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Oscillation Criteria of Third Order Nonlinear Neutral Difference Equations

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Abstract

In this paper we consider the third order nonlinear neutral difference equation of the form

$$\Delta(r_n(\Delta^2(x_n \pm p_n x_{\sigma(n)}))^{\alpha}) + f(n,\ x_{\tau(n)}) = 0,$$

we establish some sufficient conditions which ensure that every solution of this equation are either oscillatory or converges to zero. Examples are provided to illustrate the main results.

Keywords: Third order, oscillation, neutral difference equations.

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1 Introduction

In this paper, we establish oscillation criteria for third order nonlinear neutral difference equation of the form

$$\Delta\Big(r_n(\Delta^2(x_n \pm p_n x_{\sigma(n)}))^{\alpha}\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0 \tag{1}$$

where $\mathbb{N}_0 = \{n_0,\ n_0+1,\ n_0+2,\ \ldots\}$, and n_0 is a nonnegative integer subject to the following conditions:

(C_1) $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n^{1/\alpha}} = \infty$ and α is a ratio of odd positive integers;

(C_2) $\{p_n\}$ is a nonnegative real sequence with $-\mu \leq p_n \leq 1$ for $\mu \in (0,1)$;

(C_3) $\{\sigma(n)\}$ is a nonnegative sequence of integers with $\sigma(n) \leq n$ such that $\lim_{n\to\infty} \sigma(n) = \infty$;

(C_4) $\{\tau(n)\}$ is a nonnegative sequence of integers with $\tau(n) \leq n$ such that $\lim_{n\to\infty} \tau(n) = \infty$;

(C_5) $f : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$ and there is a nonnegative real sequence $\{q_n\}$ such that $\frac{f(n,\ u)}{u^{\alpha}} \geq Lq_n$, for $u \neq 0$ where $L > 0$.

By a solution of equation (1) we mean a real sequence $\{x_n\}$ and satisfying equation (1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x_n\}$ of equation (1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $N \in \mathbb{N}_0$. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, much research has been done on the oscillatory behavior of solutions of third order difference equations, see for example ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) and the references cited therein.

In ([13], [14]), the authors consider the following third order neutral difference equations of the form

$$\Delta(r_n(\Delta^2(x_n \pm p_n x_{n-\sigma}))^{\alpha}) + q_n x_{n+1-\tau}^{\alpha} = 0, \tag{2}$$

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and

$$\Delta(r_n(\Delta^2(x_n \pm p_n h(x_{n-\sigma})))^\alpha) + q_n f(x_{n+1-\tau}) = 0, \tag{3}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equations (2) and (3).
In [12], the authors studied the following third order difference equation

$$\Delta(c_n \Delta(d_n \Delta(x_n + p_n x_{n-k}))) + q_n f(x_{n-m}) = e_n \tag{4}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equation (4).
In [15], the authors considered the following third order difference equation

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-\sigma}))^\alpha) + q_n x_{n-\tau}^\alpha = 0 \tag{5}$$

and established some criteria for the oscillation and asymptotic behavior of all solutions of equation (5).
The oscillatory properties of oscillation of equation (1) was studied by the authors in [7], when $p_n \equiv 0$. Following this trend, in this paper, we establish some new sufficient conditions for the oscillation of all solutions of equation (1). In Section 2, we present some oscillation theorems and in Section 3, we provide examples to illustrate the main results.

2 Oscillation Theorems

First we consider the following difference equation

$$\Delta\Big(r_n(\Delta^2(x_n + p_n x_{\sigma(n)}))^\alpha\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0, \tag{6}$$

and establish some sufficient conditions for the oscillation and asymptotic behavior of its solutions. We begin with the following lemma.

Lemma 2.1. *Let $\{x_n\}$ be a positive solution of equation* (6)*, then the corresponding function $z_n = x_n + p_n x_{\sigma(n)}$ satisfies only of the following two cases:*

(I) $z_n > 0,\ \ \Delta z_n > 0,\ \ \Delta^2 z_n > 0;$

(II) $z_n > 0,\ \ \Delta z_n < 0,\ \ \Delta^2 z_n > 0$

for $n \geq n_1 \in \mathbb{N}_0$, where n_1 is sufficiently large.

Proof. The proof can be found in [13, 14], and hence the details are omitted. □

Lemma 2.2. *Let $\{x_n\}$ be a positive solution of equation* (6)*, and let the corresponding function $\{z_n\}$ satisfies the Case (II) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \Big[\frac{1}{r_s} \sum_{t=s}^{\infty} q_t\Big]^{1/\alpha} = \infty, \tag{7}$$

then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0$.

Proof. The proof is similar to that of Lemma 2.2 in [13], and hence the details are omitted. □

Before stating the next lemma, we define

$$A_n = \sum_{s=n_0}^{\infty} r_s^{-1/\alpha},$$
$$Q_n = (1 - p_{\tau(n)})^\alpha L q_n,$$

and

$$R_n = \sum_{s=n_0}^{n-1} Q_s \ \text{ for all } \ n \in \mathbb{N}_0.$$

Lemma 2.3. *Let $\{x_n\}$ be a positive solution of equation* (6) *and the corresponding z_n satisfies Case(I) of Lemma 2.1. Then there exists a positive real sequence $\{w_n\}$ such that*

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha}, \tag{8}$$

$$\lim_{n\to\infty} \sup[w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}] \leq c, \tag{9}$$

for some constant $c > 0$, and

$$\sum_{n=n_0}^{\infty} Q_n < \infty, \quad \sum_{n=n_0}^{\infty} A_{\tau(n)} R_{n+1}^{1+1/\alpha} < \infty. \tag{10}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (6). Assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \geq n_1 \geq n_0$. Then $z_n > x_n > 0$ and satisfies Case(I) of Lemma 2.1, for all $n \geq N \geq n_1$. From (6), we have

$$\begin{aligned} \Delta(r_n(\Delta z_n)^\alpha) &\leq -f(n, x_{\tau(n)}) \\ &\leq -x_{\tau(n)}^\alpha L q_n, \quad n \geq n_1. \end{aligned} \tag{11}$$

From the monotone nature of z_n, we have

$$x_n = z_n - p_n x_{\sigma(n)}$$

or

$$x_{\tau(n)} \geq (1 - p_{\tau(n)}) z_{\tau(n)}. \tag{12}$$

From (11) and (12), we have

$$\Delta(r_n(\Delta^2 z_n)^\alpha) \leq -(1 - p_{\tau(n)}) z_{\tau(n)}^\alpha L q_n$$

or

$$\frac{\Delta(r_n(\Delta^2 z_n)^\alpha)}{z_{\tau(n)}^\alpha} \leq -(1 - p_{\tau(n)}) L q_n. \tag{13}$$

Define

$$w_n = \frac{r_n(\Delta^2 z_n)^\alpha}{z_{\tau(n)}^\alpha}. \tag{14}$$

Then $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n = \frac{\Delta(r_n(\Delta^2 z_n)^\alpha)}{z_{\tau(n)}^\alpha} - \frac{r_{n+1}(\Delta^2 z_{n+1})^\alpha}{z_{\tau(n)}^\alpha \, z_{\tau(n+1)}^\alpha} \Delta(z_{\tau(n)}^\alpha).$$

Using (13) and (14) in the last inequality, we obtain

$$\Delta w_n \leq -(1 - p_{\tau(n)})^\alpha L q_n - w_{n+1} \frac{\Delta(z_{\tau(n)}^\alpha)}{z_{\tau(n)}^\alpha}. \tag{15}$$

By Mean Value Theorem

$$\Delta z_{\tau(n)}^\alpha = \alpha t^{\alpha-1} \Delta z_{\tau(n)},$$

where $z_{\tau(n)} \leq t \leq z_{\tau(n+1)}$. Since $\alpha \geq 1$, we have

$$\Delta z_{\tau(n)}^\alpha \geq \alpha z_{\tau(n)}^{\alpha-1} \Delta z_{\tau(n)}. \tag{16}$$

Using (16) in the inequality (15), we obtain

$$\Delta w_n \leq -Q_n - \alpha w_{n+1} \frac{\Delta z_{\tau(n)}}{z_{\tau(n)}}. \tag{17}$$

From the monotonicity property of $\{\Delta^2 z_n\}$, we have

$$\Delta z_n = \Delta z_{n_0} + \sum_{s=n_0}^{n-1} \Delta^2 z_s \geq \sum_{s=n_0}^{n-1} \Delta^2 z_s$$

or

$$\begin{aligned}\Delta z_n &\geq \sum_{s=n_0}^{n-1} r_s^{-1/\alpha}(r_s(\Delta^2 z_s)^\alpha)^{1/\alpha}\\ &\geq (r_n(\Delta^2 z_n)^\alpha)^{1/\alpha} A_n.\end{aligned}$$

Then

$$\Delta z_{\tau(n)} \geq (r_{\tau(n)}(\Delta^2 z_{\tau(n)})^\alpha)^{1/\alpha} A_{\tau(n)}. \tag{18}$$

Using (18) in the inequality (17), we get

$$\Delta w_n \leq -Q_n - \alpha w_{n+1}^{1+1/\alpha} A_{\tau(n)}$$

or

$$\Delta w_n + Q_n + \alpha w_{n+1}^{1+1/\alpha} A_{\tau(n)} \leq 0,\ n \geq N. \tag{19}$$

Summing the last inequality from N to $n-1$, we have

$$w_n \leq w_N - \sum_{s=N}^{n-1} Q_s - \sum_{s=N}^{n-1} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \text{ for } n \geq N. \tag{20}$$

We claim that $\sum_{n=N}^{\infty} Q_n < \infty$ for all $n \geq N$. Otherwise from the inequality (21), we obtain

$$w_n \leq w_N - \sum_{s=N}^{n-1} Q_s,$$

and letting limit $n \to \infty$ we obatin $w_n \to -\infty$, which contradicts the positivity of w_n. Similarly we can show that

$$\sum_{s=N}^{\infty} A_{\tau(s)} w_{s+1}^{1+1/\alpha} < \infty. \tag{21}$$

Now, letting limit as $n \to \infty$ in (20) we have

$$w_\infty - w_N + \sum_{s=N}^{\infty} Q_s + \sum_{s=N}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \leq 0$$

or

$$w_n \geq R_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha} \quad \text{for } n \geq N. \tag{22}$$

Since $Q_n > 0$ and $w_n > 0$ for $n \geq N$, we have from (19) that $\Delta w_n < 0$ and $\lim_{n\to\infty} w_n = M$, for some constant $M > 0$. Now from (19), we have

$$\Delta w_n \leq -\alpha A_{\tau(n)} w_{n+1}^{1+1/\alpha}$$

or

$$-\frac{\Delta w_n}{w_{n+1}^{1+1/\alpha}} \geq \alpha A_{\tau(n)}$$

or

$$\frac{w_n}{\alpha w_{n+1}^{1+1/\alpha}} \geq A_{\tau(n)}.$$

Taking limit supreme, we obtain

$$M \geq \lim_{n\to\infty} \sup(w_{n+1}^{1+1/\alpha} A_{\tau(n)})$$

or

$$\lim_{n\to\infty} \sup(w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}) \leq c,$$

for some constant $c > 0$. This completes the proof. □

Theorem 2.1. *Assume that*

$$\lim_{n\to\infty}\inf\frac{1}{R_n}\sum_{s=n}^{\infty}P_sR_{s+1}^{1+1/\alpha}>\frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}},\tag{23}$$

where $P_n=\alpha A_{\tau(n)}$ then every solution of equation (6) *is either oscillatory or converges to zero as $n\to\infty$.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6). Without loss of generality we may assume that $x_n>0$, $x_{\sigma(n)}>0$ and $x_{\tau(n)}>0$ for all $n\geq n_1\geq n_0$ and the corresponding $\{z_n\}$ satisfies two cases of Lemma 2.1.

Case(I). Let $\{z_n\}$ satisfies Case (I) of Lemma 2.1. From Lemma 2.3, we obtain (8), then

$$\begin{aligned}\frac{w_n}{R_n} &\geq 1+\frac{1}{R_n}\sum_{s=n}^{\infty}\alpha A_{\tau(s)}w_{s+1}^{1+1/\alpha}\\ &\geq 1+\frac{1}{R_n}\sum_{s=n}^{\infty}P_sR_{s+1}^{1+1/\alpha}\left(\frac{w_{s+1}}{R_{s+1}}\right)^{1+1/\alpha}.\end{aligned}\tag{24}$$

From the assumption of the theorem, there exists a $\beta>\frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}$,

$$\lim_{n\to\infty}\inf\frac{1}{R_n}\sum_{s=n}^{\infty}P_sR_{s+1}^{1+1/\alpha}>\beta\tag{25}$$

and let

$$\lambda=\inf_{n\geq n_0}\frac{w_n}{R_n},\tag{26}$$

then $\lambda\geq 1$. Using (25) and (26) in the inequality (24) we have

$$\lambda\geq 1+\beta\lambda^{1+1/\alpha}.$$

Therefore

$$\lambda-\beta\lambda^{1+1/\alpha}\leq\frac{\alpha}{(\alpha+1)^{\alpha}}\frac{1}{\beta^{\alpha}}.$$

Then, we get $\beta\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)/\alpha}}$, which is a contradicts to our assumption.

If $\{z_n\}$ satisfies Case(II) of Lemma 2.1, then by the condition (7) we have $\lim_{n\to\infty}x_n=0$. This completes the proof. □

Theorem 2.2. *Assume that*

$$\lim_{n\to\infty}\sup\left[A_{\tau(n)}^{\alpha/(\alpha+1)}\left(R_{n+1}+\sum_{s=n+1}^{\infty}\alpha A_{\tau(s)}R_{s+1}^{1+1/\alpha}\right)\right]=\infty\tag{27}$$

then every solution of equation (6) *is either oscillatory or converges to zero as $n\to\infty$.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (6). Without loss of generality we may assume that $x_n>0$, $x_{\sigma(n)}>0$ and $x_{\tau(n)}>0$ for all $n\geq n_1\geq n_0$ and the corresponding $\{z_n\}$ satisfies two cases of Lemma 2.1.

Case(I). Let $\{z_n\}$ satisfies Case (I) of Lemma 2.1. From Lemma 2.3, we obtain (8), then

$$w_n\geq R_n+\sum_{s=n}^{\infty}\alpha A_{\tau(s)}w_{s+1}^{1+1/\alpha}.$$

Since $w_n\geq R_n$, we have

$$w_n\geq R_n+\sum_{s=n}^{\infty}\alpha A_{\tau(s)}R_{s+1}^{1+1/\alpha}.$$

Using this in (9), we have

$$\lim_{n\to\infty}\sup\left[A_{\tau(n)}^{\alpha/(\alpha+1)}\left(R_{n+1}+\sum_{s=n+1}^{\infty}\alpha A_{\tau(s)}R_{s+1}^{1+1/\alpha}\right)\right]\leq c,$$

which is a contradiction. If $\{z_n\}$ satisfies Case(II) of Lemma 2.1, then by the condition (7) we have $\lim_{n\to\infty}x_n=0$. This completes the proof. □

Next, we consider the case $-\mu \leq p_n \leq 0$, and the equation (1) takes the form

$$\Delta\Big(r_n(\Delta^2(x_n - p_n x_{\sigma(n)}))^{\alpha}\Big) + f(n,\ x_{\tau(n)}) = 0, n \in \mathbb{N}_0. \tag{28}$$

Lemma 2.4. *Let $\{x_n\}$ be a positive solution of equation* (28) *and the corresponding $\{z_n\}$ satisfies Case(I) of Lemma 2.1. Then there exists a positive function $\{w_n\}$ such that*

$$w_n \geq Q_n + \sum_{s=n}^{\infty} \alpha A_{\tau(s)} w_{s+1}^{1+1/\alpha}, \tag{29}$$

$$\lim_{n\to\infty} \sup[w_{n+1} A_{\tau(n)}^{\alpha/(\alpha+1)}] \leq c, \tag{30}$$

for some constant $c > 0$, and

$$Q_n < \infty, \quad \sum_{s=n}^{\infty} A_{\tau(s)} Q_{s+1}^{1+1/\alpha} < \infty. \tag{31}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (28). Assume that $x_n > 0$, $x_{\sigma(n)} > 0$ and $x_{\tau(n)} > 0$ for all $n \geq n_1 \geq n_0$. Then $z_n > x_n > 0$ and satisfies Case(I) of Lemma 2.1, for all $n \geq N \geq n_1$. We have (11)

$$\Delta(r_n(\Delta z_n)^{\alpha}) \leq -x_{\tau(n)}^{\alpha} L q_n, \quad n \geq n_1. \tag{32}$$

We have two possible cases for z_n:

(i) $z_n > 0$

(ii) $z_n < 0$.

Case (i). $z_n > 0$, the proof is similar to that of Lemma 2.3 and hence the details are omitted.
Case (ii). $z_n < 0$ eventually for all $n \geq n_2 \geq n_1 \geq n_0$, then we have two cases for x_n:

(a) x_n is unbounded,

(b) x_n is bounded.

Case (a). Assume that x_n is unbounded, then

$$x_n = z_n - p_n x_{\sigma(n)} < -p_n x_{\sigma(n)} < x_{\sigma(n)}. \tag{33}$$

Since $\{x_n\}$ is unbounded, we can choose a sequence $\{x_{n_k}\}$ satisfying $\lim_{k\to\infty} x_k = \infty$ from which $\lim_{k\to\infty} x_{N_k} = \infty$ and $\max x_n = x_{N_n}$ by choosing N large such that $\sigma(N_k) > N_1$ for all $N_k > n_2$. Thus $\max x_n = x_{N_n}$. This contradicts with (33).
Case (b). Assume that $\{x_n\}$ is bounded, and we show that $x_n \to 0$ as $n \to \infty$. Since

$$\lim_{n\to\infty} \sup z_n \leq 0,$$

then we have

$$\begin{aligned}
\lim_{n\to\infty} \sup(x_n + p_n x_{\sigma(n)}) &\leq 0\\
\lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \sup p_n x_{\sigma(n)} &\leq 0\\
\lim_{n\to\infty} \sup x_n - \mu \lim_{n\to\infty} \sup x_{\sigma(n)} &\leq 0\\
(1-\mu) \lim_{n\to\infty} \sup x_n &\leq 0.
\end{aligned}$$

This shows that $x_n \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.3. *Assume that*

$$\lim_{n\to\infty} \inf \frac{1}{Q_n} \sum_{s=n}^{\infty} P_s Q_{s+1}^{1+1/\alpha} > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{34}$$

where $P_n = \alpha A_{\tau(n)}$ then every solution of equation (28) *is either oscillatory or converges to zero as $n \to \infty$.*

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted. □

Theorem 2.4. *Assume that*

$$\lim_{n\to\infty} \sup \left[A_{\tau(n)}^{\alpha/(\alpha+1)} \left(Q_{n+1} + \sum_{s=n+1}^{\infty} \alpha A_{\tau(s)} Q_{s+1}^{1+1/\alpha}\right)\right] = \infty \tag{35}$$

then every solution of equation (28) *is either oscillatory or converges to zero as* $n \to \infty$.

Proof. The proof is similar to that of Theorem 2.2 and hence the details are omitted. □

3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the third order difference equation*

$$\Delta\Big(n(\Delta^2\Big(x_n + \frac{1}{2}x_{n-2}\Big))^3\Big) + \frac{1}{(n+1)(n+2)}x_{n-3}^3 = 0, \quad n \geq 1. \tag{36}$$

Here $r_n = n,\ p_n = \frac{1}{2},\ q_n = \frac{1}{(n+1)(n+2)},\ \alpha = 3\ \sigma(n) = n-2,\ \tau(n) = n-3$ *and* $L = 1$. *It is easy to see that all conditions of Theorem 2.1 are satisfied. Hence every solution of equation* (36) *is either oscillatory or converges to zero as* $n \to \infty$.

Example 3.2. *Consider the third order difference equation*

$$\Delta\Big(\frac{1}{n}\Delta^2\Big(x_n + \frac{1}{2}x_{n-2}\Big)\Big) + \frac{1}{(n+1)(n+2)}x_{n-1} = 0, \quad n \geq 1. \tag{37}$$

Here $r_n = \frac{1}{n},\ p_n = \frac{1}{2},\ q_n = \frac{1}{(n+1)(n+2)},\ \alpha = 1\ \sigma(n) = n-2,\ \tau(n) = n-1$ *and* $L = 1$. *It is easy to see that all conditions of Theorem 2.2 are satisfied. Hence every solution of equation* (37) *is either oscillatory or converges to zero as* $n \to \infty$.

Example 3.3. *Consider the third order difference equation*

$$\Delta^3\Big(x_n - \frac{1}{3}x_{n-1}\Big) + nx_{n-2} = 0, \quad n \geq 1. \tag{38}$$

Here $r_n = 1,\ p_n = \frac{1}{3},\ q_n = n,\ \alpha = 1\ \sigma(n) = n-1,\ \tau(n) = n-2$ *and* $L = 1$. *It is easy to see that all conditions of Theorem 2.3 are satisfied. Hence every solution of equation* (38) *is either oscillatory or converges to zero as* $n \to \infty$.

Example 3.4. *Consider the third order difference equation*

$$\Delta^3\Big(x_n - \frac{1}{2}x_{n-1}\Big) + 12x_{n-2} = 0, \quad n \geq 1. \tag{39}$$

Here $r_n = 1,\ p_n = \frac{1}{2},\ q_n = 12,\ \alpha = 1\ \sigma(n) = n-1,\ \tau(n) = n-2$ *and* $L = 1$. *It is easy to see that all conditions of Theorem 2.4 are satisfied. Hence every solution of equation* (39) *is either oscillatory or converges to zero as* $n \to \infty$. *In fact* $\{x_n\} = \{(-1)^n\}$ *is one such oscillatory solution of equation* (39) *is oscillatory or converging to zero.*

We conclude this paper with the following remark.

Remark 3.1. *It would be interesting to extend the results of this paper to the equation* (1) *when* $\sum_{n=n_0}^{\infty} \frac{1}{r_n^{1/\alpha}} < \infty$.

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