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Hermite Hadamard-Fejer type inequalities for quasi convex functions via fractional integrals

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Abstract

In this paper, Hermite-Hadamard-Fejer type inequalities for quasi-convex via fractional integrals are obtained.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, quasi convex functions.

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1 Introduction

The following definition for convex functions is well know in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a,b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality. More details, one can consult ([1]-[11]).

In [3], Fejer established the following Hermite-Hadamard Fejer inequality which is the weighted generalization of Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \tag{1.2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a,b] \to \mathbb{R}$, is said quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\}, \forall x, y \in [a,b]$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$(see [10]).

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Furthermore, there exist quasi-convex functions which are not convex (see [5]).

In [8] Özdemir et. al. represented Hermite-Hadamard's inequalities for quasi-convex functions in fractional integral forms as follows:

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi convex on $[a,b]$ and $\alpha>0$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \qquad (1.3)\\ \leq\ &\frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.\end{aligned}$$

In [9] Set et. al. obtained the following lemma.

Lemma 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If $f',g\in L[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}g(a)+J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}g(b)\right]-\left[J_{\left(\frac{a+b}{2}\right)_{-}}^{\alpha}(fg)(a)+J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}(fg)(b)\right] \qquad (1.4)\\ =\ &\frac{1}{\Gamma(\alpha)}\int_a^b k(t)f'(t)\,dt\end{aligned}$$

where

$$k(t)=\begin{cases}\displaystyle\int_a^t (s-a)^{\alpha-1}g(s)ds, & t\in\left[a,\frac{a+b}{2}\right)\\ \displaystyle-\int_t^b (b-s)^{\alpha-1}g(s)ds, & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

In [11] İşcan proved the following lemma.

Lemma 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $a<b$ with $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $(a+b)/2$ then the following equality for fractional integrals holds*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\left[J_{a^{+}}^{\alpha}g(b)+J_{b^{-}}^{\alpha}g(a)\right]-\left[J_{a^{+}}^{\alpha}(fg)(b)+J_{b^{-}}^{\alpha}(fg)(a)\right] \qquad (1.5)\\ =\ &\frac{1}{\Gamma(\alpha)}\int_a^b\left[\int_a^t(b-s)^{\alpha-1}g(s)\,ds-\int_t^b(s-a)^{\alpha-1}g(s)\,ds\right]f'(t)\,dt\end{aligned}$$

with $\alpha>0$.

We give some neccessary definitions and mathematical preliminiaries of fractional calculus theory which are used throughout this paper.

Lemma 1.3. *([6],[7])For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$|a^{\alpha}-b^{\alpha}|\leq(b-a)^{\alpha}.$$

Definition 1.1. *Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^{+}}^{\alpha}f(x)$ and $J_{b^{-}}^{\alpha}f(x)$ of oder $\alpha>0$ with $\alpha\geq 0$ are defined by*

$$J_{a^{+}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\ x>a$$

and

$$J_{b^{-}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma functions by $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^{+}}^{0}f(x)=J_{b^{-}}^{0}f(x)=f(x)$.

In this paper, motivated by the recent results given in [11], [9] , we established Hermite-Hadamard-Fejer type inequalities for quasi convex functions via fractional integral.

2 Main result

Throughout this paper, let I be an interval on $\mathbb{R}$ and let $\|g\|_{[a,b],\infty} = \sup_{t\in[a,b]} g(t)$, for the continuous function $g:[a,b]\to\mathbb{R}$.

Theorem 2.3. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is quasi convex on $[a,b]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|\\ \leq\;&\frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha}(\alpha+1)\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned} \tag{2.6}$$

with $\alpha>0$.

Proof. Since is $|f'|^q$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|^q=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\sup\left\{|f'(a)|^q,|f'(b)|^q\right\} \tag{2.7}$$

Using Lemma 1.1, Power mean inequality and the quasi-convex of $|f'|^q$, it follows that

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|\\
\leq\;&\frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
\leq\;&\frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
&+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}\\
\leq\;&\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&\times\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}dt\right)^{\frac{1}{q}}+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}dt\right)^{\frac{1}{q}}\right\}\\
\leq\;&\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{\frac{1}{q}}\\
&\times\left(\|g\|_{[a,\frac{a+b}{2}],\infty}+\|g\|_{[\frac{a+b}{2},b],\infty}\right)\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\\
\leq\;&\frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)2^{\alpha}(\alpha+1)}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1}\, ds \right| dt = \int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1}\, ds \right| dt = \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)\alpha}.$$

Hence, the proof is completed.

Corollary 2.1. *If we choose $g(x)=1$ and $\alpha=1$ in the inequality (2.6), then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \le \frac{b-a}{4} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}.$$

We can state another inequality for $q>1$ as follows:

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is quasi convex on $[a,b]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} g(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} g(b) \right] - \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} (fg)(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} (fg)(b) \right] \right| \\ \le\ & \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^\alpha (\alpha p+1)^{\frac{1}{p}} \Gamma(\alpha+1)} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.1, Hölder's inequality and the quasi convexity of $|f'|^q$, it follows that

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} g(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} g(b) \right] - \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} (fg)(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} (fg)(b) \right] \right| \\ \le\ & \frac{1}{\Gamma(\alpha)} \left\{ \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s)\, ds \right| |f'(t)|\, dt \right. \\ & \left. + \int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} g(s)\, ds \right| |f'(t)|\, dt \right\} \\ \le\ & \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s)\, ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\, dt \right)^{\frac{1}{q}} \\ & + \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1} g(s)\, ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\, dt \right)^{\frac{1}{q}} \\ \le\ & \frac{\|g\|_{\infty,[a,\frac{a+b}{2}]}}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1}\, ds \right|^p \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\, dt \right)^{\frac{1}{q}} \\ & + \frac{\|g\|_{\infty,[\frac{a+b}{2},b]}}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_t^b (b-s)^{\alpha-1}\, ds \right|^p \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\, dt \right)^{\frac{1}{q}} \\ \le\ & \frac{\|g\|_\infty}{\Gamma(\alpha)} \left(\frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p} \right)^{\frac{1}{p}} \left[\left(\int_a^{\frac{a+b}{2}} \sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{a+b}{2}}^b \sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \right] \\ =\ & \frac{\|g\|_\infty (b-a)^{\alpha+1}}{2^\alpha (\alpha p+1)^{1/p} \Gamma(\alpha+1)} \left(\sup\left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

Here we use

$$\int_a^{\frac{a+b}{2}} \left|\int_a^t (s-a)^{\alpha-1} ds\right|^p dt = \int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\alpha-1} ds\right|^p dt = \frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p}$$

$$\begin{aligned}\int_a^{\frac{a+b}{2}} |f'(t)|^q dt &\leq \frac{b-a}{2}\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\\ \int_{\frac{a+b}{2}}^b |f'(t)|^q dt &\leq \frac{b-a}{2}\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}.\end{aligned}$$

Hence the inequality (2.8) is proved.

Corollary 2.2. *If we choose $g(x) = 1$ and $\alpha = 1$ in the inequality (2.8), then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is quasi convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right| \\ \leq\ & \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\sup\left\{|f'(a)|, |f'(b)|\right\}\end{aligned} \tag{2.9}$$

with $\alpha > 0$.

Proof. From Lemma 1.2, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right| |f'(t)|\,dt.\end{aligned} \tag{2.10}$$

Since $|f'|$ is quasi convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(t)| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-b}{b-a}b\right)\right| \leq \sup\left\{|f'(a)|, |f'(b)|\right\} \tag{2.11}$$

and since $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$ we write

$$\begin{aligned}\int_t^b (s-a)^{\alpha-1} g(s)\,ds &= \int_a^{a+b-t} (b-s)^{\alpha-1} g(a+b-s)\,ds \\ &= \int_a^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds.\end{aligned}$$

Then we get

$$\begin{aligned}&\left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right| \\ =\ & \left|\int_t^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds\right|\end{aligned}$$

$$\leq \begin{cases}\displaystyle\int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[a, \frac{a+b}{2}\right] \\ \displaystyle\int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[\frac{a+b}{2}, b\right]\end{cases} \tag{2.12}$$

A combination of (2.10), (2.11) and(2.12), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g\left(b\right)+J_{b^-}^{\alpha}g\left(a\right)\right]-\left[J_{a^+}^{\alpha}\left(fg\right)\left(b\right)+J_{b^-}^{\alpha}\left(fg\right)\left(a\right)\right]\right| \qquad (2.13)\\
\leq\;& \frac{1}{\Gamma\left(\alpha\right)}\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\left(b-s\right)^{\alpha-1}g\left(s\right)\right|ds\right)\left(\sup\left\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right\}\right)dt\\
&+\frac{1}{\Gamma\left(\alpha\right)}\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\left(b-s\right)^{\alpha-1}g\left(s\right)\right|ds\right)\sup\left\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right\}dt\\
\leq\;& \frac{\left\|g\right\|_{\infty}\sup\left\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right\}}{\Gamma\left(\alpha\right)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\left(b-s\right)^{\alpha-1}\right|ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\left(b-s\right)^{\alpha-1}\right|ds\right)dt\right]\\
=\;& \frac{\left\|g\right\|_{\infty}\sup\left\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right\}}{\Gamma\left(\alpha+1\right)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left[\left(b-t\right)^{\alpha}-\left(t-a\right)^{\alpha}\right]dt+\int_{\frac{a+b}{2}}^{b}\left[\left(t-a\right)^{\alpha}-\left(b-t\right)^{\alpha}\right]dt\right].
\end{aligned}$$

Since

$$\int_a^{\frac{a+b}{2}}\left(b-t\right)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}\left(t-a\right)^{\alpha}dt=\frac{\left(b-a\right)^{\alpha+1}\left(2^{\alpha+1}-1\right)}{2^{\alpha+1}\left(\alpha+1\right)} \qquad (2.14)$$

and

$$\int_a^{\frac{a+b}{2}}\left(t-a\right)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}\left(b-t\right)^{\alpha}dt=\frac{\left(b-a\right)^{\alpha+1}}{2^{\alpha+1}\left(\alpha+1\right)}. \qquad (2.15)$$

Hence, if we use (2.14) and (2.15) in (2.13), we obtaion the desired result. This completes the proof.

Remark 2.1. *In Theorem 1.5, if we take $g\left(x\right)=1$, then inequality (2.9), becomes inequality (1.3) of Theorem 1.2.*

Theorem 2.6. *Let $f:I\rightarrow\mathbb{R}$ be a differentiable mapping on I° and $f'\in L\left[a,b\right]$ with $a<b$. If $\left|f'\right|^{q}$, $q\geq1$, is quasi convex on $\left[a,b\right]$ and $g:\left[a,b\right]\rightarrow\mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g\left(b\right)+J_{b^-}^{\alpha}g\left(a\right)\right]-\left[J_{a^+}^{\alpha}\left(fg\right)\left(b\right)+J_{b^-}^{\alpha}\left(fg\right)\left(a\right)\right]\right| \qquad (2.16)\\
\leq\;& \frac{2\left(b-a\right)^{\alpha+1}\left\|g\right\|_{\infty}}{\left(\alpha+1\right)\Gamma\left(\alpha+1\right)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\sup\left\{\left|f'\left(a\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha>0$.

Proof. Using Lemma 1.2, Power mean inequality, (2.12) and the quasi convexity of $|f'|^q$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \tag{2.17}$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right. \\
&\quad\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right]^{1-\frac{1}{q}} \\
&\quad\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^q dt\right. \\
&\quad\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^q dt\right]^{\frac{1}{q}} \\
&\le \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where it is easily seen that

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\right|ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}\right|ds\right)dt \\
&= \frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right).
\end{aligned}$$

Hence if we use (2.14) and (2.15) in (2.17), we obtain the desired result. This completes the proof.

We can state another inequality for $q>1$ as follows:

Theorem 2.7. *Let $f: I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q>1$, is quasi convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

(i)

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \\
&\le \frac{2^{\frac{1}{p}}\|g\|_\infty(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.18}$$

with $\alpha>0$.

(ii)

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \\
&\le \frac{\|g\|_\infty(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.19}$$

for $0<\alpha\le 1$, where $1/p+1/q=1$.

Proof. (i) Using Lemma 1.2, Hölder's inequality, (2.12) and the quasi convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \qquad (2.20)\\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
=\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]^p dt\right)^{\frac{1}{p}}\\
&\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
\leq\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right] dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt\right)^{\frac{1}{p}}\\
&\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
\leq\ & \frac{2^{\frac{1}{p}}\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^p\leq(1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in\left[0,\frac{1}{2}\right]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^p\leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t\in\left[\frac{1}{2},1\right]$ which follows from $(A-B)^q\leq A^q-B^q$ for any $A\geq B\geq 0$ and $q\geq 1$. Hence the inequality (2.18) is proved.

(ii) The inequality (2.19) is easily proved using the inequality (2.20) and Lemma 1.3.

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