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# Line gracefulness in the context of switching of a vertex

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**Abstract**

We investigate line graceful labeling of graphs obtained by switching of vertex operation.

*Keywords:* Edge graceful labeling, line graceful labeling, graceful labeing, switching of vertex.

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## 1 Introduction

Labeling of discrete structures is a one of the potential area of research due to its potential applications. The optimal linear arrangement concern to network problems in electrical engineering and placement problems in production engineering can be formalized as a graph labeling problems as stated by Yegnanaryanan and Vaidhyanathan [13]. A dynamic survey on different graph labeling schemes with an extensive bibliography can be found in Gallian [2].
In this paper, the term "graph" means finite, connected, undirected and simple graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notation we refer to Balakrishnan and Ranganathan [1].

**Definition 1.1.** *A* graph labeling *is an assignment of numbers to the vertices or edges or both subject to certain condition(s).*

**Definition 1.2.** *A function $f$ is called* graceful labeling *of graph if $f : V(G) \to \{0, 1, 2, 3, ..., q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e = uv) = |f(x) - f(y)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.*

Most of the graph labeling techniques trace their origin with graceful labeling which was introduced independently by Rosa [7] and Golomb [4]. A variant of graceful labeling termed as edge graceful labeling is introduced by Lo [6].

**Definition 1.3.** *A graph $G = (V(G), E(G))$ is said to be* edge graceful *if there exists a bijection $f : E(G) \to \{1, 2, 3, ..., q\}$ such that the induced mapping $f^* : V(G) \to \{0, 1, ..., p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod{p}$ is bijection.*

Lo [6] derived a necessary condition for a graph to be edge graceful and also investigate edge graceful labeling of many graph families. Wilson and Risking [12] proved that the cartesian product of any number of odd cycle is edge graceful. All trees of odd order are edge graceful was conjunctured by Lee [5]. Shiu, Lee and Schaffer [8] investigated the edge gracefulness of multigraphs. Gnanajothi [3] introduced and studied line graceful labeling in her Ph.D. thesis which is little weaker than edge graceful labeling.

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**Definition 1.4.** *A mapping $f : E(G) \rightarrow \{0, 1, 2, ..., p\}$ is called* line graceful *of graph with $p$ vertices, if induced function $f^* : V(G) \rightarrow \{0, 1, 2, ..., p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod{p}$ is bijective.*

**Definition 1.5.** *The triangular snake $T_n$ is obtained from the path $P_n$ by replacing every edge of a path by a triangle $C_3$ .*

**Definition 1.6.** *A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G$.*

**Definition 1.7.** *The* helm *$H_n$ is the graph obtained from a wheel $W_n$ by attaching a pendant edge to every rim vertex.*

**Definition 1.8.** *The* fan *$f_n$ is a graph on $n+1$ vertices obtained by joining all the vertices of $P_n$ to a new vertex called the center.*

Vaidya and Kothari [9, 10, 11] have investigated many results on line gracefulness of graphs in various contexts while this paper is focus on line gracefulness on the graph obtained by switching of a vertex.

## 2 Main results

**Proposition 2.1.** *[3] If the graph is line graceful then its order is not congruent to* $2 \pmod 4$.

**Theorem 2.1.** *Switching of a pendant vertex in path $P_n$ is line graceful except $n \equiv 2 \pmod 4$.*

*Proof.* Let $v_1, v_2, \ldots, v_n$ be vertices of path $P_n$. Let $G_v$ be the graph obtained by switching pendant vertex $v$ of $P_n$. Without loss of generality let the switched vertex be $v_n$. We note that $|V(G_v)| = n$ and $|E(G_v)| = 2n - 4$. Define edge labeling $f : E(G_v) \rightarrow \{0, 1, \ldots, n-1\}$ as follows.
**Case 1:** $n \equiv 0 \pmod 4$
for odd $i$

$$f(v_iv_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \frac{n}{2} \\ \frac{i+1}{2} + 1 & \text{for } \frac{n}{2} \leq i \leq n-3 \end{cases}$$

for even $i$

$$\begin{aligned} f(v_iv_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \leq i \leq n-3 \\ f(v_{n-2}v_{n-1}) &= \tfrac{n}{2} + 2 \\ f(v_nv_i) &= 0 && \text{for } 1 \leq i \leq n-2 \end{aligned}$$

**Case 2:** $n \equiv 1 \pmod 4$
for odd $i$

$$f(v_iv_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ \frac{i+3}{2} & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-3 \end{cases}$$

for even $i$

$$\begin{aligned} f(v_iv_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \leq i \leq n-3 \\ f(v_{n-2}v_{n-1}) &= 2 \\ f(v_nv_i) &= 0 && \text{for } 1 \leq i \leq n-2 \end{aligned}$$

**Case 3:** $n \equiv 3 \pmod 4$
for odd $i$

$$f(v_iv_{i+1}) = \begin{cases} \frac{i+1}{2} & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ \frac{i+3}{2} & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq n-2 \end{cases}$$

for even $i$

$$\begin{aligned} f(v_i v_{i+1}) &= \tfrac{i+2}{2} && \text{for } 2 \le i \le n-3 \\ f(v_{n-2} v_{n-1}) &= \left\lfloor \tfrac{n}{2} \right\rfloor + 3 && \\ f(v_n v_i) &= 0 && \text{for } 1 \le i \le n-2 \end{aligned}$$

**Case 4:** $n \equiv 2\,(\mathrm{mod}\ 4)$
In this case $|V(G_v)| = n \equiv 2\,(\mathrm{mod}\ 4)$.
Then according to Proposition 2.1 $G_v$ is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_v) \rightarrow \{0, 1, ..., n-1\}$ such that $f^*(v) = \sum_{e \in E(G_v)} f(e) \pmod{(n)}$ for $n \equiv 0, 1, 3\,(\mathrm{mod}\ 4)$. Hence we proved that graph $G_v$ obtained from switching of pendant vertex in path $P_n$ is line graceful except $n \equiv 2\,(\mathrm{mod}\ 4)$.

□

**Illustration 2.1.** *Switching of vertex $v_8$ in path $P_8$ and its line graceful labeling is shown in figure* 2.

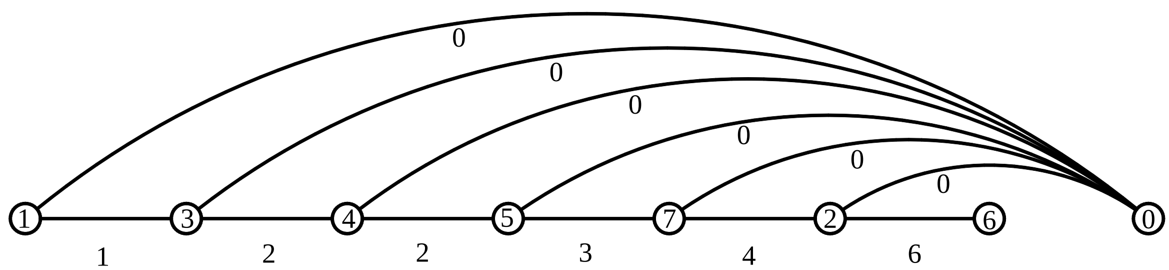


*figure* 2

**Theorem 2.2.** *Switching of vertex in cycle $C_n$ is line graceful except $n \equiv 2\,(\mathrm{mod}\ 4)$.*

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $G_{v_1}$ be the graph obtained by switching of vertex $v_1$ of cycle $C_n$. Here without loss of generality, we have switched the vertex $v_1$. Note that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$. Define edge labeling $f : E(G_{v_1}) \rightarrow \{0, 1, \ldots, n-1\}$ as follows.
**Case 1:** $n \equiv 0\,(\mathrm{mod}\ 4)$

$$f(v_1 v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

<u>for odd $i$</u>

$$f(v_i v_{i+1}) = \tfrac{i+1}{2} \quad \text{for } 3 \le i \le n-3$$

<u>for even $i$</u>

$$f(v_i v_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \le i \le \frac{n}{2} \\ \frac{i}{2} + 1 & \text{for } \frac{n}{2} < i \le n-2 \end{cases}$$

$$f(v_{n-1} v_n) = f(v_{n-2} v_{n-1}) + 2$$

**Case 2:** $n \equiv 1\,(\mathrm{mod}\ 4)$

$$f(v_1 v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

<u>for odd $i$</u>

$$f(v_i v_{i+1}) = \tfrac{i+1}{2} \quad \text{for } 3 \le i \le n-2$$

<u>for even $i$</u>

$$f(v_i v_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ \frac{i}{2} + 1 & \text{for } \left\lceil \frac{n}{2} \right\rceil \le i \le n-3 \end{cases}$$

$$f(v_{n-1} v_n) = 2$$

**Case 3:** $n \equiv 3\,(\mathrm{mod}\ 4)$

$$f(v_1 v_i) = 0 \quad \text{for } 3 \le i \le n-1$$

for odd $i$

$$f(v_iv_{i+1}) = \frac{i+1}{2} \quad \text{for } 3 \leq i \leq n-2$$

for even $i$

$$f(v_iv_{i+1}) = \begin{cases} \frac{i}{2} & \text{for } 2 \leq i \leq \lceil \frac{n}{2} \rceil \\ \frac{i}{2}+1 & \text{for } \lceil \frac{n}{2} \rceil < i \leq n-3 \end{cases}$$

$$f(v_{n-1}v_n) = \lceil \frac{n}{2} \rceil + 2$$

**Case 4:** $n \equiv 2\,(\text{mod } 4)$
In this case $|V(G_{v_1})| = n \equiv 2\,(\text{mod } 4)$.
Then according to Proposition 2.1 $G_{v_1}$ is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \rightarrow \{0, 1, ..., n-1\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n)}$ for $n \equiv 0, 1, 3\,(\text{mod } 4)$. Hence we proved that graph $G_{v_1}$ obtained from switching of vertex $v_1$ in cycle $C_n$ is line graceful except $n \equiv 2\,(\text{mod } 4)$. ☐

**Illustration 2.2.** *Switching of vertex $v_1$ in cycle $C_9$ and its line graceful labeling is shown in figure* 3.

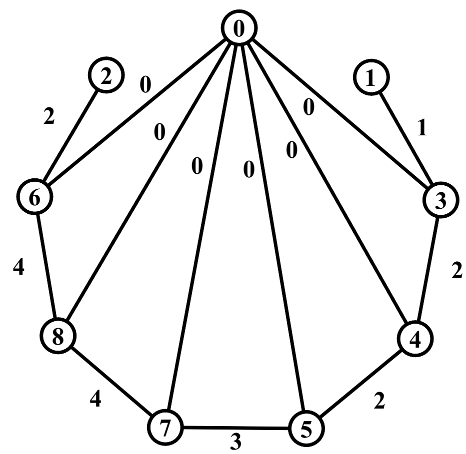


*figure* 3

**Theorem 2.3.** *For $n > 3$, switching of a rim vertex in wheel $W_n$ is line graceful except $n \equiv 1\,(\text{mod } 4)$.*

*Proof.* Let $v$ be a apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices of $W_n$ and $G_{v_1}$ be the graph obtained by switching a rim vertex $v_1$ of $W_n$. Here without loss of generality, we have switched vertex $v_1$. Observe that $|V(G_{v_1})| = n+1$ and $|E(G_{v_1})| = 3n-5$. Define edge labeling $f : E(G_{v_1}) \rightarrow \{0, 1, \ldots, n\}$. as follows.
**Case 1:** $n = 4$
The graph $G_{v_1}$ and its line graceful labeling is shown in figure 4.

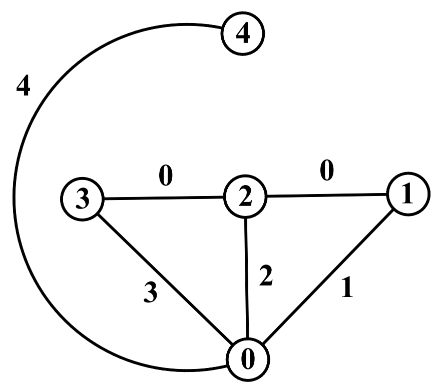


figure 4

**Case 2:** $n = 6$
The graph $G_{v_1}$ and its line graceful labeling is shown in figure 5.

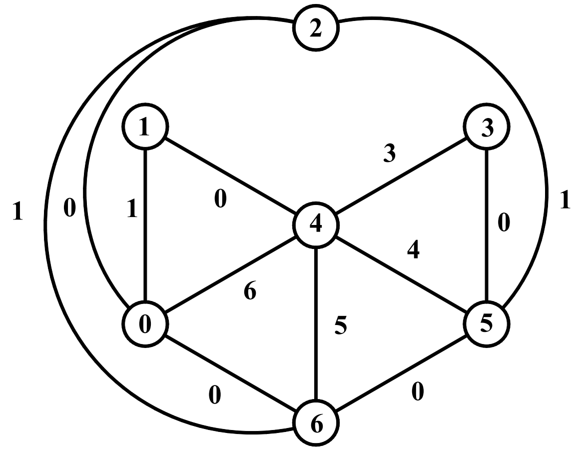


figure 5

**Case 3:** $n \equiv 0, 2 \pmod 4$

$$\begin{aligned} f(vv_i) &= i+1 \quad \text{for } 2 \le i \le n \\ f(v_iv_{i+1}) &= \begin{cases} 0 & \text{for } 2 \le i \le n-2 \\ 1 & \text{for } i = n-1 \end{cases} \\ f(v_1v_i) &= \begin{cases} 0 & \text{for } 3 \le i \le n-4,\ i = n-1 \\ 1 & n-3 \le i \le n-2 \end{cases} \end{aligned}$$

**Case 4:** $n \equiv 3 \pmod 4$

$$\begin{aligned} f(vv_i) &= i-1 \quad \text{for } 2 \le i \le n \\ f(v_1v_i) &= 0 \quad\quad \text{for } 3 \le i \le n-1 \end{aligned}$$

for $2 \le i \le \lceil \frac{n}{2} \rceil + 1$

$$f(v_iv_{i+1}) = 0$$

for $\lceil \frac{n}{2} \rceil + 2 \le i \le n-1$

$$f(v_iv_{i+1}) = \begin{cases} 1 & \text{for even } i \\ 0 & \text{for odd } i \end{cases}$$

**Case 5:** $n \equiv 1 \pmod 4$
In this case $|V(G_{v_1})| = n + 1 \equiv 2 \pmod 4$.
Then according to Proposition 2.1 $G_{v_1}$ is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \to \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n+1)}$ for $n \equiv 0, 2, 3 \pmod 4$. Hence we proved that for $n > 3$, the graph $G_{v_1}$ obtained from switching of a rim vertex $v_1$ in wheel $W_n$ is line graceful except $n \equiv 1 \pmod 4$. □

**Illustration 2.3.** *Switching of vertex $v_1$ in cycle $W_{11}$ and its line graceful labeling is shown in figure 6.*

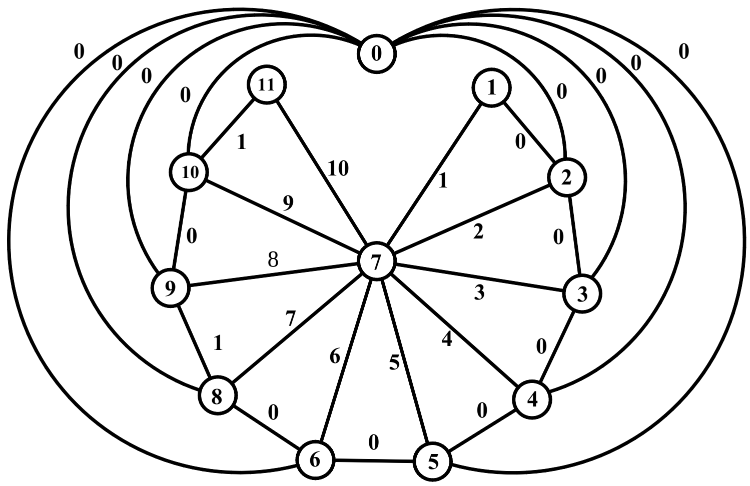

*figure* 6

**Theorem 2.4.** *Switching of apex vertex in helm $H_n$ is line graceful for all $n$.*

*Proof.* Let $v$ be a apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices and $u_1, u_2, \ldots, u_n$ be pendant vertices of helm $H_n$. $G_v$ be the graph obtained from switching apex vertex $v$ of helm. Observe that $|V(G_v)| = 2n + 1$ and $|E(G_v)| = 3n$. Define edge labeling $f : E(G_v) \to \{0, 1, \ldots, 2n\}$ as follows.

$$\begin{aligned} f(vu_i) &= n+1 \\ f(v_iv_{i+1}) &= 0 \quad \text{for } 1 \leq i \leq n-1 \\ f(v_nv_1) &= 0 \end{aligned}$$

<u>for odd $n$</u>

$$f(v_iu_i) = \left\lfloor \tfrac{n}{2} \right\rfloor + i \quad \text{for } 1 \leq i \leq n$$

<u>for even $n$</u>

$$f(v_iu_i) = \tfrac{3n}{2} + i \quad \text{for } 1 \leq i \leq n$$

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_v) \to \{0, 1, ..., 2n\}$ such that $f^*(v) = \sum_{e \in E(G_v)} f(e) \pmod{(2n+1)}$. Thus we proved that graph $G_v$ obtained by switching apex vertex of helm admits line graceful labeling for all $n$. □

**Illustration 2.4.** *Switching of apex vertex $v$ in helm $H_7$ and its line graceful labeling is shown in figure* 7.

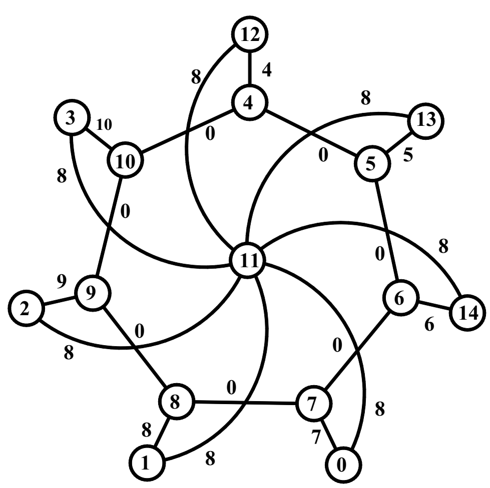

*figure* 7

**Theorem 2.5.** *Switching of vertex having degree 2 in fan $f_n$ is line graceful except $n \equiv 1 \pmod 4$.*

*Proof.* Let $v$ be the apex vertex and $v_1, v_2, \ldots, v_n$ be the vertices of $f_n$. Let $G_{v_1}$ denotes graph obtained by switching of a vertex $v_1$ having degree 2 of $f_n$. Note that $|V(G_{v_1})| = n+1$ and $|E(G_{v_1})| = 3n-5$.
We define $f : E(G_{v_1}) \to \{1, 2, \ldots, n+1\}$ as follows.
**Case 1:** $n \equiv 0, 2 \pmod 4$

$$\begin{aligned} f(vv_i) &= n \quad \text{for } 2 \le i \le n \\ f(v_i v_{i+1}) &= 0 \quad \text{for } 2 \le i \le n-1 \\ f(v_1 v_i) &= \begin{cases} i-2 & \text{for } i = 3, 4 \\ i-1 & \text{for } 5 \le i \le n \end{cases} \end{aligned}$$

**Case 2:** $n \equiv 3 \pmod 4$

$$\begin{aligned} f(vv_i) &= \left\lfloor \tfrac{n}{4} \right\rfloor \quad \text{for } 2 \le i \le n \\ f(v_i v_{i+1}) &= 0 \quad \text{for } 2 \le i \le n-1 \\ f(v_1 v_i) &= \begin{cases} i-2 & \text{for } 3 \le i \le 5 + \left\lfloor \frac{n}{4} \right\rfloor \\ i-1 & \text{for } 6 + \left\lfloor \frac{n}{4} \right\rfloor \le i \le n \end{cases} \end{aligned}$$

**Case 3:** $n \equiv 1 \pmod 4$
In this case $|V(G_{v_1})| = n+1 \equiv 2 \pmod 4$.
Then according to Proposition 2.1 $G_{v_1}$ is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(G_{v_1}) \to \{0, 1, ..., n\}$ such that $f^*(v) = \sum_{e \in E(G_{v_1})} f(e) \pmod{(n+1)}$ for $n \equiv 0, 2, 3 \pmod 4$. Hence we proved that the graph $G_{v_1}$ obtained by switching a vertex of degree 2 in fan $f_n$ is line graceful except $n \equiv 1 \pmod 4$.

□

**Illustration 2.5.** *Switching of vertex $v_1$ in fan $f_{11}$ and its line graceful labeling is shown in figure* 8.

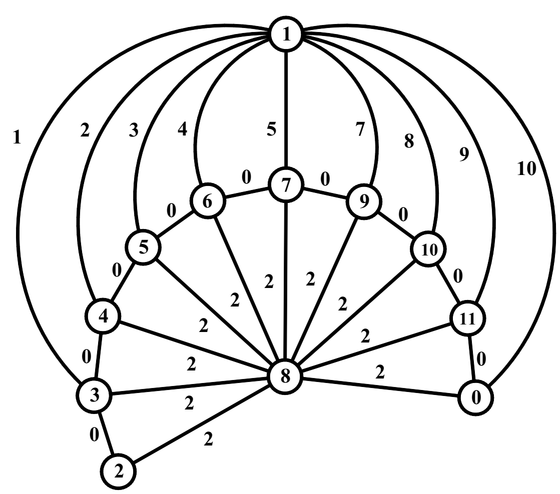


*figure* 8

# 3 Concluding Remarks

Edge gracefulness and line gracefulness of a graph are independent concepts. A graph may posses one or both of these or neither as mentioned below.

- $C_{2n+1}$ is edge graceful as well as line graceful.
- $P_n$ is neither edge graceful nor line graceful for $n \equiv 2 \pmod 4$.
- $C_{4n}$ is not edge graceful but line graceful.
- Triangular snake $T_n$ is edge graceful only for $n = 3$ while it is line graceful for all $n$.

## 4 Acknowledgment

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