MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(04)**(2022), 280–291.
http://doi.org/10.26637/mjm1004/001

The planar and outerplanar indices of Cayley graphs of finite groups

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Received 12 June 2022; Accepted 17 September 2022

Abstract. In our study, we consider the Cayley graph of finite groups and its iterated line graphs. We present a complete characterization of finite groups with planar and outerplanar indices.

AMS Subject Classifications: 05C25, 05C76, 20F65.

Keywords: Cayley graph, Finite group, Planar index, Outerplanar index.

Contents

1. Introduction and Background

Let S generates group G. We define the Cayley graph $\overrightarrow{\text{Cay}}(G,S)$ of generators S on G as follows. The vertices of $\overrightarrow{\text{Cay}}(G,S)$ are the elements of G, and there is an arc from g to gs whenever $g \in G$ and $s \in S$. The Cayley graph $\text{Cay}(G,S)$ of S on G is obtained by replacing each arc in $\text{Cay}(G,S)$ with an (undirected) edge. One can identify $\text{Cay}(G,S)$ with $\overrightarrow{\text{Cay}}(G,S\cup S^{-1})$, where $S^{-1}=\{s^{-1}; s\in S\}$.

Cayley graphs of groups enjoy a rich research history and they are a classic point of interaction of graph theory and algebra. The original definition of the Cayley graph of a group was introduced by Cayley in 1878 [1] to explain the concept of abstract groups described by a set of generators. In the last 50 years, the theory of Cayley graphs have grown into a substantial branch in algebraic graph theory. We refer the reader to [3, 6, 7, 10, 14], for more details.

It is interesting to find graphs that can be drawn respecting certain geometric or topological criteria. This work is done for some Cayley graphs on some algebraic structures. Also, there are some characterizations for these algebraic structures which their Cayley graphs can be drawn in a plane. For example see [9], and [11, 12].

A group is called planar if it admits a generating system such that the resulting Cayley graph is planar, that is, it admits a plane drawing. In 1896, Maschke characterized planar finite groups, that is groups which admit a generating system such that the resulting Cayley graph is planar.

Theorem 1.1. *[12, Maschke's Theorem] The groups and minimal generating systems in Table 1 are exactly those pairs having a planar Cayley graph.*

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0), (0,1)$
$\mathbb{Z}_2 \times \mathbb{Z}_n$	$(1,0), (0,1)$
D_3	$(123), (12), (23)$
D_4	$(1234), (13)$
D_n	$(12), (13)$ $(12\dots n), (12)$
$\mathbb{Z}_2 \times D_n$	$(1,e), (0,(12)), (0,(13))$
A_4	$(123), (12)(34)$ $(123), (234)$ $(123), (234), (13)(24)$
$\mathbb{Z}_2 \times A_4$	$(0,(123)), (1,(12)(34))$
S_4	$(123), (34)$ $(12), (23), (34)$ $(12), (1234)$ $(123), (1234)$ $(1234), (123), (34)$
$\mathbb{Z}_2 \times S_4$	$(1,(12)), (0,(23)), (0,(34))$
A_5	$(124), (23)(45)$ $(12345), (23)(45)$ $(12345), (124)$ $(12345), (124), (23)(45)$
$\mathbb{Z}_2 \times A_5$	$(1,(12)(35)), (1,(24)(35)), (1,(23)(45))$

Table 1: The planar groups and their minimal generating systems giving planar Cayley graphs

In this paper, we will focus on embeddability of the Cayley graph and its iterated line graphs into a plane. Given a graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. Also, we denote the kth iterated line graph of G by $L^k(G)$ and define it as follows: $L^k(G) = L(L^{k-1}(G))$ for $k \geqslant 1$ and $L^0(G) = G$ and $L^1(G) = L(G)$ is the line graph of G. We define the planar index of G, denoted by $\xi(G)$, as the smallest k such that $L^k(G)$ is non-planar. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$. Further, the outerplanar index of G is defined as the smallest k such that $L^k(G)$ is non-outerplanar. We denote the outerplanar index of G by $\zeta(G)$. As well as, if $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$.

This paper is organized as follows. At first, we deal with planar index of the Cayley graph of a finite group. In addition, we classify all finite groups which admit outerplanar Cayley graph. Also, we study the outerplanar index of the Cayley graph when G is a finite group.

In order to make this paper easier to follow, let recall some standard definitions and notation of group theory and graph theory we use in this paper. Let n be a positive integer. The group of integers modulo n is denoted by $\mathbb{Z}_n = \{0, \dots, n-1\}$. Also, the notation D_n stands for the dihedral group. The elements of D_n are the symmetries of the n-gon with the vertices $1, \dots, n$ and so $|D_n| = 2n$. Further, A_n and S_n are the alternating group and the symmetric group on n points, respectively. The identity element is denoted by e for all groups G except for $\mathbb{Z}_n$, where we use 0.

Now let us summarize some notations, concepts of graph theory which will be needed in the subsequent sections. For basic definitions on graphs, one may refer to [2]. Let G be a graph. Then the degree of a vertex v, denoted by deg(v), is the number of edges of G incident to v. Also, an r-regular graph is a graph where every vertex has the degree r. The maximum degree of G, denoted by $\Delta(G)$, is the maximum degree of its vertices. The

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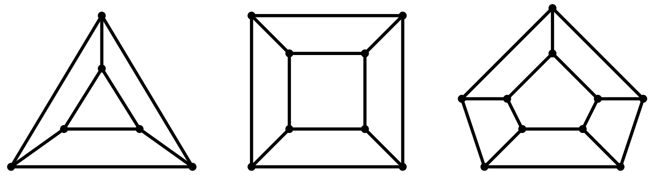

Figure 1: n-prisms

graph G is connected if for every pair of vertices, there is a path in the graph between those vertices, otherwise G is said to be disconnected. A connected component of a disconnected graph is a maximal connected subgraph of the graph. A cut vertex v is a vertex that when we removed it (with its boundary edges) from G creates more connected components than previously in G. We use the notations K_n and C_n for complete graphs and cycles on n vertices, respectively. The Cartesian product $G \times H$ of graphs G and H is a graph such that

(i) the vertex set of the graph $G \times H$ is the Cartesian product $V(G) \times V(H)$; and

(ii) any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G.

An n-prism graph is a simple graph which can be constructed as the Cartesian product of the cycle C_n with K_2. In Figure 1, 3-prism, 4-prism and 5-prism are drawn.

2. The planar index of Cay(G, S)

This section consists on classifying all finite groups with respect to planar index of their Cayley graphs. At first, we determine when $L(\text{Cay}(G, S))$ is planar. Sedláček [13], characterized graphs whose their line graph is planar. He showed that the line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leqslant 4$, and every vertex of degree 4 in G is a cut-vertex. Using Sedláček's characterization, in the following lemma, we characterizes all Cayley graphs whose their line graph is planar.

Lemma 2.1. *The groups and minimal generating systems in Table 2 are exactly those pairs which $L(Cay(G, S))$ is planar.*

Proof. By using Sedláček's characterization, if $L(\text{Cay}(G, S))$ is planar, then G is planar. So, we must only check the planar groups. By Maschke's Theorem, Theorem 1.1, we have the following cases:

Case 1. $G \cong \mathbb{Z}_n$ with $S = \{1\}$ and $G \cong D_n$ with $S = \{(12), (13)\}$. In both cases, the graph Cay(G, S) is 2-reqular graph which implies that these graphs are cycles. So, $L(\text{Cay}(G, S))$ is planar.

Case 2. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1, 0), (0, 1)\}$. In this case the Cayley graph Cay(G, S) is a cycle with 4 vertices and so $L(\text{Cay}(G, S))$ is planar.

Case 3. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ with $S = \{(1, 0), (0, 1)\}$ and $G \cong D_n$ with $S = \{(12 \dots n), (12)\}$ where $n \geqslant 3$. In both these cases, it is not hard to see that the graph Cay(G, S) is a n-prism graph. Since n-prisms are 3-regular graphs we have that $L(\text{Cay}(G, S))$ is planar.

Case 4. $G \cong D_3$ with $S = \{(123), (12), (23)\}$. Since $|S \cup S^{-1}| = 4$, the graph Cay(G, S) is a 4-reqular graph. This graph is drawn in Figure 2. By this figure, we see that the graph Cay(G, S) has a vertex of degree 4 which is not a cut vertex. Hence $L(\text{Cay}(G, S))$ is not planar.

Case 5. $G \cong D_4$ with $S = \{(1234), (13)\}$. The graph Cay(G, S) is a 4-prism graph. So it is a 3-regular graph and we have that $L(\text{Cay}(G, S))$ is planar.

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0),(0,1)$
$\mathbb{Z}_2 \times \mathbb{Z}_n$ where $n \geqslant 3$	$(1,0),(0,1)$
D_3	$(123),(12),(23)$
D_4	$(1234),(13)$
D_n	$(12),(13)$ $(12\dots n),(12)$
$\mathbb{Z}_2 \times D_n$	$(1,e),(0,(12)),(0,(13))$
A_4	$(123),(12)(34),$
$\mathbb{Z}_2 \times A_4$	$(0,(123)),(1,(12)(34))$
S_4	$(123),(34)$ $(12),(23),(34)$ $(12),(1234)$
$\mathbb{Z}_2 \times S_4$	$(1,(12)),(0,(23),(0,(34))$
A_5	$(124),(23)(45)$ $(12345),(23)(45)$
$\mathbb{Z}_2 \times A_5$	$(1,(12)(35)),(1,(24)(35)),(1,(23)(45))$

Table 2: The groups and their minimal generating systems which the line of their cayley graphs are planar

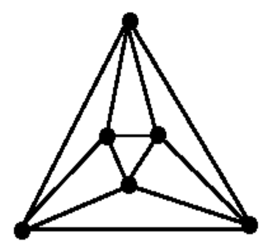

Figure 2: Cay$(D_3,\{(123),(12),(23)\})$

Case 6. $G \cong \mathbb{Z}_2 \times D_n$ and $S=\{(1,e),(0,(12)),(0,(13))\}$. The Cayley graph Cay$(G,S)$ is a $2n$-prism. Therefore this graph is a 3-regular graph which implies that $L($Cay$(G,S))$ is planar.

Case 7. $G \cong A_4$ and $S=\{(123),(12)(34)\}$. Since $|S \cup S^{-1}|=3$, the Cayley graph Cay(G,S) is a 3-regular graph (Figure 3) and so $L($Cay$(G,S))$ is planar.

Case 8. $G \cong A_4$ and $S=\{(123),(234)\}$. By Figure 4, Cayley graph Cay(G,S) is a 4-regular graph and it has a vertex of degree 4 which is not a cut vertex. So $L($Cay$(G,S))$ is not planar.

Case 9. $G \cong A_4$ and $S=\{(123),(234),(13)(24)\}$. Since $|S \cup S^{-1}|=5$, Cayley graph Cay(G,S) is a 5-regular graph and so $L($Cay$(G,S))$ is not planar.

Case 10. $G \cong \mathbb{Z}_2 \times A_4$ with $S=\{(0,(123)),(1,(12)(34))\}$ and $G \cong S_4$ with $S=\{(123),(34)\}$. Since $|S \cup S^{-1}|=3$, in both cases the Cayley graph Cay(G,S) is a 3-regular graph. It is not hard to see that

$$\text{Cay}(\mathbb{Z}_2 \times A_4,\{(0,(123)),(1,(12)(34))\}) \cong \text{Cay}(S_4,\{(123),(34)\}).$$

The graph is pictured in Figure 5. Hence we can conclude that $L($Cay$(G,S))$ is planar.

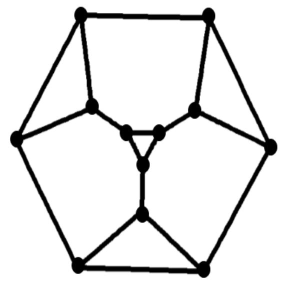

Figure 3: Cay$(A_4, \{(123), (12)(34)\})$

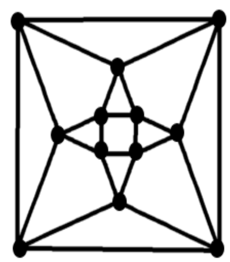

Figure 4: Cay$(A_4, \{(123), (234)\})$

Case 11. $G \cong S_4$ with $S = \{(12), (23), (34)\}$ or $S = \{(12), (1234)\}$. It is easy to see that the Cayley graph Cay(G, S) is a 3-regular graph and they are isomorphic to Figure 6. Hence $L($Cay$(G, S))$ is planar.

Case 12. $G \cong S_4$ with $S = \{(123), (1234)\}$. It is easy to see that the Cayley graph Cay(G, S) is a 4-regular graph and none of the vertices is a cut vertex. So $L($Cay$(G, S))$ is not planar.

Case 13. $G \cong S_4$ with $S = \{(1234), (123), (34)\}$. Since $|S \cup S^{-1}| = 5$, the Cayley graph Cay(G, S) is a 5-regular graph. Therefore $L($Cay$(G, S))$ is not planar.

Case 14. $G \cong \mathbb{Z}_2 \times S_4$ with $S = \{(1, (12)), (0, (23)), (0, (34))\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph which is pictured in Figure 7. Therefore $L($Cay$(G, S))$ is planar.

Case 15. $G \cong A_5$ with $S = \{(124), (23)(45)\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph (Figure 8). Therefore $L($Cay$(G, S))$ is planar.

Case 16. $G \cong A_5$ with $S = \{(12345), (23)(45)\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph which is drawn in Figure 9. Hence $L($Cay$(G, S))$ is planar.

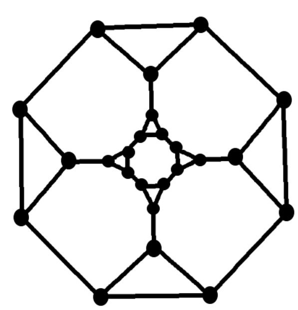

Figure 5: Cay$(\mathbb{Z}_2 \times A_4, \{(0, (123)), (1, (12)(34))\})$

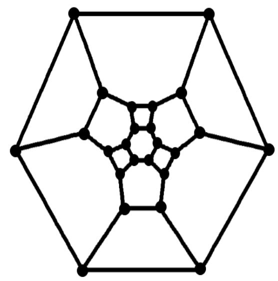

Figure 6: Cay$(S_4, \{(12), (23), (34)\}) \cong$ Cay$(S_4, \{(12), (1234)\})$

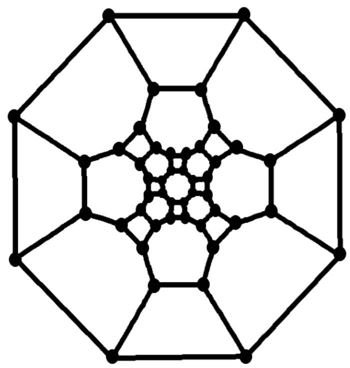

Figure 7: Cay$(\mathbb{Z}_2 \times S_4, \{(1, (12)), (0, (23)), (0, (34))\})$

Case 17. $G \cong A_5$ with $S = \{(12345), (124)\}$. Since $|S \cup S^{-1}| = 4$, the Cayley graph Cay(G, S) is a 4-regular graph. It is easy to see that none of the vertices of this graph is a cut vertex. Therefore $L($Cay$(G, S))$ is not planar.

Case 18. $G \cong A_5$ with $S = \{(12345), (124), (23)(45)\}$. Since $|S \cup S^{-1}| = 5$, the Cayley graph Cay(G, S) is a 5-regular graph, which implies that $L($Cay$(G, S))$ is not planar.

Case 19. $G \cong \mathbb{Z}_2 \times A_5$ with $S = \{(1, (12)(35)), (1, (24)(35)), (1, (23)(45))\}$. Since $|S \cup S^{-1}| = 3$, the Cayley graph Cay(G, S) is a 3-regular graph(Figure 10), which implies that $L($Cay$(G, S))$ is planar.

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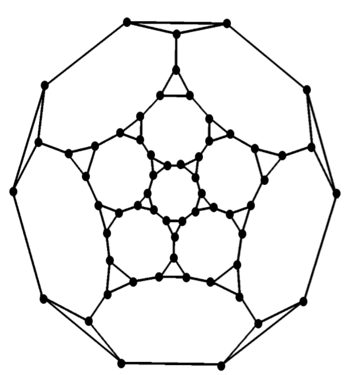

Figure 8: Cay$(A_5, \{(124), (23)(45)\})$

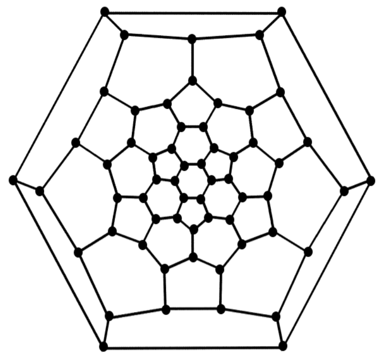

Figure 9: Cay$(A_5, \{(12345), (23)(45)\})$

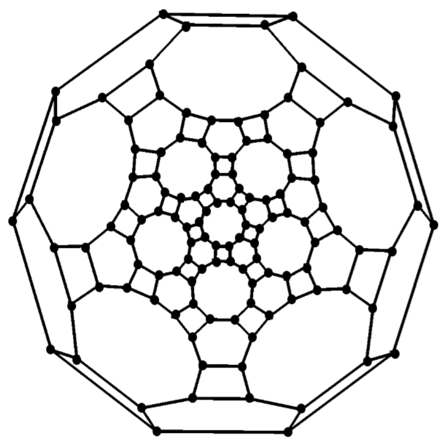

Figure 10: Cay$(\mathbb{Z}_2 \times A_5, \{(1, (12)(35)), (1, (24)(35)), (1, (23)(45))\})$

In the rest of this section, we deal with planar index of Cay(G, S). It was shown in [13] that if G is non-planar, then $L(G)$ is also non-planar. Also, if H is a subgraph of G, in [4, Lemma 4], it was shown that $\xi(G) \leqslant \xi(H)$, and hence the planar index of a graph is the minimum of the planar indices of its connected components. Further, in [4], the authors gave a full characterization of connected graphs with respect to their planar index.

Theorem 2.2. *[4, Theorem 10] Let G be a connected graph. Then:*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or* $K_{1,3}$.

(iii) $\xi(G) = 1$ *if and only if G is planar and either* $\Delta(G) \geqslant 5$ *or G has a vertex of degree* 4 *which is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if* $L(G)$ *is planar and G contains one of the graphs* H_i *in Figure 11 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if G is one of the graphs* X_k *or* Y_k *(Figure 11) for some* $k \geqslant 2$.

(vi) $\xi(G) = 3$ *otherwise.*

In the next theorem we classify the Cayley graphs Cay(G, S) of finite groups with respect to their planar index.

Theorem 2.3. *Let G be a finite group. Then:*

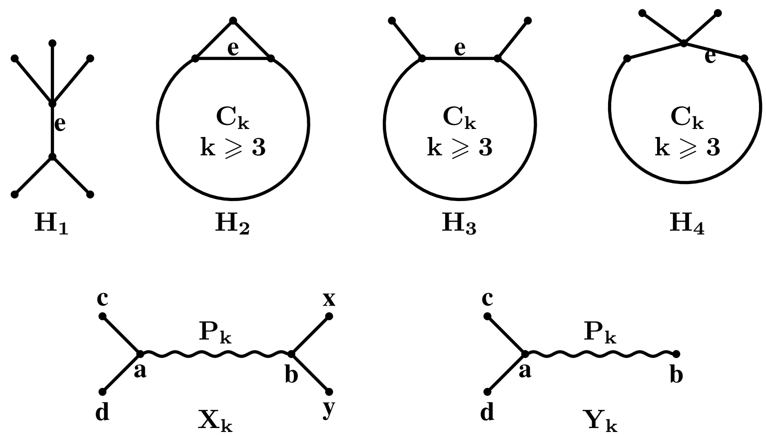

Figure 11: Figures of Theorem 2.2

Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0),(0,1)$
D_n	$(12),(13)$

Table 3: The groups and their minimal generating systems which $\xi(\text{Cay}(G,S)) = \infty$.

(i) $\xi(Cay(G,S)) = 0$ *if and only if* $Cay(G,S)$ *is non-planar.*

(ii) $\xi(Cay(G,S)) = \infty$ *if and only if G and S are as in Table 3.*

(iii) $\xi(Cay(G,S)) = 1$ *if and only if G and S are as in Table 4.*

(iv) $\xi(Cay(G,S)) = 2$ *if and only if G and S are as in Table 5.*

Proof. We know $\xi(\text{Cay}(G,S)) = 0$ if $\text{Cay}(G,S)$ is non-planar. Thus we may assume that $\text{Cay}(G,S)$ is planar. So, by Maschke's Theorem, we must consider the groups and minimal generating sets which were stated in Table 1. By comparing Tables 1 and 2, we can conclude that $\xi(\text{Cay}(G,S)) = 1$ for the groups and minimal generating sets of Table 4. Now, by Lemma 2.1 and Table 2, we have the following cases:

Case 1. Let $G \cong \mathbb{Z}_n$ with $S = \{1\}$, $G \cong D_n$ with $S = \{(12),(13)\}$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1,0),(0,1)\}$. Since the Cayley graphs of these groups and their generating sets are cycles, we can conclude that $\xi(\text{Cay}(G,S)) = \infty$.

Case 2. Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ with $S = \{(1,0),(0,1)\}$, $G \cong D_n$ with $S = \{(12\ldots n),(12)\}$ and $G \cong \mathbb{Z}_2 \times D_n$ and

Group	Minimal generating systems
A_4	$(123),(234)$
	$(123),(234),(13)(24)$
S_4	$(123),(1234)$
	$(1234),(123),(34)$
A_5	$(12345),(124)$
	$(12345),(124),(23)(45)$

Table 4: The groups and their minimal generating systems which $\xi(\text{Cay}(G,S)) = 1$.

Group	Minimal generating system
$\mathbb{Z}_2 \times \mathbb{Z}_n$ where $n \geqslant 3$	$(1,0),(0,1)$
D_3	$(123),(12),(23)$
D_4	$(1234),(13)$
D_n	$(12\ldots n),(12)$
$\mathbb{Z}_2 \times D_n$	$(1,e),(0,(12)),(0,(13))$
A_4	$(123),(12)(34)$
$\mathbb{Z}_2 \times A_4$	$(0,(123)),(1,(12)(34))$
S_4	$(123),(34)$ $(12),(23),(34)$
$\mathbb{Z}_2 \times S_4$	$(1,(12)),(0,(23),(0,(34))$
A_5	$(124),(23)(45)$ $(12345),(23)(45)$ $(12345),(124)$
$\mathbb{Z}_2 \times A_5$	$(1,(12)(35)),(1,(24)(35)),(1,(23)(45))$

Table 5: The groups and their minimal generating systems which $\xi(\text{Cay}(G,S)) = 2$.

$S = \{(1,e),(0,(12)),(0,(13))\}$. In these cases, the graph Cay(G,S) is a prism and so they have H_3 as a subgraph. Hence $\xi(L(\text{Cay}(G,S)) = 2$.

Case 3. Assume that $G \cong D_3$ and $S = \{(123),(12),(23)\}$ be the minimal generating set of it. By Figure 2, the Cayley graph Cay(G,S) has H_4 as a subgraph which implies that $\xi(L(\text{Cay}(G,S)) = 2$.

Case 4. Suppose that $G \cong A_4$ and $S = \{(123),(12)(34)\}$. By Figure 3, the Cayley graph of this group has H_3 as a subgraph. Hence $\xi(L(\text{Cay}(G,S)) = 2$.

Case 5. $G \cong A_4$ and $S = \{(123),(234)\}$. By Figure 4, the Cayley graph of this group has H_2 as a subgraph. Hence $\xi(L(\text{Cay}(G,S)) = 2$.

Case 6. $G \cong \mathbb{Z}_2 \times A_4$ with $S = \{(0,(123)),(1,(12)(34))\}$ and $G \cong S_4$ with $S = \{(123),(34)\}$. By Figure 5, the Cayley graph of this group has H_2 as a subgraph. Hence $\xi(L(\text{Cay}(G,S)) = 2$.

Case 7. $G \cong S_4$ with $S = \{(12),(23),(34)\}$ or $S = \{(12),(1234)\}$. It is easy to see that the Cayley graph Cay(G,S) has H_3 as a subgraph (Figure 6). Hence $\xi(L(\text{Cay}(G,S)) = 2$.

Case 8. $G \cong \mathbb{Z}_2 \times S_4$ with $S = \{(1,(12)),(0,(23)),(0,(34))\}$. Since the Cayley graph Cay(G,S) has a subgraph isomorphic to H_3 (Figure 7). Therefore $\xi(\text{Cay}(G,S)) = 2$.

Case 9. $G \cong A_5$ with $S = \{(124),(23)(45)\}$. By Figure 8, it is easy to see that the Cayley graph Cay(G,S) has a subgraph which is isomorphic to H_3. Thus $\xi(\text{Cay}(G,S)) = 2$.

Case 10. $G \cong A_5$ with $S = \{(12345),(23)(45)\}$. It is easy to see that the Cayley graph Cay(G,S) has H_3 as a subgraph (Figure 9). Hence $\xi(\text{Cay}(G,S)) = 2$.

Case 11. $G \cong \mathbb{Z}_2 \times A_5$ with $S = \{(1,(12)(35)),(1,(24)(35)),(1,(23)(45))\}$. By Figure 10, the Cayley graph Cay(G,S) has a subgraph isomorphic to H_3 which implies that $\xi(\text{Cay}(G,S)) = 2$

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Group	Minimal generating systems
$\mathbb{Z}_n$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0), (0,1)$
D_n	$(12), (13)$

Table 6: The groups and their minimal generating systems giving outerplanar Cayley graphs.

3. Outerplanar index of Cayley graphs

In this section, we study the outerplanar index of the Cayley graphs of finite groups. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. At first, we deal with the investigation of when Cayley graphs of finite groups are outerplanar.

Lemma 3.1. *The groups and minimal generating systems in Table 6 are exactly those pairs having an outerplanar Cayley graph.*

Proof. It is well known that every outerplanar graph is planar. So, we must only check planar groups. By Maschke's Theorem, we must consider the groups and minimal generating sets which were stated in Table 1. If $G \cong \mathbb{Z}_n$ with $S = \{1\}$, $G \cong D_n$ with $S = \{(12), (13)\}$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $S = \{(1,0), (0,1)\}$, since the Cayley graphs of these groups and their generating sets are cycles, we can conclude Cay(G,S) is outerplanar. For other groups and their minimal generating sets of Table 1, it is easy to see that their Cayley graphs have a subgraph homeomorphic to $K_{2,3}$ which implies that Cay(G,S) is not outerplanar. ■

Recall that the outerplanar index of a graph G, which is denoted by $\zeta(G)$, is the smallest integer k such that the kth iterated line graph of G is non-outerplanar. If $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$. In [5], the authors gave a full characterization of all graphs with respect to their outerplanar index which is stated in the following theorem.

Theorem 3.2. *Let G be a connected graph. Then:*

(i) $\zeta(G) = 0$ *if and only if G is non-outerplanar.*

(ii) $\zeta(G) = \infty$ *if and only if G is a path, a cycle, or* $K_{1,3}$.

(iii) $\zeta(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to* $K_{1,4}$ *or* $K_1 + P_3$ *in Figure 12.*

(iv) $\zeta(G) = 2$ *if and only if* $L(G)$ *is planar and G has a subgraph isomorphic to one of the graphs* G_2 *and* G_3 *in Figure 12.*

(v) $\zeta(G) = 3$ *if and only if* $G \in I(d_1, d_2, \ldots, d_t)$ *where* $d_i \geqslant 2$ *for* $i = 2, \ldots, t-1$, *and* $d_1 \geqslant 1$ *(Figure 12).*

Theorem 3.3. *Let G be a finite group. Then:*

(i) $\zeta(G) = \infty$ *if and only if G and S are as in Table 6.*

(ii) $\zeta(Cay(G,S)) = 0$ *otherwise.*

Proof. It follows from Lemma 3.1 and Theorem 3.2. ■

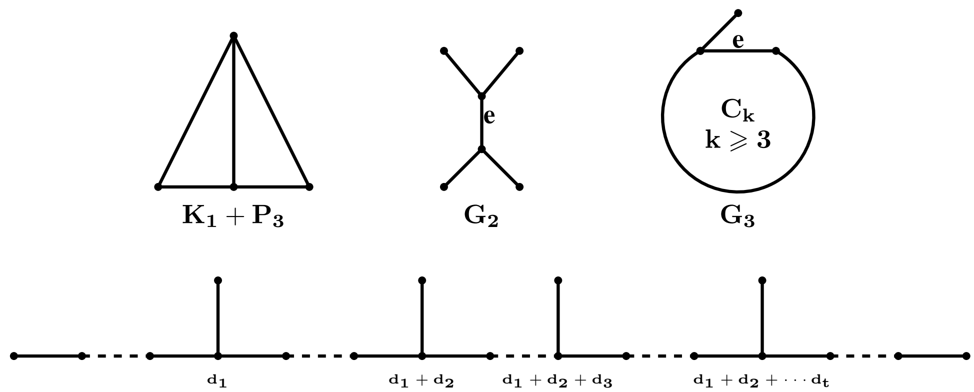

Figure 12: Figures of Theorem 3.2

4. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

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