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Some Results for the Bessel transform

Mohamed El Hamma[a,*] , Radouan Daher[b] and Salah El Ouadih[c]

[a,b,c] Department of Mathematics, Faculty of Sciences Aïn Chock & University of Hassan II, Casablanca, Morocco.

Abstract

In this paper, using a Bessel generalized translation , we prove the estimates for the Bessel transform in the space $L^2_p(\mathbb{R}_+)$ on certain classes of functions.

Keywords: Bessel operator; Bessel transform; Bessel generalized translation .

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1 Introduction and preliminaries

Integral transforms and their inverses (e.g., the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathemtical physics, and computational mathematics (see, e.g.,[3, 8]).

In [7], E.C. Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition for the Fourier transform, namely we have

Theorem 1.1. *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*

1. $\|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,
2. $\int_{|\lambda|\geq r} |\mathcal{F}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow +\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

The main aim of this paper is to establish a generalization of Theorem 1.1 in the Bessel transform setting by means of the Bessel generalized translation. We point out that similar results have been established in the context of noncompact rank 1 Riemannian symmetric spaces and of Jacobi transform (see [2, 6]).

In this section, we give some definition and preliminaries concerning the Bessel transform. Everywhere below p is a real number, $p \geq -\frac{1}{2}$.

Let
$$D = \frac{d^2}{dx^2} + \frac{(2p+1)}{x}\frac{d}{dx}$$
be the Bessel differential operator. We introduce the normalized Bessel function of the first kind j_p defined by

$$j_p(z) = \Gamma(p+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)}\left(\frac{z}{2}\right)^{2n}, z \in \mathbb{C}, \tag{1.1}$$

where $\Gamma(x)$ is the gamma-function (see[4]). The function $y = j_p(x)$ satisfies the differential equation

*Corresponding author.

E-mail address: m_elhamma@yahoo.fr (Mohamed El Hamma), rjdaher024@gmail.com(Radouan Daher), salahwadih@gmail.com(Salah El Ouadih).

$$Dy + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_p(x)$ is infinitely differentiable, even, and, moreover entire analytic.

From (1.1) we see that

$$\lim_{z \longrightarrow 0} \frac{j_p(z) - 1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Longrightarrow |j_p(z) - 1| \geq c|z|^2 \tag{1.2}$$

From [1], we have

$$|j_p(x)| \leq 1. \tag{1.3}$$

and

$$1 - j_p(x) = O(x^2),\ 0 \leq x \leq 1. \tag{1.4}$$

Assume that $L^2_p(\mathbb{R}_+)$, $p \geq -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}_+$ with the finite norm

$$\|f\| = \|f\|_{2,p} = \left(\int_0^\infty |f(x)|^2 x^{2p+1} dx \right)^{1/2}$$

Given $f \in L^2_p(\mathbb{R}_+)$, the Bessel transform is defined by

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_p(\lambda t) t^{2p+1} dt,\ \lambda \in \mathbb{R}_+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^p \Gamma(p+1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_p(\lambda t) \lambda^{2p+1} d\lambda.$$

From [3], we have the Parseval's identity

$$\int_0^\infty |\widehat{f}(\lambda)|^2 \lambda^{2p+1} d\lambda = 2^{2p} \Gamma^2(p+1) \int_0^\infty |f(t)|^2 t^{2p+1} dt.$$

In $L^2_p(\mathbb{R}_+)$, consider the Bessel generalized translation T_h (see [3, p. 121])

$$T_h f(x) = c_p \int_0^\pi f(\sqrt{x^2 + h^2 - 2xhcost}) sin^{2p} t dt,\ \ p \geq -\frac{1}{2},\ h > 0,$$

where

$$c_p = \left(\int_0^\pi sin^{2p} t dt \right)^{-1} = \frac{\Gamma(p+1)}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})}$$

From [5], we note importants properties of Bessel transform

$$\widehat{(Df)}(\lambda) = (-\lambda^2)\widehat{f}(\lambda). \tag{1.5}$$

and

$$\widehat{(T_h f)}(\lambda) = j_p(\lambda h)\widehat{f}(\lambda). \tag{1.6}$$

We define the differences of first and higher orders as

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{E}) f(x)$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{T}_h - \mathrm{E})^k f(x) = \sum_{i=1}^{\infty} (-1)^{k-i} \binom{k}{i} \mathrm{T}_h^i f(x), \tag{1.7}$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1} f(x))$, $i = 1, 2, .., k$; k=1,2,.... and E is the unit operator in the space $\mathrm{L}_p^2(\mathbb{R}_+)$.

2 Main results

Lemma 2.1. *For $f \in \mathrm{L}_p^2(\mathbb{R}_+)$. Then*

$$\|\Delta_h^k \mathrm{D}^r f(x)\|^2 = \int_0^\infty t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt$$

Proof From formula (1.5), we have

$$\widehat{(\mathrm{D}^r f)}(t) = (-1)^r t^{2r} \widehat{f}(t);\ r = 0, 1, 2, ... \tag{2.8}$$

We use formulas (1.6) and (2.8), we conclude that

$$\widehat{\mathrm{T}_h^i \mathrm{D}^r f}(t) = (-1)^r j_p^i(th) t^{2r} \widehat{f}(t);\ 1 \leq i \leq k. \tag{2.9}$$

Or, from formulas (1.7) and (2.9) the image $\Delta_h^k \mathrm{D}^r f(x)$ under the Bessel transform has the form

$$\widehat{\Delta_h^k \mathrm{D}^r f}(t) = (-1)^r (j_p(th) - 1)^k t^{2r} \widehat{f}(t).$$

By Parseval's identity, we have the result.

Our main result is as follows

Theorem 2.2. *Let $f \in \mathrm{L}_p^2(\mathbb{R}_+)$. Then the following are equivalents*

1. $\|\Delta_h^k \mathrm{D}^r f(x)\| = O(h^\alpha)$ *as* $h \longrightarrow 0$, $(0 < \alpha < k)$
2. $\int_s^\infty t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt = O(s^{-2\alpha})$ *as* $s \longrightarrow +\infty$

Proof 1) $\Longrightarrow$ 2) Suppose that

$$\|\Delta_h^k \mathrm{D}^r f(x)\| = O(h^\alpha) \text{ as } h \longrightarrow 0$$

From Lemma 2.1, we have

$$\|\Delta_h^k \mathrm{D}^r f(x)\|^2 = \int_0^\infty t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt.$$

By formula (1.2), we obtain

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt \geq \frac{c^{2k} \eta^{4k}}{2^{4k}} \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt.$$

There exists then a positive constant C such that

$$\begin{aligned} \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\widehat{f}(t)|^2 t^{2p+1} dt &\leq C \int_0^\infty t^{4r} |j_p(th) - 1|^{2k} |\widehat{f}(t)|^2 t^{2p+1} dt \\ &\leq C h^{2\alpha} \end{aligned}$$

Then

$$\int_s^{2s} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha})$$

for all $s > 0$.

Moreover, we have

$$\begin{aligned}\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt &= \sum_{j=0}^{\infty}\int_{2^j s}^{2^{j+1}s} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt \\ &\leq C\sum_{j=0}^{\infty}(2^j s)^{-2\alpha} \\ &\leq Cs^{-2\alpha}.\end{aligned}$$

This proves that

$$\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha}) \text{ as } s \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_s^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(s^{-2\alpha}) \text{ as } s \longrightarrow +\infty.$$

We have to show that

$$\int_0^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt = O(h^{2\alpha}) \text{ as } h \longrightarrow 0.$$

We write

$$\int_0^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{1/h} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} t^{4r}|j_p(th)-1|^{2k}|\widehat{f}(t)|^2 t^{2p+1}dt$$

From formula (1.3), we obtain

$$\mathrm{I}_2 \leq 4^k\int_{1/h}^{\infty} t^{4r}|\widehat{f}(t)|^2 t^{2p+1}dt = O(h^{2\alpha}) \text{ as } h \longrightarrow 0.$$

Set

$$\psi(t) = \int_t^{\infty} x^{4r}|\widehat{f}(x)|^2 x^{2p+1}dx$$

From formula (1.4) and integration by parts, we have

$$\begin{aligned}\mathrm{I}_1 &= -\int_0^{1/h}|j_p(th)-1|^{2k}|\psi'(t)dt \\ &\leq -h^{2k}\int_0^{1/h} t^{2k}\psi'(t)dt \\ &\leq -\psi(\frac{1}{h}) + 2kh^{2k}\int_0^{1/h} t^{2k-1-2\alpha}dt\end{aligned}$$

Or, we see that $\alpha < k$ the integral exists. Then

$$\begin{aligned} \mathrm{I}_1 &\leq \frac{2k}{2k-2\alpha} h^{2k} h^{-2k+2\alpha} \\ &\leq Ch^{2\alpha} \end{aligned}$$

and this ends the proof.

Corollary 2.1. *Let* $f \in \mathrm{L}^2_p(\mathbb{R}_+)$, $(p \geq -\frac{1}{2})$, *and let*

$$\|\Delta^k_h \mathrm{D}^r f(x)\| = O(h^\alpha) \text{ as } h \longrightarrow 0.$$

Then

$$\int_s^\infty |\widehat{f}(t)|^2 t^{2p+1} dt = O(s^{-4r-2\alpha}) \text{ as } s \longrightarrow +\infty$$

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