba}{2}) = \frac{\pi_\alpha(ab + ba)}{2} = \frac{\pi_\alpha(ab) + \pi_\alpha(ba)}{2} \\ &= \quad \frac{\pi_\alpha(a)\pi_\alpha(b) + \pi_\alpha(b)\pi_\alpha(a)}{2} = \frac{a_\alpha b_\alpha + b_\alpha a_\alpha}{2} = a_\alpha \circ b_\alpha. \end{aligned}$$

Now, applying to each J_α, $\alpha \in \Lambda$ Civin-Yood theorem from [2] we conclude that each J_α, $\alpha \in \Lambda$ is a two-sided *-ideal of A_α, i.e. for arbitrary $a_\alpha \in J_\alpha$ and $b_\alpha \in A_\alpha$ it follows that $a_\alpha b_\alpha, b_\alpha a_\alpha, a_\alpha^* \in J_\alpha$.

Let now $a \in J$ and $b \in A$ be arbitrary elements from J and A respectively. Then for each $\alpha \in \Lambda$,

$$J_\alpha \ni \pi_\alpha(a)\pi_\alpha(b) = \pi_\alpha(ab),$$

which implies that there exists a unique element $ab \in J$. Similarly we obtain that $ba \in J$.

At the same time for each $\alpha \in \Lambda$, even though generally speaking a^* exists in A, because

$$(\pi_\alpha(a))^* = \pi_\alpha(a^*) = a_\alpha^* \in J_\alpha,$$

which implies that $a^* \in J$. □

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